RIMS Kôkyûroku Bessatsu
B10 (2008), 133–148

Construction of the fundamental solution and curvature of manifolds

By

Chisato Iwasaki*

Abstract

We obtain a generalization of a local version of the Gauss-Bonnet-Chen theorem for manifolds with boundary. The above theorem is proved by only calculating the main term of the fundamental solution, if we introduce a new weight of symbols of pseudo-differential operators.

§ 1. Introduction

In this paper we give, by means of symbolic calculus of pseudo-differential operators, an extension theorem of a local version of the Gauss-Bonnet-Chern theorem on manifolds with boundary given in C.Iwasaki [11].

Let M be a Riemannian manifold and let $\chi(M)$ be the Euler characteristic of M. Let dv and $d\sigma$ be a volume element of M and a surface element of its boundary ∂M respectively. The Gauss-Bonnet-Chern theorem which was proved by S.Chern [2] , [3] is stated as follows: This theorem means that topological quantity can be represented by the geometric quantities.

The Gauss-Bonnet-Chern Theorem
Let M be a Riemannian manifold of dimension n.
(1) For M without boundary we have

$$\chi(M) = \int_M C_n(x, M)dv.$$

Received January 30, 2008. Revised May 5, 2008. Accepted May 9, 2008.
2000 Mathematics Subject Classification(s): Primary 58J35; Secondary 58J40, 58J20
Key Words: symbolic calculus, fundamental solution, curvature
*University of Hyogo, Hyogo, 671-2201, Japan.

(2) If M has boundary, then

$$\chi(M) = \int_M C_n(x, M)dv + \int_{\partial M} D_{n-1}(x)d\sigma.$$

The precise definitions of $C_n(x, M)$ and $D_{n-1}(x)$ are given in Section 3 and Section 5 respectively. We note that if n is odd, then

$$D_{n-1}(x) = \frac{1}{2}C_{n-1}(x, \partial M)$$

holds.

Analytical proofs are based on the following formula (cf.V.K.Patodi [15]):

$$\chi(M) = \int_M \sum_{p=0}^{n}(-1)^p \operatorname{tr} e_p(t, x, x)dv, \tag{1.1}$$

where $e_p(t, x, y)$ denotes the kernel of the fundamental solution $E_p(t)$ for the heat equation for Δ_p on differential p-forms $A^p(M) = \Gamma(\wedge^p T^*(M))$ and $\operatorname{tr} e_p(t, x, x)$ means the trace of operator $e_p(t, x, x)$ on $\wedge^p T_x^*(M)$.

If M has no boundary, $E_p(t)$ is the fundamental solution for the Cauchy problem, that is,

$$E_p(t)f(x) = \int_M e_p(t, x, y)\varphi(y)dv_y, \quad \varphi \in A^p(M)$$

satisfies for $0 < T < \infty$

$$\begin{aligned} (\frac{d}{dt} + \Delta_p)E_p(t) = 0 \qquad & \text{in } (0, T) \times M, \\ E_p(0) = I \qquad & \text{in } M. \end{aligned}$$

If M has boundary ∂M, $E_p(t)$ satisfies the following equations instead of the above equations

$$\begin{aligned} (\frac{d}{dt} + \Delta_p)E_p(t) = 0 \qquad & \text{in } (0, T) \times M, \\ B_p E_p(t) = 0 \qquad & \text{on } (0, T) \times \partial M, \\ E_p(0) = I \qquad & \text{in } M, \end{aligned}$$

with some boundary operator B_p (See Section 4).

So, we may say that a local version of the Gauss-Bonnet-Chern theorem holds, if we have

$$\sum_{p=0}^{n}(-1)^p \operatorname{tr} e_p(t, x, x) = C_n(x, M) + 0(\sqrt{t}) \tag{1.2}$$

as t tends to 0.

The author has proved (1.2) for $x \in M \backslash \partial M$ and has also shown for $x \in \partial M$

$$\sum_{p=0}^{n}(-1)^p \operatorname{tr}e_p(t,x,x) = 2D_{n-1}(x)\frac{1}{\sqrt{t}} + O(1) \qquad \text{as } t \to 0 \tag{1.3}$$

in [11]. The author has shown a way to constructing the fundamental solution of which the main term is enough to show (1.2) and (1.3). It is constructed by technique of pseudodifferential operators of new weights on symbols.

In this paper, a genaralization of a local version of the Gauss-Bonnet-Chern theorem on manifolds with boundary is obtained. Before stating our theorems, we introduce some notations.

We denote $\mathcal{I}$ and $\mathcal{I}_0$ the set of index

$$\mathcal{I} = \{I = (i_1, i_2, \cdots, i_r) : 0 \le r \le n,\ 1 \le i_1 < \cdots < i_r \le n\},$$

$$\mathcal{I}_0 = \{I = (i_1, i_2, \cdots, i_r) : 0 \le r \le n-1,\ 1 \le i_1 < \cdots < i_r \le n-1\}$$

and

$$\binom{a}{b} = 0 \text{ if } a < b, \text{ or } b < 0, \quad \binom{0}{0} = 1.$$

In the rest of this paper fix an integer ℓ such that $0 \le \ell \le n$.

Theorem 1.1 (Main Theorem). *Let M be a Riemannian manifold of dimension n with boundary and let $E_p(t)$ be the fundamental solution on $\Gamma(\wedge^p T^*(M))$. Suppose that f_p are constants with arbitrary constants $\{k_j\}_{j=\ell+1,\cdots,n}$ as follows:*

$$f_p = (-1)^p\left\{\binom{n-p}{n-\ell} + \sum_{j=\max\{p,\ell+1\}}^{n} k_j \binom{n-p}{n-j}\right\} \quad (0 \le p \le n). \tag{1.4}$$

(1) If $x \in M \backslash \partial M$, we have

$$\sum_{p=0}^{n} f_p \operatorname{tr}e_p(t,x,x) = C_\ell(x) t^{-\frac{n}{2}+\frac{\ell}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}) \qquad \text{as } t \to 0,$$

where $C_\ell(x)$ is given in Definition 3.3.

(2) If $x \in \partial M$, we have

$$\sum_{p=0}^{n} f_p \operatorname{tr}e_p(t,x,x) = 2D_{\ell-1}(x) t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{1}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}}) \qquad \text{as } t \to 0,$$

where $D_{\ell-1}(x)$ is given in Definition 5.1.

(3) We have

$$\int_M \sum_{p=0}^{n} f_p \,\mathrm{tr} e_p(t,x,x)\psi(x)dv$$
$$= \left(\int_M C_\ell(x,M)\psi(x)dv + \int_{\partial M} D_{\ell-1}(x)\psi(x)d\sigma\right) t^{-\frac{n}{2}+\frac{\ell}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}})$$

as $t \to 0$ for any $\psi(x) \in C^\infty(\overline{M})$.

Remark. Assume $\ell = n$. Then $f_p = (-1)^p$ for all p of (1.4). So statements (1) and (2) of Main Theorem are a local version of the Gauss-Bonnet-Chern theorem which was proved in [11]. The statement (3) is the Gauss-Bonnet-Chern theorem.

Remark. If M has no boundary, $D_{\ell-1}$ vanishes. So (1) of Main Theorem has been obtained in C.Iwasaki [12].

Remark. Assume $k_j = 0$ for all j. Then $f_p = (-1)^p\binom{n-p}{n-\ell}$ $(0 \le p \le \ell)$, $f_p = 0(\ell+1 \le p \le n)$. In this case, under the assumption M has no boundary, (1) of the Main theorem coincides with the result given in [5].

Since B_p is a coercive boundary operator for Δ_p, it is well-known that $e_p(t,x,y)$ is regular for either $x \neq y$ or $t > 0$. On the diagonal set, $e_p(t,x,y)$ has singularity with respect to t as follows:

$$e_p(t,x,x) \sim c_0(x)t^{-\frac{n}{2}} + c_1(x)t^{-\frac{n}{2}+\frac{1}{2}} + \cdots + \cdots \qquad t \to 0.$$

In [11] it is proved that the singularity of $\sum_{p=0}^{n}(-1)^p \mathrm{tr}\, e_p(t,x,x)$ at any point x in $M\backslash\partial M$ vanishes by an algebraic theorem on linear spaces stated in H.L.Cycon-R.G.Froese-W.Kirsch-B.Simon [4]. This theorem of the form suitable for our discussion is given in Section 2 in this paper.

Our point is that one can prove Main Theorem by only calculating the main term of the symbol of the fundamental solution, introducing a new weight of symbols of pseudodifferential operators.

The plan of this papaer is as follows. In section 2 we review some algebraic theorems, which are proved in [12]. In section 3 the asymptotic expansion of the fundamental solution for the Cauchy problem is discussed. This argument is similar to one of [12]. Our boundary operator B_p will be explained in section 4. See [11] for the detailed argument. Section 5 is devoted to the sketch of a proof of (2) and (3) of Main Theorem.

§ 2. Algebraic properties for the calculation of the trace

Let V be a vector space of dimension n with an inner product and let $\wedge^p(V)$ be its anti-symmetric p tensors. Set $\wedge^*(V) = \sum_{p=0}^n \wedge^p(V)$. Fix an orthonormal basis $\{v_1, \cdots, v_n\}$ of V. Let a_i^* be a linear transformation on $\wedge^*(V)$ defined by $a_i^* v = v_i \wedge v$ and let a_i be the adjoint operator of a_i^* on $\wedge^*(V)$.

We note that $\{a_i^*, a_j\}_{1\le i,j\le n}$ satisfy the following relations.

$$a_i a_j + a_j a_i = 0, \quad a_i^* a_j^* + a_j^* a_i^* = 0, \quad a_i a_j^* + a_j^* a_i = \delta_{ij}.$$

Definition 2.1. Set

$$\beta_\phi = 1, \beta_j = a_j^* a_j - a_j a_j^* \text{ for } 1 \le j \le n$$

and

$$\beta_I = \beta_{i_1} \cdots \beta_{i_k} \text{for } I = (i_1, \cdots, i_k) \in \mathcal{I}.$$

Let Ψ_p be the projection of $\wedge^*(V)$ on $\wedge^p(V)$ and let $\Gamma_0 = 1$, $\Gamma_k = \sum_{I\in\mathcal{I},\sharp(I)=k} \beta_I$.

The following assertion is essentially due to [15].

Proposition 2.2. *We have for any $I = (i_1, \cdots, i_k) \in \mathcal{I}$ the following assertions:*

(1)If $p < k$, then

$$\operatorname{tr}[\beta_I a_{j_1} a_{j_2} \cdots a_{j_p} a_{h_1}^* a_{h_2}^* \cdots a_{h_p}^*] = 0.$$

(2)Suppose $p = k$ and $\{j_1, j_2, \cdots, j_k\} \ne \{i_1, i_2, \cdots, i_k\}$ or $\{h_1, h_2, \cdots, h_k\}$ $\ne \{i_1, i_2, \cdots, i_k\}$. Then

$$\operatorname{tr}[\beta_I a_{j_1} a_{j_2} \cdots a_{j_p} a_{h_1}^* a_{h_2}^* \cdots a_{h_p}^*] = 0.$$

(3) Let π, σ be elements of the permutation group of degree k. Then we have

$$\operatorname{tr}[\beta_I a_{i_{\pi(1)}}^* a_{i_{\sigma(1)}} a_{i_{\pi(2)}}^* a_{i_{\sigma(2)}} \cdots a_{i_{\pi(k)}}^* a_{i_{\sigma(k)}}] = 2^{n-k}\operatorname{sign}(\pi)\operatorname{sign}(\sigma).$$

The following theorem is the key algebraic argument of this paper.

Theorem 2.3. *Let f_p $(0 \le p \le n)$ be constants of the form*

$$f_p = (-1)^p \Big\{ \binom{n-p}{n-\ell} + \sum_{j=\max(\ell+1,p)}^{n} k_j \binom{n-p}{n-j} \Big\}$$

with any constants k_j $(\ell+1 \le j \le n)$*. Then the following equation holds*

$$\sum_{q=0}^{n} f_q \Psi_q = (-1)^{\ell} 2^{\ell-n} \Gamma_\ell + \sum_{p=\ell+1}^{n} \alpha_p \Gamma_p$$

with α_p $(\ell+1 \le p \le n)$ *defined by*

$$\alpha_p = (-1)^p \Big\{ 2^{\ell-n} \binom{p}{\ell} + \sum_{j=\ell+1}^{p} \binom{p}{j} k_j \Big\}.$$

Especially
(1) If $\ell = n$*, then*

$$\sum_{q=0}^{n} f_q \Psi_q = (-1)^n \Gamma_n$$

holds if and only if

$$f_p = (-1)^p \quad \text{for any } p.$$

(2) If $\alpha_p = (-1)^p 2^{\ell-n} \binom{p}{\ell}$ $(\ell+1 \le p \le n)$*, then*

$$f_p = \begin{cases} (-1)^p \binom{n-p}{n-\ell}, & (0 \le p \le \ell); \\ 0, & (\ell+1 \le p \le n). \end{cases}$$

(3) If $\alpha_p = (-1)^{\ell} 2^{\ell-n} \binom{p}{\ell}$ $(\ell+1 \le p \le n)$*, then*

$$f_p = \begin{cases} 0, & (0 \le p \le n-\ell-1); \\ (-1)^{n-\ell+p} \binom{p}{n-\ell}; & (n-\ell \le p \le n). \end{cases}$$

§ 3. The proof of statement (1) of Main Theorem

We give a rough review of a proof of the statement of (1) of Main Theorem. This argument is similar to that of [12].

Let M be a smooth Riemannian manifold of dimension n with a Riemannian metric g. Let $X_1, X_2, \cdots, X_n$ be a local orthonormal frame of $T(M)$ in a local patch U. And let $\omega^1, \omega^2, \cdots, \omega^n$ be its dual frame. The differential d and its dual ϑ acting on $\Gamma(\wedge^p T^*(M))$ are written as follows,using the Levi-Civita connection ∇ (See Appendix A of [14]):

$$d = \sum_{j=1}^{n} e(\omega^j) \nabla_{X_j}, \qquad \vartheta = -\sum_{j=1}^{n} \imath(X_j) \nabla_{X_j},$$

where we use the following notations.
Notations.

$$e(\omega^j)\omega = \omega^j \wedge \omega,\ \imath(X_j)\omega(Y_1,\cdots,Y_{p-1}) = \omega(X_j,Y_1,\cdots,Y_{p-1}).$$

The Laplacian $\Delta = d\vartheta + \vartheta d$ on $\sum_{p=0}^n \Gamma(\wedge^p T^*(M))$ has Weitzenböck's formula;

$$\Delta = -\{\sum_{j=1}^n \nabla_{X_j}\nabla_{X_j} - \sum_{j=1}^n \nabla_{(\nabla_{X_j}X_j)} + \sum_{i,j=1}^n e(\omega))\imath(X_j)R(X_i,X_j)\}.$$

Let $c_{i,j}^k$ and R_{mkij} be the coefficients of the connection and the curvature transformation

$$\nabla_{X_i}X_j = \sum_{k=1}^n c_{i,j}^k X_k, \quad R(X_i,X_j)X_k = \sum_{m=1}^n R_{mkij}X_m \quad 1 \le i,j,k,m \le n.$$

We use the following notations in the rest of this paper:

$$\begin{aligned} &a_j^* = e(\omega*), \qquad a_k = \iota(X_k). \\ &a_I = a_{i_1}a_{i_2}\cdots a_{i_p}, \qquad a_I^* = a_{i_p}^*\cdots a_{i_1}^*, \\ &\omega^I = \omega^{i_1}\wedge\omega^{i_2}\wedge\cdots\wedge\omega^{i_p} \qquad \text{for } I=(i_1,i_2,\cdots,i_p)\in\mathcal{I}. \end{aligned}$$

We note that the coefficients $c_{i,j}^k$ satisfy

$$c_{i,j}^k = -c_{i,k}^j, \quad [X_i,X_j] = \sum_{k=1}^n (c_{i,j}^k - c_{j,i}^k)X_k$$

because of the fact that our connection is the Riemannian connection. By the above notations we have for $\varphi_J\omega^J \in A^*(M)$

$$\nabla_{X_j}(\varphi_J\omega^J) = (X_j\varphi_J)\omega^J - \varphi_J G_J(\omega^J),$$

where

$$G_j = \sum_{k,m=1}^n c_{j,k}^m a_k^* a_m.$$

We have also

$$R(X_i,X_j)(\varphi_J\omega^J) = -\sum_{k,m=1}^n \varphi_J R_{mkij} a_k^* a_m(\omega^J).$$

Then we have the representation of Δ in a local chart U

$$\Delta = -\{\sum_{j=1}^n (X_jI - G_j)^2 - \sum_{i,j=1}^n c_{i,i}^j (X_jI - G_j) - \sum_{i,j,k,m=1}^n R_{mkij}a_i^*a_ja_k^*a_m\} \tag{3.1}$$

on $A^*(M)$. Here I is the identity operator on $\wedge^*(T^*(M))$.

Take a local coordinate $\{x_1, \cdots, x_n\}$ of U. Let $\{\xi_1, \cdots, \xi_n\}$ be its dual coordinate. $\mathcal{A} = (\mathcal{A}_{ij})$ denotes a matrix whose elements are $\mathcal{A}_{ij} = a_i^* a_j \quad 1 \leq i, j \leq n$.

Definition 3.1. A subset K^m of $S^m_{1,0}$ is given by $K^m = \{p(x, \xi : \mathcal{A})$; polynomials with respect to ξ and $\mathcal{A}_{i,j}, (i, j = 1, 2, \cdots, n)$ of order m with coefficients in $\mathcal{B}(\mathbf{R}^n)\}$. We define a pseudo-differential operator $P = p(x, D : \mathcal{A})$ acting on $A^*(M)$ of a symbol $\sigma(P) = p(x, \xi : \mathcal{A}) = \sum_{I,J} p_{I,J}(x, \xi) a_I^* a_J \in K^m$ as follows.

$$p(x, D : \mathcal{A})(\varphi_K \omega^K) = \sum_{I,J} p_{I,J}(x, D) \varphi_K a_I^* a_J(\omega^K).$$

In our case, by (3.1) the symbol of Δ is of the form $\sigma(\Delta) = r_2 + r_1$, where

$$r_2 = -\sum_{j=1}^{n} (\alpha_j I - G_j)^2 + R,$$

with

$$R = \sum_{i,j,k,m=1}^{n} R_{mkij} a_i^* a_j a_k^* a_m \quad \text{and} \quad r_1 \in K^1$$

It is clear that r_2 belongs to K^2.

If we review the results in [11], we see the following facts. The above representation of Δ signifies that the fundamental solution$E_p(t)(e_p(t, x, y))$ have a common representation for any p. So we use $E(t)(e(t, x, y))$. In our notation, $E_p(t) = \Psi_p E(t)$. It is sufficient to construct an asymptotic solution for the fundamental solution locally because we can reduce constructing the fundamental solution to solving an integral equation on the manifold M of Volterra type. Our fundamental solution is of the form

$$E(t) = U(t) + V(t),$$

where $U(t)$ is the fundamental solution for the Cauchy problem and $V(t)$ is smooth with respect t off the boundary. See details in Section 5 .

We will give a sketch of the argument for $U(t)$ in [12]. Now consider the following Cauchy problem:

$$\begin{aligned} (\frac{d}{dt} + \Delta) U(t) = 0 \qquad & \text{in } (0, T) \times \mathbf{R}^n, \\ \lim_{t \to 0} U(t) = I \qquad & \text{in } \mathbf{R}^n. \end{aligned}$$

The fundamental solution $U(t)$ constructed in [11] is an integral operator

$$U(t)\varphi(x) = \int_M u(t, x, y) \varphi(y) dv_y, \quad \varphi \in A^*(M)$$

and $U(t)$ has the following expansion:

$$U(t) \sim \sum_{j=0} u_j(t, x, D), \tag{3.2}$$

where $u_j(t, x, D)$ are pseudodifferential operators with parameter t. The following fact is obtained in p.255 of [11]. The main part of $U(t)$ is obtained as a pseudo-differential operator with symbol $u_0(t, x, \xi) := e^{-r_2 t}$. So we have

$$\begin{aligned} \tilde{u}_0(t, x, x) &:= (2\pi)^{-n} \int_{\mathbf{R}^n} u_0(t, x, \xi) d\xi \\ &= (\frac{1}{2\sqrt{\pi t}})^n \sqrt{\det g} e^{-tR} \Big(1 + 0(\sqrt{t})\Big). \end{aligned} \tag{3.3}$$

We shall calculate

$$\text{tr } (\beta_I \tilde{u}_0(t, x, x)) dx = (\frac{1}{2\sqrt{\pi t}})^n \text{tr } (\beta_I e^{-tR}) dv (1 + 0(\sqrt{t})), \tag{3.4}$$

for $I \in \mathcal{I}$, $\sharp(I) = r$.

Using $e^{-tR} = \sum_{k=0}^{\infty} \{\frac{(-R)^k}{k!} t^k\}$, we have by Proposition 2.2

$$\text{tr } (\beta_I e^{-tR}) = \begin{cases} \text{tr}\big(\beta_I (-R)^m\big) \frac{t^m}{m!} + 0(t^{m+1}), & \text{if } r = 2m \ ; \\ 0(t^{\frac{r+1}{2}}), & \text{if } r \text{ is odd.} \end{cases} \tag{3.5}$$

We have the following proposition for the terms of the right hand side of the above equation.

Proposition 3.2. *For $I = (i_1, i_2, \cdots, i_r) \in \mathcal{I}$ $(r = 2m)$*

$$\begin{aligned} \text{tr}(\beta_I (-R)^m) = 2^{n-r-m} & \sum_{\pi, \sigma \in S_r} \text{sign}(\pi) \text{sign}(\sigma) \\ \times R_{i_{\pi(1)} i_{\pi(2)} i_{\sigma(1)} i_{\sigma(2)}} & \cdots R_{i_{\pi(r-1)} i_{\pi(r)} i_{\sigma(r-1)} i_{\sigma(r)}}. \end{aligned}$$

By Theorem 2.3 we obtain

$$\begin{aligned} \text{tr } (\sum_{p=0}^{n} f_p \Psi_p u(t, x, x)) = (-1)^{\ell} 2^{\ell - n} & \sum_{I \in \mathcal{I}, \sharp(I) = \ell} \text{tr } (\beta_I u(t, x, x)) \\ & + \sum_{p=\ell+1}^{n} \alpha_p \text{tr } (\Gamma_p u(t, x, x)). \end{aligned}$$

So we have by (3.4)$\sim$ (3.5) and Proposition 3.2

$$\text{tr } (\sum_{p=0}^{n} f_p \Psi_p u(t, x, x)) = C_\ell(x, M) t^{-\frac{n}{2} + \frac{\ell}{2}} + 0(t^{-\frac{n}{2} + \frac{\ell}{2} + \frac{1}{2}}) \quad x \in M$$

with $C_\ell(x, M)$ given by

Definition 3.3.
(1-i) If ℓ is odd, then $C_\ell(x,M)=0$
(1-ii)If ℓ is even ($\ell=2m$), then

$$C_\ell(x,M)=\sum_{I\in\mathcal{I},\sharp(I)=\ell} C_I(x,M),$$

where

$$C_I(x,M)=(\frac{1}{2\sqrt{\pi}})^n\frac{1}{m!}(\frac{1}{2})^m\sum_{\pi,\sigma\in S_\ell}\mathrm{sign}(\pi)\mathrm{sign}(\sigma)$$
$$\times R_{i_{\pi(1)}i_{\pi(2)}i_{\sigma(1)}i_{\sigma(2)}}\cdots\cdots R_{i_{\pi(\ell-1)}i_{\pi(\ell)}i_{\sigma(\ell-1)}i_{\sigma(\ell)}}$$

for $I=(i_1,i_2,\cdots,i_\ell)\in\mathcal{I}$.

§ 4. The construction of the fundamental solution with boundary

The main part of the construction of the fundamental solution or its asymptotics lies in constructing that in a local chart (cf.C.Iwasaki [10]). So it is enough to show a method of construction of the fundamental solution in $\mathbf{R}^n_+$. The study in [10] is applicable for the construction of the fundamental solution for our initial-boundary value problem . In this case we introduce the symbol class $\mathcal{J}_s$ according to [11], as we used K^m instead of $S^m_{1,0}$ in Section 3.

First of all, we write down the boundary operator B_p for $A^p(M)$ in a local chart. Take a local patch Ω near ∂M such that ∂M is defined by $\{\rho=0\}$ in Ω and $M\cap\Omega\subset\{\rho\geq 0\}$. Assume that $\omega^n=cd\rho$ with some function c on M. Note that X_j $(1\leq j\leq n-1)$ are tangential to ∂M.

Choose local coordinates $\{x_1,\cdots,x_n\}$ in Ω such that $M\cap\Omega=\{(x',x_n);x'\in\mathcal{U},x_n\geq 0\},\Gamma=\partial M\cap\Omega=\{(x',0);x'\in\mathcal{U}\}$ and $X_n|_\Gamma=\frac{d}{dx_n}$.

The boundary condition is as follows:

$$\varphi\in\mathrm{Dom}(\vartheta),\qquad d\varphi\in\mathrm{Dom}(\vartheta),$$

where

$$\mathrm{Dom}(\vartheta)=\{\varphi=\sum_J\varphi_J\omega^J:\ \varphi_J|_\Gamma=0 \text{ if } n\in J\}.$$

Note that $\mathrm{Dom}(\vartheta)$ is the set of all ψ of $A^*(M)$ which satisfy

$$\int_M(d\varphi,\psi)dv=\int_M(\varphi,\vartheta\psi)dv,\ \text{for any } \varphi\in A^*(M).$$

We can write our boundary condition in the following form:

$$\varphi_J|_\Gamma = 0 \text{ if } n \in J, \quad B(\sum_{n \notin J} \varphi_J \omega^J)|_\Gamma = 0,$$

where

$$B = \frac{d}{dx_n} - \gamma + b$$

with the following b and γ.

Definition 4.1. (1) Set

$$\gamma = \gamma(x' : \mathcal{A}) = G_n|_\Gamma = \sum_{1 \leq j,k \leq n} (c^j_{n,k}|_\Gamma) a^*_k a_j, \tag{4.1}$$

$$b = b(x' : \mathcal{A}) = -\sum_{1 \leq j,k \leq n-1} (c^n_{j,k}|_\Gamma) a^*_j a_k + \sum_{j=n \text{ or } k=n} (c^j_{n,k}|_\Gamma) a^*_k a_j \tag{4.2}$$

(2) Set $\mathcal{P} = a^*_n a_n, \ \mathcal{Q} = a_n a^*_n = I - \mathcal{P}$.

Definition 4.2. For a pair (j,k) of integer j and nonpositive integer k we define a function

$$\tilde{v}_{j,k}(t, x_n, y_n; b, \gamma) = e^{\gamma(x_n - y_n)} \tilde{w}_{j,k}(t, x_n + y_n; b),$$

with b and γ given in Definition 4.1, where $\{\tilde{w}_{j,k}\}$ are symbols defined in Definition 7 of [10] as follow:

$$\tilde{w}_{j,0}(t,\omega) = \frac{1}{\sqrt{\pi}} (\frac{1}{2\sqrt{t}})^{j+1} h_j(\frac{\omega}{2\sqrt{t}}), \text{ if } \quad j \geq 0,$$
$$\tilde{w}_{j,0}(t,\omega) = -\frac{1}{\sqrt{\pi}} (\frac{1}{2\sqrt{t}})^{j+1} \int_0^\infty e^{-(\sigma + \frac{\omega}{2\sqrt{t}})^2} \frac{(-\sigma)^{-j-1}}{(-j-1)!} d\sigma, \text{ if } \quad j \leq -1,$$

For $k \leq -1$

$$\tilde{w}_{j,k}(t,\omega;b) = -\frac{1}{\sqrt{\pi}} (\frac{1}{2\sqrt{t}})^{j+k+1} \int_0^\infty e^{-(\sigma + \frac{\omega}{2\sqrt{t}})^2 + 2b\sqrt{t}\sigma} \frac{(-\sigma)^{-k-1}}{(-k-1)!} \times h_j(\sigma + \frac{\omega}{2\sqrt{t}}) d\sigma, \text{ if } j \geq 0;$$

$$\tilde{w}_{j,k}(t,\omega;b) = \frac{1}{\sqrt{\pi}} (\frac{1}{2\sqrt{t}})^{j+k+1} \int_0^\infty \frac{(-\tau)^{-j-1}}{(-j-1)!} d\tau \times \int_0^\infty e^{-(\sigma + \tau + \frac{\omega}{2\sqrt{t}})^2 + 2b\sqrt{t}\sigma} \frac{(-\sigma)^{-k-1}}{(-k-1)!} d\sigma, \text{ if } j \leq -1,$$

where $h_j(\sigma) = \{(\frac{d}{d\sigma})^j e^{-\sigma^2}\} e^{\sigma^2}$.

Set $r_2|_\Gamma = (\xi_n + i\gamma)^2 + \beta(x', \xi' : \mathcal{A})$. Then we have

$$\beta = \beta(x', \xi' : \mathcal{A}) = -\sum_{j=1}^{n-1}\Big((\alpha_j|_\Gamma)I - G_j|_\Gamma\Big)^2 + R|_\Gamma.$$

Definition 4.3. $\mathcal{J}_s$ is the set of all finite sums of the functions of the following form

$$g(t, x_n, y_n, x', \xi' : \mathcal{A}) = t^d (x_n)^p q(sx', \xi' : \mathcal{A}) \tilde{v}_{j,k}(t, x_n, y_n; b(x' : \mathcal{A}), \gamma(x' : \mathcal{A})) \times e^{-\beta(x', \xi' : \mathcal{A})t}$$

$$(d, p, j, k \in \mathbf{Z}, d \geq 0, p \geq 0, k \leq 0, q \in K^m(\mathbf{R}^{n-1}) \text{ with } m = s + 2d + p - j - k).$$

For a symbol $g(t, x_n, y_n, x', \xi', \mathcal{A}) \in \mathcal{J}_s$ we define an integral-pseudo-differential operator G as follows:

$$(G\varphi)(t, x', x_n : \mathcal{A}) = \int_0^\infty g(t, x_n, y_n, x', D' : \mathcal{A})\varphi(\cdot, y_n)dy_n.$$

So the kernel $\tilde{g}$ of operator G is given by

$$\tilde{g}(t, x', x_n, y', y_n : \mathcal{A}) = (2\pi)^{-(n-1)} \int_{R^{n-1}} e^{i(x'-y')\bullet\xi'} g(t, x_n, y_n, x', \xi' : \mathcal{A})d\xi'.$$

By the argument of page 276 of [11] an asymptotic $E_N(t)$ of the fundamental solution for the mixed problem is obtained of the form for any integer N

$$E_N(t) = U_0 + U_1 + \cdots + U_N + V_1 + V_0 + \cdots + V_{-N},$$

where $U_j = u_j(t; x, D)$ are pseudo-differential operators given in (3.2) and V_j are integral operators whose symbols $v_j(t, x_n, y_n, x', \xi' : \mathcal{A}) \in \mathcal{J}_j$.

Especially the symbol of operator V_1 which is the main part of operator $V(t)$ is obtained as follows:

$$v_1(t, x_n, y_n, x', \xi' : \mathcal{A}) = 2\mathcal{Q}\tilde{v}_{1,-1}(t, x_n, y_n; b, \gamma)e^{-\beta(x', \xi' : \mathcal{A})t}. \tag{4.3}$$

In the rest of this paper $\tilde{v}_j(t, x', x_n, y', y_n : \mathcal{A})$ denotes the kernel of the integral operator V_j, that is,

$$\tilde{v}_j(t, x', x_n, y', y_n : \mathcal{A}) = (2\pi)^{-(n-1)} \int_{R^{n-1}} e^{i(x'-y')\bullet\xi'} v_j(t, x_n, y_n, x', \xi' : \mathcal{A})d\xi'.$$

§ 5. The sketch of a proof of statement (2),(3) of Main Theorem

We will calculate tr $(\sum_{p=0}^{n} f_p\Psi_p V_j(t,x,x))(j\leq 1)$. We prepare some lemmas and the definition of $D_{\ell-1}(x)$.

Let $\hat{R}(W,Z,X,Y)$ be the Riemannnian curvature tensors induced on ∂M.

We give the definition of $D_{\ell-1}(x)$ on $x\in\partial M$.

Definition 5.1. For $x\in\partial M$ $D_{\ell-1}(x)$ is defined as follows:

$$D_{\ell-1}(x)=\sum_{I\in\mathcal{I}_0,\sharp(I)=\ell-1} D_I(x),$$

(i) If ℓ is odd ($\ell=2m+1$), then

$$D_I(x)=\frac{1}{2}(\frac{1}{2\sqrt{\pi}})^{n-1}\frac{1}{m!}(\frac{1}{2})^m\sum_{\pi,\sigma\in S_{\ell-1}}\mathrm{sign}(\pi)\mathrm{sign}(\sigma)$$
$$\times\hat{R}_{i_{\pi(1)}i_{\pi(2)}i_{\sigma(1)}i_{\sigma(2)}}\cdots\cdots\hat{R}_{i_{\pi(\ell-2)}i_{\pi(\ell-1)}i_{\sigma(\ell-2)}i_{\sigma(\ell-1)}}$$

for $I=(i_1,i_2,\cdots,i_{\ell-1})\in\mathcal{I}_0$.

(ii)If ℓ is even ($\ell=2m$), then

$$D_I(x)=\frac{1}{2}(\frac{1}{2\sqrt{\pi}})^{n-1}\sum_{k=0}^{m-1}\frac{d_{m-k-1}}{k!}(\frac{1}{2})^k\sum_{\pi,\sigma\in S_{\ell-1}}\mathrm{sign}(\pi)\mathrm{sign}(\sigma)$$
$$\times\hat{R}_{i_{\pi(1)}i_{\pi(2)}i_{\sigma(1)}i_{\sigma(2)}}\cdots\quad\cdots\hat{R}_{i_{\pi(2k-1)}i_{\pi(2k)}i_{\sigma(2k-1)}i_{\sigma(2k)}}$$
$$\times c^n_{\pi(2k+1),\sigma(2k+1)}c^n_{\pi(2k+2),\sigma(2k+2)}\cdots\quad\cdots c^n_{\pi(\ell-1),\sigma(\ell-1)}$$

for $I=(i_1,i_2,\cdots,i_{\ell-1})\in\mathcal{I}_0$ with $d_k=\frac{(-1)^k}{k!(k+\frac{1}{2})\sqrt{\pi}}$.

Now the calculation of tr $\big(\beta_I\tilde{v}_1(t,x',0)\big)$ is obtained as follows.

Lemma 5.2. *Let $\hat{g}$ be the Riemannain metric induced on ∂M.*
(1) Let $I=(i_1,i_2,\cdots,i_{l-1},n)\in\mathcal{I}$ and $I_0=(i_1,i_2,\cdots,i_{l-1})\in\mathcal{I}_0$
(1-i) We have

$$\mathrm{tr}\,\big(\beta_I\tilde{v}_1(t,x',0,x',0)\big)=(-1)^\ell 2^{n-\ell+1}t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{1}{2}}D_{I_0}(x')\sqrt{\det g(x',0)}+0(t^{-\frac{n}{2}+\frac{\ell}{2}}).$$

(1-ii) We have for any $\psi(x',x_n)\in C^\infty(\mathbf{R}^n)$ and for any positive constant ε

$$\int_0^\varepsilon \mathrm{tr}\big(\beta_I\tilde{v}_1(t,x',x_n,x',x_n)\big)\psi(x',x_n)dx_n$$
$$=(-1)^\ell 2^{n-\ell}t^{-\frac{n}{2}+\frac{\ell}{2}}\psi(x',0)D_{I_0}(x')\sqrt{\det\hat{g}(x')}+0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}).$$

(2) Let $I \in \mathcal{I}_0, \sharp(I) = \ell$ or $I \in \mathcal{I}, \sharp(I) \geq \ell + 1$
(2-i) We have

$$\mathrm{tr}\left(\beta_I \tilde{v}_1(t, x', 0, x', 0)\right) = 0(t^{-\frac{n}{2}+\frac{\ell}{2}}).$$

(2-ii) We have for any $\psi(x', x_n) \in C^\infty(\mathbf{R}^n)$ and for any positive constant ε

$$\int_0^\varepsilon \mathrm{tr}\left(\beta_I \tilde{v}_1(t, x', x_n, x', x_n)\right)\psi(x', x_n) dx_n = 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}).$$

The proof is omitted. Proposition 2.2, (4.2) and (4.3) are essential to give the proof.

Corollary 5.3. *For any integer N we have*

$$\sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{v}_1)(t, x, x)$$
$$= \begin{cases} 0(t^N), & \text{if } x_n > 0; \\ 2t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{1}{2}} D_{\ell-1}(x')\sqrt{\det \hat{g}(x', 0)} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}}), & \text{if } x_n = 0. \end{cases}$$

We have for any $\psi(x', x_n) \in C^\infty(\mathbf{R}^n)$ and for any positive constant ε

$$\int_0^\varepsilon \sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{v}_1)(t, x', x_n, x', x_n)\psi(x', x_n) dx_n$$

$$= t^{-\frac{n}{2}+\frac{\ell}{2}} D_{\ell-1}(x')\sqrt{\det \hat{g}(x', 0)}\psi(x', 0) + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}).$$

Proof. By Theorem 2.3 and Lemma 5.2 we have the assertion. □

In order to calculate the trace of $\sum_{p=0}^{n} f_p \Psi_p V_j \quad (j = 0, -1, -2 \cdots, -N)$ we have the following lemma which is obtained by the similar method to the above corollary.

Lemma 5.4. *For any integer $j \leq 0$ we have for any N*

$$\sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{v}_j)(t, x, x) = \begin{cases} 0(t^N), & \text{if } x_n > 0; \\ 0(t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{j}{2}}), & \text{if } x_n = 0. \end{cases}$$

We have for any $\psi(x', x_n) \in C^\infty(\mathbf{R}^n)$ and for any positive constant ε

$$\int_0^\varepsilon \sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{v}_j)(t, x', x_n, x', x_n)\psi(x', x_n) dx_n = 0(t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{j}{2}+\frac{1}{2}}).$$

Proof of (2) and (3) of Main Theorem. If we study the asymptotic behavior of the fundamental solution, it is sufficient to consider the fundamental solution locally by an argument similar to that employed for the case M is closed, which was proved in [11], for example. In a local patch we have

$$e(t,x,x)dv = \tilde{u}(t,x,x)dx + \tilde{v}(t,x,x)dx,$$

$$\tilde{u}(t,x,y) = \sqrt{\det g(x)}\,u(t,x,y),$$

where $u(t,x,y)$ is obtained in Section 3.

By Corollary 5.3 and Lemma 5.4 we obtain

Lemma 5.5. *For any integer N we have*

$$\sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{v})(t,x,x)$$
$$= \begin{cases} 0(t^N), & \textit{if } x_n > 0; \\ 2t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{1}{2}} D_{\ell-1}(x')\sqrt{\det \hat{g}(x',0)} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}}), & \textit{if } x_n = 0. \end{cases}$$

We have for any $\psi(x',x_n) \in C^\infty(\mathbf{R}^n)$ and for any positive constant ε

$$\int_0^\varepsilon \sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{v})(t,x',x_n,x',x_n)\psi(x',x_n)dx_n$$
$$= t^{-\frac{n}{2}+\frac{\ell}{2}} D_{\ell-1}(x')\sqrt{\det \hat{g}(x',0)}\,\psi(x',0) + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}).$$

From the above lemma and

$$e_p(t,x,x) = \Psi_p(\tilde{u}(t,x,x) + \tilde{v}(t,x,x))\frac{1}{\sqrt{\det g}}$$

it is easy to show for any N

$$\begin{aligned}
\sum_{p=0}^{n} f_p \mathrm{tr}\, e_p(t,x,x) &= \frac{1}{\sqrt{\det g}} \sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p \tilde{u})(t,x,x) + 0(t^N) \quad x \in M\backslash\partial M, \\
&= \sum_{p=0}^{n} f_p \mathrm{tr}(\Psi_p u)(t,x,x) + 0(t^N) \quad x \in M\backslash\partial M, \\
&= C_\ell(x) t^{-\frac{n}{2}+\frac{\ell}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}) \quad x \in M\backslash\partial M, \\
\sum_{p=0}^{n} f_p \mathrm{tr}\, e_p(t,x,x) &= 2D_{\ell-1}(x) t^{-\frac{n}{2}+\frac{\ell}{2}-\frac{1}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}}) \quad x \in \partial M.
\end{aligned}$$

and

$$\int_M \sum_{p=0}^{n} f_p \mathrm{tr}\, e_p(t,x,x)\psi(x)dv$$
$$= \Big(\int_M C_\ell(x)\psi(x)dv + \int_{\partial M} D_{\ell-1}(x')\psi(x')d\sigma\Big)t^{-\frac{n}{2}+\frac{\ell}{2}} + 0(t^{-\frac{n}{2}+\frac{\ell}{2}+\frac{1}{2}}).$$

So the proof is complete.

References

[1] N.Berline, E.Getzler and M.Vergne, *Heat Kernels and Dirac Operators.* Springer-Verlag, 1992.

[2] S.Chern, *A simple instrinsic proof of the Gauss Bonnet formula for closed Riemannian manifolds.* Ann.of Math.45 (1944), 747-752.

[3] S.Chern, *On the curvature integra in a Riemannian manifold.* Ann.of Math.46 (1945), 674-684.

[4] H.l.Cycon,R.G.Froese,W.Kirsch and B.Simon, *Schrödinger operators.* Texts and Monographs in Physics, Springer,1987.

[5] P.Günther and R.Schimming, *Curvature and spectrum of compact Riemannian manifolds.* J.Diff.Geom. **12** (1977), 599-618.

[6] E.Getzler, *The local Atiyah-Singer index theorem.* Critical phenomena, radom systems, gauge theories, K.Sterwalder and R.Stora,eds. Les Houches, Sessin XLIII, (1984), 967-974, North-Holland.

[7] E.Getzler, *A short proof of the local Atiyah-Singer index Theorem.* Topology **25** (1986),111-117.

[8] P.B.Gilkey, *Invariance Theory,The Heat Equation,and the Atiyah-Singer Index Theorem.* Publish or Perish, Inc.,1984.

[9] P.B.Gilkey, *The boundary integrand in the formula for the signature and Euler charactertistic of a Riemannian manifold with boundary.* Adv.in Math., **15** (1975), 334-360.

[10] C.Iwasaki, *The asymptotic expansion of the fundamental solution for initial-boundary value problems and its application.* Osaka J .Math. **31** (1994), 663-728.

[11] C.Iwasaki, *A proof of the Gauss-Bonnet-Chern Theorem by the symbol calculus of pseudo-differential operators.* Japanese J. Math.**21** (1995), 235-285.

[12] C.Iwasaki, *Symbolic Calculus of Pseudo-differential Operators and Curvature of Manifolds.* Operator Theory:Advances and Applications,**172** (2007), 51-66.

[13] S.Kobayashi and K.Nomizu, *Foundations of Differential Geometry.* **I,II**, John Wiley & Sons, 1963.

[14] S.Murakami, *Manifolds.* Kyoritsusshuppan,1969 (in Japanese).

[15] V.K.Patodi, *Curvature and the eigenforms of the Laplace operator.* J.Differential Geometry **5** (1971), 233-249.