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Summability of formal solutions for singular first-order linear PDEs with holomorphic coefficients

By

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Abstract

This article is concerned with the study of the Borel summability of divergent power series solutions for singular first-order linear partial differential equations of nilpotent type. In order to assure the Borel summability of divergent solutions, global analytic continuation properties for coefficients are required despite the fact that the domain of the Borel sum is local.

§ 1. Introduction and Main Result

In this paper we study the following first-order linear partial differential equation with two complex variables:

$$\{A(x,y)D_x + B(x,y)D_y + 1\}\, u(x,y) = f(x,y), \tag{1.1}$$

where x, $y \in \mathbb{C}$, $D_x = \partial/\partial x$, $D_y = \partial/\partial y$. The coefficients A, B and f are holomorphic at $(x,y) = (0,0) \in \mathbb{C}^2$.

Throughout this paper we always assume the following three fundamental conditions:

$$A(x,0) \equiv 0, \tag{1.2}$$

$$\frac{\partial A}{\partial y}(0,0) \neq 0, \tag{1.3}$$

$$B(x,0) \equiv \frac{\partial B}{\partial y}(x,0) \equiv 0. \tag{1.4}$$

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Remark 1. Conditions (1.2) and (1.4) imply $A(0,0) = B(0,0) = 0$, which means that (1.1) is *singular at the origin*. Moreover, it follows from (1.2)–(1.4) that the Jacobi matrix $\partial(A,B)/\partial(x,y)|_{(x,y)=(0,0)}$ is a nilpotent matrix

$$\begin{pmatrix} 0 & (\partial A/\partial y)(0,0) \\ 0 & 0 \end{pmatrix}.$$

In this sense, our equation is called of *nilpotent type*.

First of all, let us consider the existence of formal power series solutions $u(x,y) = \sum_{m,n=0}^{\infty} u_{mn} x^m y^n$ around $(x,y) = (0,0)$. Then, under the above conditions we can prove the unique existence of $u(x,y)$. Moreover, we see that it takes the form of $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n$, where $u_n(x)$ are holomorphic in a common neighborhood of $x = 0$. However, because of the singularity of (1.1) at the origin, this formal power series solution $u(x,y)$ with respect to y-variable diverges in general and the rate of divergence is characterized in terms of the Gevrey index (cf. Definition 1.1, (1)–(3) and Theorem 1.2). So, we are interested in the Borel summability of such a divergent solution (cf. Definition 1.1, (4)–(6)). *Our main purpose in this paper is to obtain the conditions under which the divergent solution is Borel summable.*

The content of this paper is as follows. In §1.1, we give the definitions of divergent power series of the Gevrey type and the Borel summability. Moreover, we state the theorem which assures the unique existence of divergent power series solutions (Theorem 1.2). From §1.2, we consider the problem of the Borel summability. In §1.2, we introduce some results obtained in [4, 5] (cf. Theorems 1.3 and 1.4). Main Theorem in this paper is a generalization of Theorem 1.3, and it is stated in §1.3. In order to assure the Borel summability of the divergent solution, global analytic continuation properties for coefficients are required. The proof of Main Theorem is given in §2–§4. In §2, the proof of Main Theorem is reduced to that of a global solvability of the initial value problem of some convolution equation. §3 and §4 constitute the main part of the proof. We transform the convolution equation obtained in §2 into some integral equation, and prove the global solvability of that integral equation by applying an iteration method.

§ 1.1. Definition and Fundamental Result

Definition 1.1. (1) $\mathcal{O}[R]$ denotes the ring of holomorphic functions on the closed ball $B(R) = \{x \in \mathbb{C};\ |x| \le R\}$, where R is a positive number.

(2) The ring of formal power series in y ($\in \mathbb{C}$) over the ring $\mathcal{O}[R]$ is denoted as $\mathcal{O}[R][[y]]$:

$$\mathcal{O}[R][[y]] = \left\{ u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n;\ u_n(x) \in \mathcal{O}[R] \right\}. \tag{1.5}$$

(3) We say that $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n$ ($\in \mathcal{O}[R][[y]]$) *belongs to* $\mathcal{O}[R][[y]]_2$, if there exist some positive constants C and K such that

$$\max_{|x|\le R} |u_n(x)| \le CK^n n! \tag{1.6}$$

for all $n = 0, 1, 2, \ldots$. The suffix 2 of $\mathcal{O}[R][[y]]_2$ expresses the Gevrey index of power series. Elements of $\mathcal{O}[R][[y]]_2$ are divergent power series in general.

(4) For $\theta \in \mathbb{R}$, $\kappa > 0$ and $0 < \rho \le +\infty$, the sector $S(\theta,\kappa,\rho)$ in the universal covering space of $\mathbb{C} \setminus \{0\}$ is defined by

$$S(\theta,\kappa,\rho) = \Big\{y;\ |\arg(y) - \theta| < \frac{\kappa}{2},\ 0 < |y| < \rho\Big\}. \tag{1.7}$$

We refer to θ, κ and ρ as the *bisecting direction*, the *opening angle* and the *radius* of $S(\theta,\kappa,\rho)$, respectively.

(5) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ and let $U(x,y)$ be a holomorphic function on $X = B(R)\times S(\theta,\kappa,\rho)$. Then we say that $U(x,y)$ *has* $u(x,y)$ *as an asymptotic expansion of the Gevrey order* 2 *in* X if the following asymptotic estimates hold: there exist some positive constants C and K such that

$$\max_{|x|\le R} \left| U(x,y) - \sum_{n=0}^{N-1} u_n(x)y^n \right| \le CK^N N! |y|^N, \tag{1.8}$$

for all $y \in S(\theta,\kappa,\rho)$ and $N = 1, 2, \ldots$. Then we write this as

$$U(x,y) \cong_2 u(x,y) \text{ in } X.$$

(6) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$. We say that $u(x,y)$ *is Borel summable in a direction* θ if there exists a holomorphic function $U(x,y)$ on $X = B(r)\times S(\theta,\kappa,\rho)$ for some $0 < r \le R$ and $\kappa > \pi$ which satisfies $U(x,y) \cong_2 u(x,y)$ in X. A given divergent power series $u(x,y) \in \mathcal{O}[R][[y]]_2$ is not necessarily Borel summable in general. However, if $u(x,y)$ is Borel summable in a direction θ, then we see that the above holomorphic function $U(x,y)$ is unique (cf. Balser[1, 2]). So we call this unique $U(x,y)$ *the Borel sum of* $u(x,y)$ *in a direction* θ.

Now we already know the following fact, which will be fundamental in the argument below.

Theorem 1.2 ([3]). *Let us assume* (1.2)–(1.4). *Then* (1.1) *has a unique formal power series solution* $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ *for some* $R > 0$.

Remark 2. The Gevrey index 2 of formal solutions $u(x,y)$ (that is, estimates (1.6)) is optimal. For example, let us consider the following simple equation:

$$\alpha y D_x u(x,y) + u(x,y) = f(x), \tag{1.9}$$

where α is a constant satisfying $\alpha \neq 0$. Equation (1.9) has a unique formal solution $u(x,y) = \sum_{n=0}^{\infty} (-\alpha)^n f^{(n)}(x) y^n$. Hence, if $f(x) = 1/(1-x)$, for example, it holds that $u_n(x) \equiv (-\alpha)^n f^{(n)}(x) = (-\alpha)^n n!(1-x)^{n+1}$. Therefore, in this case, the Gevrey index of $u(x,y)$ is exactly 2.

On the basis of Theorem 1.2, we can study the coming problem; the Borel summability of the formal solution: *When is the formal solution Borel summable in a given bisecting direction θ?* For simplicity, throughout this paper we consider the case where

$$\frac{\partial A}{\partial y}(x,0) \equiv \alpha \quad \text{(constant)}. \tag{1.10}$$

We remark that $\alpha \neq 0$ by (1.3).

§ 1.2. Known Results

Before stating the main result in this paper, in this subsection we introduce some results obtained in [4, 5]. First of all let us rewrite the equation (1.1). It follows from (1.2)–(1.4) and (1.10) that (1.1) is rewritten in the following form:

$$\{\alpha + \beta(x,y)\} y D_x u(x,y) + \{a(x) + b(x,y)\} y^2 D_y u(x,y) + u(x,y) = f(x,y), \tag{1.11}$$

where each coefficient is holomorphic at the origin. Moreover β and b satisfy

$$\beta(x,0) \equiv b(x,0) \equiv 0. \tag{1.12}$$

In [4, 5] we studied the case where

$$a(x) \equiv a \quad \text{(constant)}, \tag{1.13}$$

and obtained the conditions under which the formal solution is Borel summable. Here we state those conditions.

Case (I): $a = 0$.

In this case, (1.11) is written as follows:

$$\{\alpha + \beta(x,y)\} y D_x u(x,y) + b(x,y) y^2 D_y u(x,y) + u(x,y) = f(x,y). \tag{1.14}$$

Let us give assumptions. First we define the region $\Omega_{r,\theta,\kappa}$ by

$$\Omega_{r,\theta,\kappa} = B(r) \cup S(\theta + \arg\alpha + \pi, \kappa, +\infty). \tag{1.15}$$

(A1) $\beta(x,y)$, $b(x,y)$ and $f(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \leq c\}$ for some positive r, κ and c. Moreover, they satisfy the following estimates there:

$$\sup_{x\in\Omega_{r,\theta,\kappa},\, |y|\le c} |\beta(x,y)| < \infty; \tag{1.16}$$

$$\max_{|y|\le c} |b(x,y)| \le \frac{K}{(1+|x|)^p}, \qquad x\in\Omega_{r,\theta,\kappa}; \tag{1.17}$$

$$\max_{|y|\le c} |f(x,y)| \le Ce^{\delta|x|}, \qquad x\in\Omega_{r,\theta,\kappa}, \tag{1.18}$$

where K, C and δ are positive constants independent of $x\in\Omega_{r,\theta,\kappa}$ and y with $|y|\le c$. p is the constant satisfying $p>1$.

In [4] we obtained the following theorem.

Theorem 1.3 ([4]). *Under assumption* (A1) *the formal solution* $u(x,y)$ *of* (1.14) *is Borel summable in the direction* θ.

Remark 3. By applying Cauchy's integral formula, we see that (1.16) and (1.17) are equivalent to the following estimates (1.19) and (1.20), respectively. There exist some positive constants K and L such that

$$\left|\frac{\partial^m\beta}{\partial y^m}(x,0)\right| \le KL^m m!, \qquad x\in\Omega_{r,\theta,\kappa}, \qquad m=1,2,\dots; \tag{1.19}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x,0)\right| \le KL^m m!\frac{1}{(1+|x|)^p}, \qquad x\in\Omega_{r,\theta,\kappa}, \qquad m=1,2,\dots. \tag{1.20}$$

In the next case $a\ne 0$, we will give the conditions in such forms as (1.19) and (1.20).

Case (II): $a\ne 0$.

In this case, (1.11) is written as follows:

$$\{\alpha+\beta(x,y)\}yD_xu(x,y)+\{a+b(x,y)\}y^2D_yu(x,y)+u(x,y)=f(x,y). \tag{1.21}$$

Let us give assumptions. We define the region $\Phi_{r,\theta,\kappa}$ by

$$\Phi_{r,\theta,\kappa} = \left\{-\frac{\alpha}{a}\log(1+a\tau);\ \tau\in B(r)\cup S(\theta,\kappa,+\infty)\right\}. \tag{1.22}$$

In order to ensure the well-definedness of $\Phi_{r,\theta,\kappa}$, we always assume

(A1)′ $\theta\ne\arg(-1/a)$.

For the inhomogeneity term $f(x,y)$ we assume the following.

(A2)′ $f(x,y)$ can be continued analytically to $\Phi_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\le c\}$ for some positive r, κ and c. Moreover, it has the following estimate there. There exist some positive constants C and δ such that

$$\max_{|y|\le c}|f(x,y)| \le C\exp\left[\delta\left|\exp\left(-\frac{a}{\alpha}x\right)\right|\right], \qquad x\in\Phi_{r,\theta,\kappa}. \tag{1.23}$$

For the coefficients $\beta(x,y)$ and $b(x,y)$, we impose the following conditions.

(A3)′ $\beta(x,y)$ and $b(x,y)$ can be continued analytically to $\Phi_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\le c\}$. Moreover, there exist some positive constants K and L, which are independent of m, and $p_m<m$ such that

$$\left|\frac{\partial^m\beta}{\partial y^m}(x,0)\right|\le KL^m m!\left|\exp\left(-\frac{a}{\alpha}x\right)\right|^{p_m},\qquad x\in\Phi_{r,\theta,\kappa},\qquad m=1,2,\ldots; \tag{1.24}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x,0)\right|\le KL^m m!\left|\exp\left(-\frac{a}{\alpha}x\right)\right|^{p_m},\qquad x\in\Phi_{r,\theta,\kappa},\qquad m=1,2,\ldots. \tag{1.25}$$

Finally, we assume that

(A4)′ $\inf\{m-p_m;\ m=1,2,\ldots\}>0$.

In [5] we obtained the following theorem.

Theorem 1.4 ([5]). *Under assumptions* (A1)′–(A4)′ *the formal solution* $u(x,y)$ *of* (1.21) *is Borel summable in the direction* θ.

§ 1.3. Main Result

In [4, 5] we studied the case where $a(x)$ is the constant. In this paper, we study the case where $a(x)$ is the linear function. Precisely, we consider the case where

$$a(x)=ax. \tag{1.26}$$

In this case, (1.11) is written as follows:

$$\{\alpha+\beta(x,y)\}yD_xu(x,y)+\{ax+b(x,y)\}y^2D_yu(x,y)+u(x,y)=f(x,y). \tag{1.27}$$

Assumptions. First let us consider the following initial value problem:

$$\frac{d\xi}{d\tau}=\exp\left(\frac{\alpha a}{2}\xi^2\right),\qquad \xi(0)=0. \tag{1.28}$$

Then we assume the following:

(Assumption 1) (1.28) has a holomorphic solution $\xi=F(\tau)$ on the region $B(r)\cup S(\theta,\kappa,+\infty)$ for some $r>0$ and $\kappa>0$.

It is obvious that $F(\tau)$ is unique, if it exists.

Next, let us define the region $\Xi_{r,\theta,\kappa}$ consisting of the image of F by

$$\Xi_{r,\theta,\kappa}=\{F(\tau);\ \tau\in B(r)\cup S(\theta,\kappa,+\infty)\}, \tag{1.29}$$

and let us assume the following:

(Assumption 2) $\sup_{\xi \in \Xi_{r,\theta,\kappa}} \left| \exp\left(\frac{\alpha a}{2}\xi^2\right)\right| < \infty.$

Next, in order to state assumptions for coefficients, we define the region $\Omega_{r,\theta,\kappa}$ by

$$\Omega_{r,\theta,\kappa} = -\alpha \cdot \Xi_{r,\theta,\kappa} = \{-\alpha \cdot \xi;\ \xi \in \Xi_{r,\theta,\kappa}\}. \tag{1.30}$$

For the inhomogeneity term $f(x, y)$ we assume the following.

(Assumption 3) $f(x, y)$ can be continued analytically to $\Omega_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$ for some $c > 0$. Moreover, it has the following estimate there. There exist some positive constants C and δ such that

$$\max_{|y|\le c} |f(x, y)| \le C \exp\left[\delta \left|\mathscr{F}\left(-\frac{1}{\alpha}x\right)\right|\right], \qquad x \in \Omega_{r,\theta,\kappa}, \tag{1.31}$$

where $\mathscr{F}$ is the entire function defined by

$$\mathscr{F}(\xi) = \int_0^{\xi} \exp\left(-\frac{\alpha a}{2}\zeta^2\right) d\zeta. \tag{1.32}$$

Finally, we impose the following conditions for the coefficients $\beta(x, y)$ and $b(x, y)$:

(Assumption 4) $\beta(x, y)$ and $b(x, y)$ can be continued analytically to $\Omega_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$. Moreover, there exist some positive constants K, $L > 0$ and $p > 1$ such that

$$\left|\frac{\partial^m \beta}{\partial y^m}(x, 0)\right| \le KL^m m!|E(x)|^m, \qquad x \in \Omega_{r,\theta,\kappa}, \qquad m = 1, 2, \ldots; \tag{1.33}$$

$$\left|\frac{\partial^m \beta}{\partial y^m}(x, 0) \cdot ax\right| \le \frac{KL^m m!|E(x)|^{m+1}}{\{1 + |\mathscr{F}(-(1/\alpha)x)|\}^p}, \qquad x \in \Omega_{r,\theta,\kappa}, \qquad m = 1, 2, \ldots; \tag{1.34}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x, 0)\right| \le \frac{KL^m m!|E(x)|^{m+1}}{\{1 + |\mathscr{F}(-(1/\alpha)x)|\}^p}, \qquad x \in \Omega_{r,\theta,\kappa}, \qquad m = 1, 2, \ldots, \tag{1.35}$$

where $E(x)$ is the entire function defined by

$$E(x) = \exp\left(-\frac{a}{2\alpha}x^2\right). \tag{1.36}$$

Let us state the main result in this paper.

Main Theorem. *Under assumptions* (Assumption 1)–(Assumption 4) *the formal solution $u(x, y)$ of* (1.27) *is Borel summable in the direction θ.*

Remark 4. If $a = 0$, then the equation (1.27) is same as (1.14). In this case, (Assumption 1) and (Assumption 2) are always satisfied ($\exp\left((\alpha a/2)\xi^2\right) \equiv 1$). (1.31) in (Assumption 3) is equivalent to (1.18) in (A1) ($\mathscr{F}(\xi) = \xi$). Moreover, (1.33) and (1.35) are equivalent to (1.16) and (1.17) in (A1), respectively (cf. Remark 3; $E(x) \equiv 1$), and (1.34) is always satisfied. Consequently, Main Theorem gives one of the generalizations of Theorem 1.3.

§ 2. Formal Borel Transform of Equations

In this section, we reduce the proof of Main Theorem to that of a global solvability of the initial value problem of some convolution equation. First we give some preliminaries.

Definition 2.1. For $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$, we define the convergent power series $\mathcal{B}[u](x,\eta)$ in a neighborhood of $(x,\eta) = (0,0)$ by

$$\mathcal{B}[u](x,\eta) = \sum_{n=0}^{\infty} u_n(x) \frac{\eta^n}{n!}. \tag{2.1}$$

We call $\mathcal{B}[u](x,\eta)$ the *formal Borel transform of* $u(x,y)$.

When we would like to check the Borel summability of a given formal power series $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$, the following theorem plays a fundamental role in general.

Theorem 2.2 ([1, 2]). *For a formal power series* $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$, *let us put* $v(x,\eta) = \mathcal{B}[u](x,\eta)$. *Then the following two conditions* (i) *and* (ii) *are equivalent.*

(i) $u(x,y)$ *is Borel summable in a direction* θ.

(ii) $v(x,\eta)$ *can be continued analytically to* $B(r_0) \times S(\theta, \kappa_0, +\infty)$ *for some* $r_0 > 0$ *and* $\kappa_0 > 0$, *and has the following exponential growth estimate for some positive constants* C *and* δ:

$$\max_{|x| \le r_0} |v(x,\eta)| \le C e^{\delta|\eta|}, \qquad \eta \in S(\theta, \kappa_0, +\infty). \tag{2.2}$$

When condition (i) *or* (ii) (*therefore both*) *is satisfied, the Borel sum* $U(x,y)$ *of* $u(x,y)$ *in the direction* θ *is given by*

$$U(x,y) = \frac{1}{y} \int_0^{\infty(\theta)} e^{-\eta/y} v(x,\eta)\, d\eta. \tag{2.3}$$

Thus, in order to prove Main Theorem, it is sufficient to prove that the formal Borel transform $v(x,\eta) = \mathcal{B}[u](x,\eta)$ of the formal solution $u(x,y)$ satisfies the above condition (ii) under assumptions (Assumption 1)–(Assumption 4). In order to do that, first let us write down the equation which $\mathcal{B}[u](x,\eta)$ should satisfy. By operating the formal Borel transform to (1.27), we obtain the following equality:

$$\begin{aligned}(2.4)\quad &\alpha\int_0^\eta \mathcal{B}[D_x u](x,t)\,dt+\int_0^\eta \mathcal{B}[\beta](x,\eta-t)\mathcal{B}[D_x u](x,t)\,dt\\ &+ax\int_0^\eta \mathcal{B}[yD_y u](x,t)\,dt+\int_0^\eta \mathcal{B}[b](x,\eta-t)\mathcal{B}[yD_y u](x,t)\,dt+\mathcal{B}[u](x,\eta)\\ &=\mathcal{B}[f](x,\eta),\end{aligned}$$

where $\mathcal{B}[\beta](x,\eta)$, $\mathcal{B}[b](x,\eta)$ and $\mathcal{B}[f](x,\eta)$ are the formal Borel transforms of $\beta(x,y)=\sum_{n=1}^\infty \beta_n(x)y^n$, $b(x,y)=\sum_{n=1}^\infty b_n(x)y^n$ and $f(x,y)=\sum_{n=0}^\infty f_n(x)y^n$, respectively, that is,

$$\mathcal{B}[\beta](x,\eta)=\sum_{n=1}^\infty \beta_n(x)\frac{\eta^n}{n!},\qquad \mathcal{B}[b](x,\eta)=\sum_{n=1}^\infty b_n(x)\frac{\eta^n}{n!},\qquad \mathcal{B}[f](x,\eta)=\sum_{n=0}^\infty f_n(x)\frac{\eta^n}{n!}.$$

(2.4) is obtained by applying the following equality:

$$\begin{aligned}\mathcal{B}[y^{m+n+1}](\eta)&=\frac{1}{(m+n+1)!}\eta^{m+n+1}=B(m+1,n+1)\frac{\eta^{m+n+1}}{m!n!}\quad\text{(Beta integral)}\\ &=\int_0^1(1-s)^m s^n\,ds\cdot\frac{\eta^{m+n+1}}{m!n!}=\int_0^\eta(\eta-t)^m t^n\,dt\cdot\frac{1}{m!n!}\\ &=\int_0^\eta \mathcal{B}[y^m](\eta-t)\mathcal{B}[y^n](t)\,dt.\end{aligned}$$

Next, let us calculate $\mathcal{B}[D_x u](x,\eta)$ and $\mathcal{B}[yD_y u](x,\eta)$. It is clear that $\mathcal{B}[D_x u](x,\eta)=D_x\mathcal{B}[u](x,\eta)$. On $\mathcal{B}[yD_y u](x,\eta)$, it follows from the same argument as the above that $\mathcal{B}[yD_y u](x,\eta)=\int_0^\eta \mathcal{B}[D_y u](x,t)\,dt$. Moreover, by noting the commutative diagram

$$\begin{array}{ccc} y^n & \xrightarrow{\text{Borel tr.}} & \dfrac{\eta^n}{n!}\\ {\scriptstyle D_y}\big\downarrow & & \big\downarrow{\scriptstyle D_\eta\eta D_\eta}\\ ny^{n-1} & \xrightarrow[\text{Borel tr.}]{} & n\dfrac{\eta^{n-1}}{(n-1)!}\end{array}$$

we have $\mathcal{B}[D_y u](x,\eta)=D_\eta\eta D_\eta\mathcal{B}[u](x,\eta)$. Hence, it holds that

$$(2.5)\quad \mathcal{B}[yD_y u](x,\eta)=\int_0^\eta \mathcal{B}[D_y u](x,t)\,dt=\int_0^\eta D_t t D_t\mathcal{B}[u](x,t)\,dt=\eta D_\eta\mathcal{B}[u](x,\eta).$$

By adopting (2.5) we obtain

$$
\begin{aligned}
&\int_0^\eta \mathcal{B}[b](x,\eta-t)\mathcal{B}[yD_yu](x,t)\,dt \\
&\quad= \int_0^\eta \mathcal{B}[b](x,\eta-t)\cdot t\cdot D_t\mathcal{B}[u](x,t)\,dt \\
&\quad= \big[\mathcal{B}[b](x,\eta-t)\cdot t\cdot \mathcal{B}[u](x,t)\big]_{t=0}^{\eta} - \int_0^\eta \frac{\partial}{\partial t}\big\{\mathcal{B}[b](x,\eta-t)\cdot t\big\}\cdot \mathcal{B}[u](x,t)\,dt \\
&\quad= \int_0^\eta \mathcal{B}[b]_\eta(x,\eta-t)\cdot t\cdot \mathcal{B}[u](x,t)\,dt - \int_0^\eta \mathcal{B}[b](x,\eta-t)\mathcal{B}[u](x,t)\,dt.
\end{aligned}
$$

Therefore, we see that $\mathcal{B}[u](x,\eta)$ is a solution of the following equation:

$$
\begin{aligned}
(2.6)\qquad &\alpha\int_0^\eta v_x(x,t)\,dt + \int_0^\eta \mathcal{B}[\beta](x,\eta-t)v_x(x,t)\,dt + ax\int_0^\eta t\cdot v_\eta(x,t)\,dt \\
&\quad + \int_0^\eta \mathcal{B}[b]_\eta(x,\eta-t)\cdot t\cdot v(x,t)\,dt - \int_0^\eta \mathcal{B}[b](x,\eta-t)v(x,t)\,dt + v(x,\eta) \\
&\quad = \mathcal{B}[f](x,\eta),
\end{aligned}
$$

Finally, let us operate D_η to (2.6) from the left. Then we see that $\mathcal{B}[u](x,\eta)$ is a solution of the following initial value problem:

$$
(2.7)\quad
\begin{cases}
\mathscr{L}v(x,\eta) = -\displaystyle\int_0^\eta \mathcal{B}[\beta]_\eta(x,\eta-t)v_x(x,t)\,dt - \mathcal{B}[b]_\eta(x,0)\cdot\eta\cdot v(x,\eta) \\
\qquad\qquad\quad -\displaystyle\int_0^\eta \mathcal{B}[b]_{\eta\eta}(x,\eta-t)\cdot t\cdot v(x,t)\,dt + \int_0^\eta \mathcal{B}[b]_\eta(x,\eta-t)v(x,t)\,dt \\
\qquad\qquad\quad + g(x,\eta), \\
v(x,0) = f(x,0),
\end{cases}
$$

where $\mathscr{L}$ is the first-order linear partial differential operator defined by

$$
(2.8)\qquad \mathscr{L} = \alpha D_x + (1+ax\eta)D_\eta,
$$

and $g(x,\eta) = \mathcal{B}[f]_\eta(x,\eta)$. It is easy to prove that $\mathcal{B}[u](x,\eta)$ is the unique locally holomorphic solution of (2.7). Hence, Main Theorem will be proved by showing that the solution $v(x,\eta)$ of (2.7) satisfies condition (ii) in Theorem 2.2.

§ 3. Proof of Main Theorem

Let us start the proof of Main Theorem.

Proof. First of all, we transform the convolution equation (2.7) into the integral equation. We apply the following formula. The solution $w(x,\eta)$ of the initial value problem of the following first-order linear partial differential equation

$$\begin{cases} \mathscr{L}w(x,\eta) = k(x,\eta), \\ w(x,0) = l(x) \end{cases} \tag{3.1}$$

is given by

$$w(x,\eta) = l\big(-\alpha\cdot\mathscr{A}(x,\eta)\big) + \int_0^\eta k\big(-\alpha\cdot\mathscr{A}(x,\eta-z), \mathscr{E}(x,\eta-z)\cdot z\big)\cdot\mathscr{E}(x,\eta-z)\,dz, \tag{3.2}$$

where

$$\mathscr{A}(x,\eta) = F\left(\mathscr{F}\Big(-\frac{1}{\alpha}x\Big) + \Big\{\exp\Big(-\frac{a}{2\alpha}x^2\Big)\Big\}\cdot\eta\right), \tag{3.3}$$

$$\mathscr{E}(x,\eta) = \frac{\exp\big\{(\alpha a/2)\cdot\mathscr{A}(x,\eta)^2\big\}}{\exp\big\{(a/2\alpha)x^2\big\}}. \tag{3.4}$$

By (3.2), we see that (2.7) is equivalent to the following equation:

$$\begin{aligned} v(x,\eta) = {} & f\big(-\alpha\cdot\mathscr{A}(x,\eta),0\big) \\ & + \int_0^\eta g\big(-\alpha\cdot\mathscr{A}(x,\eta-z), \mathscr{E}(x,\eta-z)\cdot z\big)\cdot\mathscr{E}(x,\eta-z)\,dz \\ & + \sum_{i=1}^4 J_i v(x,\eta), \end{aligned} \tag{3.5}$$

where each operator J_i ($i = 1, 2, 3, 4$) is given by

$$\begin{aligned} J_1v(x,\eta) = {} & -\int_0^\eta\int_0^{\mathscr{E}(x,\eta-z)\cdot z} \mathcal{B}[\beta]_\eta\big(-\alpha\cdot\mathscr{A}(x,\eta-z), \mathscr{E}(x,\eta-z)\cdot z - t\big) \\ & \times v_x\big(-\alpha\cdot\mathscr{A}(x,\eta-z),t\big)\,dt\,\mathscr{E}(x,\eta-z)\,dz, \\ J_2v(x,\eta) = {} & -\int_0^\eta \mathcal{B}[b]_\eta\big(-\alpha\cdot\mathscr{A}(x,\eta-z),0\big)\cdot\mathscr{E}(x,\eta-z)\cdot z \\ & \times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z), \mathscr{E}(x,\eta-z)\cdot z\big)\cdot\mathscr{E}(x,\eta-z)\,dz, \\ J_3v(x,\eta) = {} & -\int_0^\eta\int_0^{\mathscr{E}(x,\eta-z)\cdot z} \mathcal{B}[b]_{\eta\eta}\big(-\alpha\cdot\mathscr{A}(x,\eta-z), \mathscr{E}(x,\eta-z)\cdot z - t\big)\cdot t \\ & \times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),t\big)\,dt\,\mathscr{E}(x,\eta-z)\,dz, \\ J_4v(x,\eta) = {} & \int_0^\eta\int_0^{\mathscr{E}(x,\eta-z)\cdot z} \mathcal{B}[b]_\eta\big(-\alpha\cdot\mathscr{A}(x,\eta-z), \mathscr{E}(x,\eta-z)\cdot z - t\big) \\ & \times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),t\big)\,dt\,\mathscr{E}(x,\eta-z)\,dz \end{aligned}$$

Furthermore, let us practice an integration by substitution, change the order of integrals and an integration by parts. Then we see that (3.5) is equivalent to the following integral equation:

$$v(x,\eta) = f\big(-\alpha\cdot\mathscr{A}(x,\eta),0\big) + \int_0^\eta g\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot z\big)\cdot\mathscr{E}(x,\eta-z)\,dz + \sum_{i=1}^{7} I_i v(x,\eta), \tag{3.6}$$

where each operator I_i ($i = 1, \ldots, 7$) is given by

$$\begin{aligned}
I_1v(x,\eta) &= -\frac{1}{\alpha}\int_0^\eta \mathcal{B}[\beta]_\eta(x,\eta-z)v(x,z)\,dz,\\
I_2v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta \mathscr{E}(x,\eta-z)\cdot\{1-a\alpha\cdot\mathscr{A}(x,\eta-z)\cdot\mathscr{E}(x,\eta-z)\cdot z\}\\
&\quad\times\mathcal{B}[\beta]_\eta\big(-\alpha\cdot\mathscr{A}(x,\eta-z),0\big)\cdot v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot z\big)\,dz,\\
I_3v(x,\eta) &= \frac{1}{\alpha}\int_0^\eta\int_0^z \mathscr{E}(x,\eta-z)^2\cdot\{1-a\alpha\cdot\mathscr{A}(x,\eta-z)\cdot\mathscr{E}(x,\eta-z)\cdot z\}\\
&\quad\times\mathcal{B}[\beta]_{\eta\eta}\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot(z-s)\big)\\
&\quad\times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot s\big)\,ds\,dz,\\
I_4v(x,\eta) &= \int_0^\eta\int_0^z \mathscr{E}(x,\eta-z)^2\cdot\mathcal{B}[\beta]_{x\eta}\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot(z-s)\big)\\
&\quad\times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot s\big)\,ds\,dz,\\
I_5v(x,\eta) &= -\int_0^\eta \mathscr{E}(x,\eta-z)^2\cdot z\cdot\mathcal{B}[b]_\eta\big(-\alpha\cdot\mathscr{A}(x,\eta-z),0\big)\\
&\quad\times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot z\big)\,dz,\\
I_6v(x,\eta) &= -\int_0^\eta\int_0^z \mathscr{E}(x,\eta-z)^3\cdot s\cdot\mathcal{B}[b]_{\eta\eta}\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot(z-s)\big)\\
&\quad\times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot s\big)\,ds\,dz,\\
I_7v(x,\eta) &= \int_0^\eta\int_0^z \mathscr{E}(x,\eta-z)^2\cdot\mathcal{B}[b]_\eta\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot(z-s)\big)\\
&\quad\times v\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot s\big)\,ds\,dz.
\end{aligned}$$

In order to prove that the solution $v(x,\eta)$ of (3.6) satisfies condition (ii) in Theorem 2.2, we employ the iteration method. Let us define $\{v_n(x,\eta)\}_{n=0}^\infty$ inductively as follows:

$$v_0(x,\eta) = f\big(-\alpha\cdot\mathscr{A}(x,\eta),0\big) + \int_0^\eta g\big(-\alpha\cdot\mathscr{A}(x,\eta-z),\mathscr{E}(x,\eta-z)\cdot z\big)\cdot\mathscr{E}(x,\eta-z)\,dz, \tag{3.7}$$

$$v_{n+1}(x,\eta) = v_0(x,\eta) + \sum_{i=1}^{7} I_i v_n(x,\eta) \qquad (n \geq 0). \tag{3.8}$$

Next, let us define $\{w_n(x,\eta)\}_{n=0}^{\infty}$ by $w_0(x,\eta) = v_0(x,\eta)$ and $w_n(x,\eta) = v_n(x,\eta) - v_{n-1}(x,\eta)$ $(n \geq 1)$.

Here we break the proof, and provide the notation needed in stating the key lemma later. Since $\mathscr{F}(0) = 0$, we can take $r_0 > 0$, $\kappa_0 > 0$ and $l > 0$ such that

$$\begin{aligned} &|x| \leq r_0, \ \ \zeta \in S(\theta, \kappa_0, +\infty) - le^{i\theta} \\ &\qquad \Longrightarrow \ \mathscr{F}\Big(-\frac{1}{\alpha}x\Big) + \Big\{\exp\Big(-\frac{a}{2\alpha}x^2\Big)\Big\} \cdot \zeta \in B(r) \cup S(\theta, \kappa, +\infty), \end{aligned} \tag{3.9}$$

where $r > 0$ and $\kappa > 0$ are the constants given in (Assumption 1). Therefore, it holds that

$$\begin{aligned} &|x| \leq r_0, \ \ \zeta \in S(\theta, \kappa_0, +\infty) - le^{i\theta} \\ &\qquad \Longrightarrow \ \mathscr{A}(x,\zeta) = F\left(\mathscr{F}\Big(-\frac{1}{\alpha}x\Big) + \Big\{\exp\Big(-\frac{a}{2\alpha}x^2\Big)\Big\} \cdot \zeta\right) \in \varXi_{r,\theta,\kappa}. \end{aligned} \tag{3.10}$$

Hence, we obtain

$$|x| \leq r_0, \ \ \zeta \in S(\theta, \kappa_0, +\infty) - le^{i\theta} \quad \Longrightarrow \quad -\alpha \cdot \mathscr{A}(x,\zeta) \in \varOmega_{r,\theta,\kappa}. \tag{3.11}$$

It follows from (1.33)–(1.35), (3.11) and Cauchy's integral formula that there exist some positive constants C_0 and M_0 satisfying:

$$\begin{cases} \max\limits_{|x|\leq r_0} \big|\beta_m\big(-\alpha \cdot \mathscr{A}(x,\zeta)\big) \cdot \mathscr{E}(x,\zeta)^m\big| \leq C_0 {M_0}^m, \\ \max\limits_{|x|\leq r_0} \left|\left[\dfrac{\partial}{\partial\zeta}\Big\{\beta_m\big(-\alpha \cdot \mathscr{A}(x,\zeta)\big)\Big\}\right] \cdot \mathscr{E}(x,\zeta)^m\right| \leq C_0 {M_0}^m, \\ \max\limits_{|x|\leq r_0} \big|\beta_m\big(-\alpha \cdot \mathscr{A}(x,\zeta)\big) \cdot a\alpha \cdot \mathscr{A}(x,\zeta) \cdot \mathscr{E}(x,\zeta)^{m+1}\big| \leq \dfrac{C_0 {M_0}^m}{(1+|\zeta|)^p}, \\ \max\limits_{|x|\leq r_0} \big|b_m\big(-\alpha \cdot \mathscr{A}(x,\zeta)\big) \cdot \mathscr{E}(x,\zeta)^{m+1}\big| \leq \dfrac{C_0 {M_0}^m}{(1+|\zeta|)^p}, \end{cases} \tag{3.12}$$

for all $m = 1, 2, \ldots$ and all $\zeta \in S(\theta, \kappa_0, +\infty) - l'e^{i\theta}$, where $l' = l/2$. Finally, let us take a monotonically decreasing positive sequence $\{l_n\}_{n=0}^{\infty}$ so that

$$l' = \sum_{n=0}^{\infty} l_n. \tag{3.13}$$

Then we obtain the following lemma.

Lemma 3.1. *$w_n(x,\eta)$ is continued analytically to $B(r_0) \times \{S(\theta, \kappa_0, +\infty) - (l' - \sum_{j=0}^{n} l_j)e^{i\theta}\}$. Moreover, we have the following estimate. For some positive constants $C_1 > 0$ and $\delta_0 > \max\{1, M_0\}$,*

(3.14)

$$\max_{|x|\leq r_0} |w_n(x,\eta)| \leq C_1 e^{\delta_0|\eta|} {M_1}^n \Big(1 + \frac{1}{p-1}\Big)^n \frac{|\eta|^n}{n!}, \quad \eta \in S(\theta, \kappa_0, +\infty) - \Big(l' - \sum_{j=0}^{n} l_j\Big)e^{i\theta},$$

where

$$M_1 = 4C_0 \left\{ \Big(\frac{2}{|\alpha|} + 1 \Big) \sum_{m=1}^{\infty} \frac{{M_0}^m}{{\delta_0}^{m-1}} \right\}.$$

We prove Lemma 3.1 in §4. For the present, we admit it and let us continue the proof of Main Theorem. It follows from Lemma 3.1 that

$$\begin{aligned} \max_{|x|\le r_0} \sum_{n=0}^{\infty} |w_n(x,\eta)| &\le C_1 e^{\delta_0|\eta|} \sum_{n=0}^{\infty} {M_1}^n \Big(1 + \frac{1}{p-1}\Big)^n \frac{|\eta|^n}{n!} \\ &= C_1 e^{\delta_1|\eta|} \end{aligned}$$

for $\eta \in S(\theta, \kappa_0, +\infty)$, where $\delta_1 = \delta_0 + M_1(1 + 1/(p-1))$. This shows that $v_n(x,\eta)$ $(= \sum_{k=0}^{n} w_k(x,\eta))$ converges to the solution $V(x,\eta)$ of (3.6) uniformly on $B(r_0) \times S(\theta, \kappa_0, +\infty)$. Consequently, $V(x,\eta)$ is an analytic continuation of $v(x,\eta)$, and it holds that

$$\max_{|x|\le r_0} |V(x,\eta)| \le C_1 e^{\delta_1|\eta|}, \qquad \eta \in S(\theta, \kappa_0, +\infty).$$

It follows from the above argument that $v(x,\eta)$ satisfies condition (ii) in Theorem 2.2. This completes the proof of Main Theorem. □

§ 4. Proof of Lemma 3.1

Let us prove Lemma 3.1. It is proved by the induction with respect to n.

Proof. In the case $n = 0$, the lemma follows from (Assumption 1)–(Assumption 3). If we assume the analytic continuation property for $w_n(x,\eta)$, then we can prove that for $w_{n+1}(x,\eta)$ from (3.8), (3.11) and the analytic continuation properties for $\beta(x,y)$ and $b(x,y)$ stated in (Assumption 4). In order to prove (3.14), we give the relation between $w_n(x,\eta)$ and $w_{n+1}(x,\eta)$ in a different form from (3.8). We note the following fact: in general let

$$P(x,y) = \sum_{m=0}^{\infty} P_m(x) y^m \quad \text{and} \quad \mathcal{B}[P](x,\eta) = \sum_{m=0}^{\infty} P_m(x) \frac{\eta^m}{m!}.$$

Then it follows from an integration by parts that

$$\begin{aligned} &\int_0^{\eta} \mathcal{B}[P]\big(x, Z(\eta - t)\big) Q(x,t)\, dt \\ &= \sum_{m=0}^{\infty} \frac{P_m(x) Z^m}{m!} \int_0^{\eta} (\eta - t)^m \cdot Q(x,t)\, dt \\ &= \sum_{m=0}^{\infty} P_m(x) Z^m \int_0^{\eta} \int_0^{\eta_1} \cdots \int_0^{\eta_m} Q(x, \eta_{m+1})\, d\eta_{m+1} \cdots d\eta_2\, d\eta_1. \end{aligned} \tag{4.1}$$

By applying (4.1) to (3.8), we obtain the following relation between $w_n(x,\eta)$ and $w_{n+1}(x,\eta)$:

$$w_{n+1}(x,\eta)=\sum_{i=1}^{5}\mathscr{I}_i w_n(x,\eta), \tag{4.2}$$

where

$$\begin{aligned}
&\mathscr{I}_1 w_n(x,\eta)\\
&\quad= I_1 w_n(x,\eta)\\
&\quad= -\frac{1}{\alpha}\sum_{m=1}^{\infty}\beta_m(x)\int_0^{\eta}\int_0^{\eta_1}\cdots\int_0^{\eta_{m-1}} w_n(x,\eta_m)\,d\eta_m\cdots d\eta_2\,d\eta_1,\\
&\mathscr{I}_2 w_n(x,\eta)\\
&\quad= I_2 w_n(x,\eta)+I_3 w_n(x,\eta)\\
&\quad= \frac{1}{\alpha}\sum_{m=1}^{\infty}\int_0^{\eta}\beta_m\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1)\big)\cdot\mathscr{E}(x,\eta-\eta_1)^m\\
&\qquad\times\{1-a\alpha\cdot\mathscr{A}(x,\eta-\eta_1)\cdot\mathscr{E}(x,\eta-\eta_1)\cdot\eta_1\}\\
&\qquad\times\int_0^{\eta_1}\cdots\int_0^{\eta_{m-1}} w_n\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1),\mathscr{E}(x,\eta-\eta_1)\cdot\eta_m\big)\,d\eta_m\cdots d\eta_2\,d\eta_1,\\
&\mathscr{I}_3 w_n(x,\eta)\\
&\quad= I_4 w_n(x,\eta)\\
&\quad= -\frac{1}{\alpha}\sum_{m=1}^{\infty}\int_0^{\eta}\frac{\partial}{\partial\zeta}\Big\{\beta_m\big(-\alpha\cdot\mathscr{A}(x,\zeta)\big)\Big\}\Big|_{\zeta=\eta-\eta_1}\cdot\mathscr{E}(x,\eta-\eta_1)^m\\
&\qquad\times\int_0^{\eta_1}\cdots\int_0^{\eta_m} w_n\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1),\mathscr{E}(x,\eta-\eta_1)\cdot\eta_{m+1}\big)\,d\eta_{m+1}\cdots d\eta_2\,d\eta_1,\\
&\mathscr{I}_4 w_n(x,\eta)\\
&\quad= I_5 w_n(x,\eta)+I_6 w_n(x,\eta)\\
&\quad= -\sum_{m=1}^{\infty}\int_0^{\eta} b_m\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1)\big)\cdot\mathscr{E}(x,\eta-\eta_1)^{m+1}\\
&\qquad\times\int_0^{\eta_1}\cdots\int_0^{\eta_{m-1}}\eta_m\cdot w_n\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1),\mathscr{E}(x,\eta-\eta_1)\cdot\eta_m\big)\,d\eta_m\cdots d\eta_2\,d\eta_1,\\
&\mathscr{I}_5 w_n(x,\eta)\\
&\quad= I_7 w_n(x,\eta)\\
&\quad= \sum_{m=1}^{\infty}\int_0^{\eta} b_m\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1)\big)\cdot\mathscr{E}(x,\eta-\eta_1)^{m+1}\\
&\qquad\times\int_0^{\eta_1}\cdots\int_0^{\eta_m} v\big(-\alpha\cdot\mathscr{A}(x,\eta-\eta_1),\mathscr{E}(x,\eta-\eta_1)\cdot\eta_{m+1}\big)\,d\eta_{m+1}\cdots d\eta_2\,d\eta_1.
\end{aligned}$$

Finally, let us apply estimates (3.12) to (4.2). Then, we can prove the estimate (3.14) inductively. □

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