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Finding Rigged Configurations From Paths

By

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Abstract

We review reformulation of the map from tensor product of crystals to the rigged configurations in terms of the energy function of affine crystals. Especially, we give intuitive picture of the inverse scattering formalism for the periodic box-ball systems formulated by Kuniba–Takagi–Takenouchi.

§ 1. Introduction

In this lecture note, we review reformulation of the bijection of Kerov–Kirillov–Reshetikhin [KKR] (and extension due to [KR], see also [KSS])

$$\phi : \text{Paths} \longmapsto \text{Rigged Configurations}$$

in terms of the crystal bases theory [K] and its application to the periodic box-ball systems following [S2] and [KS3].

The bijection ϕ was originally introduced in order to show the so-called $X = M$ formula (see [O, S4] for reviews) by using its statistic preserving property. Recently, another application of the bijection ϕ to the box-ball systems [TS, T] was found [KOSTY]. In this context, the bijection ϕ plays the role of the inverse scattering formalism [GGKM, AC] for the box-ball systems.

The original definition of the bijection ϕ is described by purely combinatorial language such as box adding or removing procedures. Purpose of this note is to clarify what the representation theoretic origin of the bijection ϕ is. Motivated by the connection with the box-ball systems, consider the following isomorphism under the affine combinatorial R:

$$u_l[0] \otimes b_1[0] \otimes \cdots \otimes b_L[0] \simeq b'_1[-E_{l,1}] \otimes \cdots \otimes b'_L[-E_{l,L}] \otimes u'_l[E_l], \tag{1.1}$$

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where $u_l \in B_l$ is the highest element and $u'_l \in B_l$, $b_k, b'_k \in B_{\lambda_k}$. This is nothing but time evolution of the box-ball systems [HHIKTT, FOY]. Then, E_l here is related to shape of the rigged configuration (see Eq.(4.2)) and it is the conserved quantity of the box-ball systems introduced in [FOY]. By using this property, we introduce a table containing purely algebraic data (local energy distribution) from which we can read off which letter 2 of path belongs to which row of the rigged configuration. As the result, we can reconstruct the map ϕ by using purely representation theoretic procedure. Recently, the formalism is extended to general elements of tensor products of the Kirillov–Reshetikhin crystals of $\mathfrak{sl}_n$ type [S3]. Again, this is achieved by extension of Eq.(4.2). This shows that the formalism here is quite natural.

The plan of this note is as follows. In Section 2, we prepare minimal foundations of crystal theory. In Section 3, we define the local energy distribution. In Section 4, we present our main result. In Section 5, we explain some of applications of our formalism for the box-ball system with periodic boundary condition [YT, YYT] along with review of the inverse scattering formalism for the periodic box-ball system [KTT]. In Section 6 (Appendix), we consider tensor product of highest paths in terms of the rigged configurations.

§ 2. Combinatorial R and energy function

Crystal. Let B_k be the crystal of k-fold symmetric powers of the vector (or natural) representation of $U_q(\mathfrak{sl}_2)$. As the set, it is

$$B_k = \{(x_1, x_2) \in \mathbb{Z}^2_{\geq 0} \,|\, x_1 + x_2 = k\}. \tag{2.1}$$

We usually identify elements of B_k as the semi-standard Young tableaux

$$(x_1, x_2) = \boxed{\overbrace{1\cdots 1}^{x_1}\overbrace{2\cdots 2}^{x_2}}, \tag{2.2}$$

i.e., the number of letters i contained in a tableau is x_i. For example, the highest element $u_l \in B_l$ is $u_l = (l, 0) = \boxed{111\cdots 1}$.

For two crystals B_k and B_l of $U_q(\mathfrak{sl}_2)$, one can define the tensor product $B_k \otimes B_l = \{b \otimes b' \mid b \in B_k, b' \in B_l\}$. Then we have a unique isomorphism $R : B_k \otimes B_l \xrightarrow{\sim} B_l \otimes B_k$, i.e. a unique map which commutes with actions of the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$. We call this map combinatorial R and usually write the map R simply by $\simeq$. We call elements of tensor product of crystals *paths*.

Affinization. Consider the affinization of the crystal B [KMN[2]]. As the set, it is

$$\mathrm{Aff}(B) = \{b[d] \mid b \in B,\, d \in \mathbb{Z}\}. \tag{2.3}$$

Integers d of $b[d]$ are called modes. For the tensor product $b_1[d_1] \otimes b_2[d_2] \in \mathrm{Aff}(B_k) \otimes \mathrm{Aff}(B_l)$, we can lift the above definition of the combinatorial R as follows:

$$b_1[d_1] \otimes b_2[d_2] \overset{R}{\simeq} b_2'[d_2 - H(b_1 \otimes b_2)] \otimes b_1'[d_1 + H(b_1 \otimes b_2)], \tag{2.4}$$

where $b_1 \otimes b_2 \simeq b_2' \otimes b_1'$ is the combinatorial R defined in the above.

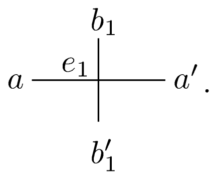
Explicit expressions. There is piecewise linear formula to obtain the combinatorial R and the energy function [HHIKTT]. This is suitable for computer programming. For the affine combinatorial $R : x[d] \otimes y[e] \simeq \tilde{y}[e - H(x \otimes y)] \otimes \tilde{x}[d + H(x \otimes y)]$, we have

$$\begin{aligned} &\tilde{x}_i = x_i + Q_i(x,y) - Q_{i-1}(x,y), \qquad \tilde{y}_i = y_i + Q_{i-1}(x,y) - Q_i(x,y),\\ &H(x \otimes y) = Q_0(x,y),\\ &Q_i(x,y) = \min(x_{i+1}, y_i), \end{aligned} \tag{2.5}$$

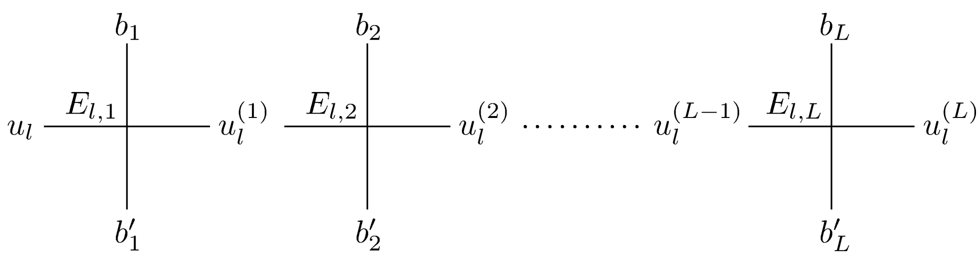
where we have expressed $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$. All indices i should be considered as $i \in \mathbb{Z}/2\mathbb{Z}$. There is another graphical method due to Nakayashiki–Yamada [NY] (see [S5] for generalizations). It is useful when we are going to prove mathematical statements.

§ 3. Local energy distribution

We express the isomorphism $a \otimes b_1 \simeq b_1' \otimes a'$ (with the energy function $e_1 := H(a \otimes b_1)$) by the following vertex diagram:



Definition 3.1. *(1) For a given path $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$, we define local energy $E_{l,j}$ by $E_{l,j} := H(u_l^{(j-1)} \otimes b_j)$. Here $u_l^{(j-1)}$ are defined in the following diagram (we set $u_l^{(0)} := u_l$):*



We define $E_{0,j} = 0$ *for all* $1 \le j \le L$. *We also use the notation* $E_l := \sum_{j=1}^{L} E_{l,j}$ *which coincides with the conserved quantity in [FOY].*
(2) We define operator T_l *by* $T_l(b) = b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L$ *(see the above diagram).* □

Lemma 3.2. *For a given path* $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L$, *we have* $E_{l,j} - E_{l-1,j} = 0$ *or 1, for all* $l > 0$ *and for all* $1 \le j \le L$. □

Definition 3.3. *The local energy distribution is a table containing* $(E_{l,j} - E_{l-1,j} = 0, 1)$ *at the position* (l, j), *i.e., at the* l *th row and the* j *th column.* □

§ 4. Results

§ 4.1. Statement

Theorem 4.1. *Let* $b = b_1 \otimes b_2 \otimes \cdots \otimes b_L \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}$ *be an arbitrary path.* b *can be highest weight or non-highest weight. Set* $N = E_1(b)$. *We determine the pair of numbers* (μ_1, r_1), (μ_2, r_2), $\cdots$, (μ_N, r_N) *by the following procedure from Step 1 to Step 4. Then the resulting* $(\lambda, (\mu, r))$ *coincides with the (unrestricted) rigged configuration* $\phi(b)$.

1. *Draw the local energy distribution for* b.
2. *Starting from the rightmost 1 in the* $l = 1$ *st row, pick one 1 from each successive row. The one in the* $(l + 1)$ *th row must be weakly right of the one selected in the* l *th row. If there is no such 1 in the* $(l + 1)$ *th row, the position of the lastly picked 1 is called* (μ_1, j_1). *Change all selected 1 into 0.*
3. *Repeat Step 2 for* $(N - 1)$ *times to further determine* (μ_2, j_2), $\cdots$, (μ_N, j_N) *thereby making all 1 into 0.*
4. *Determine* $r_1, \cdots, r_N$ *by*

$$r_k = \sum_{i=1}^{j_k - 1} \min(\mu_k, \lambda_i) + E_{\mu_k, j_k} - 2 \sum_{i=1}^{j_k} E_{\mu_k, i}. \tag{4.1}$$

Sketch of proof. The key formula is

$$E_l = \sum_{i=1}^{N} \min(\mu_i, l). \tag{4.2}$$

The rest is combinatorial arguments whose details we left to [S2]. □

In order to depict the rigged configurations, we usually use Young diagrammatic expression whose rows have lengths $\mu_1, \cdots, \mu_N$, and we put integers $r_1, \cdots, r_N$ on the right of rows $\mu_1, \cdots, \mu_N$, respectively. Integers r_k are called *riggings* corresponding to μ_k. The letters 1 in the local energy distribution precisely records combinatorial procedure of ϕ, i.e., letter 1 at l th row, k th column of the local energy distribution corresponds to box addition to l th column of some row of rigged configuration.

The groups μ_i obtained here represent solitons (moving at velocity μ_i with respect to T_∞) contained in a path. In fact, we have the following property [KOSTY].

Proposition 4.2. *Given the (unrestricted) rigged configuration corresponding to b:*

$$b \xrightarrow{\phi} \big(\lambda, (\mu_j, r_j)_{j=1}^N\big). \tag{4.3}$$

Then, corresponding to $T_l(b)$, we have

$$T_l(b) \xrightarrow{\phi} \big(\lambda, (\mu_j, r_j + \min(\mu_j, l))_{j=1}^N\big). \tag{4.4}$$

□

For the proof, see Proposition 2.6 of [KOSTY] (see also Proposition 2.3 of [S2]).

Remark. A crystal interpretation of the inverse map ϕ^{-1} is known [S1] for the cases $B^{1,s_1} \otimes \cdots \otimes B^{1,s_L}$. This formalism gives recursive description of ϕ^{-1} with respect to rank of $\mathfrak{sl}_n$ and, owing to this property, it is substantially used in [KSY]. However, our formalism seems to have different origin from [S1], since it can be generalized [S3] to wider class of representations $B^{r_1,s_1} \otimes \cdots \otimes B^{r_L,s_L}$. In this generalization, the procedure is done almost separately with respect to the rank. □

§ 4.2. Example

Consider the following path:

$$b = \boxed{1111} \otimes \boxed{11} \otimes \boxed{2} \otimes \boxed{1122} \otimes \boxed{1222} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{122} \tag{4.5}$$

Corresponding to Step 1, the local energy distribution is given by the following table (j stands for column coordinate of the table).

	1111	11	2	1122	1222	1	2	122
$E_{1,j} - E_{0,j}$	0	0	1	0	1	0	1	0
$E_{2,j} - E_{1,j}$	0	0	0	1	0	0	0	1
$E_{3,j} - E_{2,j}$	0	0	0	1	0	0	0	0
$E_{4,j} - E_{3,j}$	0	0	0	0	1	0	0	0
$E_{5,j} - E_{4,j}$	0	0	0	0	1	0	0	0
$E_{6,j} - E_{5,j}$	0	0	0	0	0	0	0	1
$E_{7,j} - E_{6,j}$	0	0	0	0	0	0	0	0

Following Step 2 and Step 3, letters 1 contained in the above table are found to be classified into 3 groups, as indicated in the following table.

	1111	11	2	1122	1222	1	2	122
$E_{1,j}-E_{0,j}$			3		2^*		1	
$E_{2,j}-E_{1,j}$				3				1^*
$E_{3,j}-E_{2,j}$				3				
$E_{4,j}-E_{3,j}$					3			
$E_{5,j}-E_{4,j}$					3			
$E_{6,j}-E_{5,j}$								3^*
$E_{7,j}-E_{6,j}$								

From the above table, we see that the cardinalities of groups 1, 2 and 3 are 2, 1 and 6, respectively. Also, in the above table, positions of (μ_1,j_1), (μ_2,j_2) and (μ_3,j_3) are indicated by asterisks. Their explicit locations are $(\mu_1,j_1)=(2,8)$, $(\mu_2,j_2)=(1,5)$ and $(\mu_3,j_3)=(6,8)$.

Now we evaluate riggings r_i according to Eq.(4.1).

$$\begin{aligned}
r_1 &= \sum_{i=1}^{8-1}\min(2,\lambda_i)+E_{2,8}-2\sum_{i=1}^{8}E_{2,i}\\
&=(2+2+1+2+2+1+1)+1-2(0+0+1+1+1+0+1+1)\\
&=2,\\
r_2 &= \sum_{i=1}^{5-1}\min(1,\lambda_i)+E_{1,5}-2\sum_{i=1}^{5}E_{1,i}\\
&=(1+1+1+1)+1-2(0+0+1+0+1)\\
&=1,\\
r_3 &= \sum_{i=1}^{8-1}\min(6,\lambda_i)+E_{6,8}-2\sum_{i=1}^{8}E_{6,i}\\
&=(4+2+1+4+4+1+1)+2-2(0+0+1+2+3+0+1+2)\\
&=1.
\end{aligned}$$

Therefore we obtain $(\mu_1,r_1)=(2,2)$, $(\mu_2,r_2)=(1,1)$ and $(\mu_3,r_3)=(6,1)$. This coincides with the calculation based on the original combinatorial definition of the map ϕ.

§ 5. Application to periodic box-ball system

§ 5.1. Definition

In this section, we consider application of Theorem 4.1 to the periodic box-ball system (pBBS). Many part of this section is contained in [KTT, KS1]. We exclusively treat $\mathfrak{sl}_2$ type path b of the form $b \in B_1^{\otimes L}$. The pBBS is the BBS with periodic boundary condition and its definition rely on the following fact.

Proposition 5.1. *Define $v_l \in B_l$ by*

$$u_l \otimes b \overset{R}{\simeq} T_l(b) \otimes v_l. \tag{5.1}$$

Then we have

$$v_l \otimes b \overset{R}{\simeq} b^* \otimes v_l, \tag{5.2}$$

where $b^* \in B_1^{\otimes L}$. □

For the proof, see Proposition 2.1 of [KTT] and the comment following it.

Definition 5.2. *We define operator of pBBS $\bar{T}_l$ by $\bar{T}_l(b) = b^* \in B_1^{\otimes L}$, where b^* is obtained in the right hand side of Eq.(5.2).* □

Note that $\bar{T}_1$ is merely the cyclic shift operator on a path.

Example 5.3. *The time evolutions $b, \bar{T}_l(b), \dots, \bar{T}_l^9(b)$ of the state b on the top line are listed downward for $l = 2$ and 3. The system size is $L = 14$. We omit the symbol $\otimes$ and frames of tableaux.*

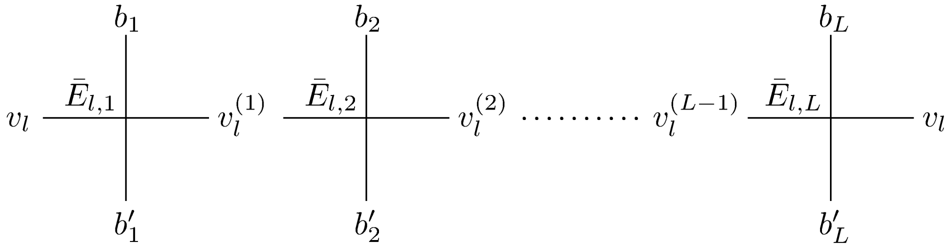
evolution under $\bar{T}_2$	*evolution under $\bar{T}_3$*
1 1 2 1 1 2 2 1 1 1 1 2 2 2	1 1 2 1 1 2 2 1 1 1 1 2 2 2
2 2 1 2 1 1 1 2 2 1 1 1 1 2	2 2 1 2 2 1 1 2 2 1 1 1 1 1
1 2 2 1 2 2 1 1 1 2 2 1 1 1	1 1 2 1 1 2 2 1 1 2 2 2 1 1
1 1 1 2 1 2 2 2 1 1 1 2 2 1	2 1 1 2 1 1 1 2 2 1 1 1 2 2
2 1 1 1 2 1 1 2 2 2 1 1 1 2	1 2 2 1 2 2 1 1 1 2 2 1 1 1
1 2 2 1 1 2 1 1 1 2 2 2 1 1	1 1 1 2 1 1 2 2 2 1 1 2 2 1
1 1 1 2 2 1 2 1 1 1 1 2 2 2	2 2 1 1 2 1 1 1 1 2 2 1 1 2
2 2 1 1 1 2 1 2 2 1 1 1 1 2	1 1 2 2 1 2 2 1 1 1 1 2 2 1
1 2 2 2 1 1 2 1 1 2 2 1 1 1	2 1 1 1 2 1 1 2 2 2 1 1 1 2
1 1 1 2 2 2 1 2 1 1 1 2 2 1	1 2 2 1 1 2 1 1 1 1 2 2 2 1

There are three solitons with amplitudes $3, 2$ and 1 traveling to the right. □

§ 5.2. Basic procedures

In the rest of the note, we exclusively consider the path $b \in B_1^{\otimes L}$ where number of $\boxed{2}$ is equal to or less than that of $\boxed{1}$. The other case follows from this case by virtue of Proposition 2.3 of [KTT]. In order to analyze the pBBS by using Theorem 4.1, we follow the following procedures.

1. Instead of using u_l, use v_l of Eq.(5.1) to calculate the local energy distribution. We express energy function appearing here as $\bar{E}_{l,j}$ and $\bar{E}_l = \sum_j \bar{E}_{l,j}$. See the following diagram:



2. Pick one of the lowest 1. Pick 1 in lth row which is weakly left of the already selected 1 in $(l+1)$th row. If there is no such 1, return to the rightmost column and search 1.

3. Find a vertical line such that no soliton cross the line. We call the line *seam*. By using cyclic shift $\bar{T}_1$, move the seam to the left end of the path. Write this procedure as $b_+ = \bar{T}_1^d(b)$.

4. Apply Theorem 4.1 to b_+. (In order to obtain the local energy distribution here, we only have to rotate periodic version of the local energy distribution obtained in Step 1 according to $\bar{T}_1^d$).

We can always find seam of a path due to the following simple property and Proposition 6.2.

Lemma 5.4. *For arbitrary element $b \in B_1^{\otimes L}$, there exists integer d such that $b_+ = \bar{T}_1^d(b)$ is highest weight.* □

This assertion is proved by elementary argument. See Example 3.2 of [KTT].

The b_+ in this lemma can be used in Step 3 of the above procedure. Consider the path $b_+^{\otimes n}$ and calculate the local energy distribution using u_l (not v_l). Combining Eq.(5.2) and the argument used in Proposition 6.2, we can show that the local energy distribution is n times repetition of that of b_+.

§ 5.3. Action variables

Definition 5.5. *Given $b \in B_1^{\otimes L}$, choose the highest element b_+ such that there exists integer d such that $b = \bar{T}_1(b_+)$. Apply the bijection ϕ and obtain $\phi(b_+) = \big((1^L), (\mu, J)\big)$. Then μ is called action variable of b.* □

Proposition 5.6. *For any $l \in \mathbb{Z}_{\geq 1}$, the action variable of $\bar{T}_l(b)$ is equal to that of b.*

Sketch of proof. Follows from the relation $\bar{E}_k(\bar{T}_l(b)) = \bar{E}_k(b)$ which is the consequence of the Yang–Baxter relation for the affine crystals. See Theorem 2.2 of [KTT] for more details. □

§ 5.4. Definition of the angle variables

Thanks to Proposition 5.1, our basic strategy is to embed suitably cut periodic path b into the usual infinite system as $b \otimes b \otimes \cdots \otimes b$. For this purpose, it is convenient to use the highest path obtained by $b_+ = \bar{T}_1^d(b)$ with suitable d. However, this correspondence between b and (d, b_+) is not unique in general. In the following, we give prescriptions to cope with this ambiguity.

5.4.1. Notations We fix some notations used in the following arguments. Let b_+ be a highest element of $B_1^{\otimes L}$ and the corresponding rigged configuration be

$$b_+ \xrightarrow{\phi} \big((1^L), (\mu_i, J_i)_{i=1}^N\big). \tag{5.3}$$

Denote the multiplicity of k in $(\mu_i)_{i=1}^N$ by m_k, and the riggings corresponding to length k rows by $J_1^{(k)} \leq J_2^{(k)} \leq \cdots \leq J_{m_k}^{(k)}$. Let the distinct lengths of rows of (μ_i) be $k_1 < k_2 < \cdots < k_s$. We denote the set of distinct lengths of rows of (μ_i) as $H = \{k_1, \ldots, k_s\}$. Finally, we define the set of all possible riggings as follows:

$$\mathrm{Rig}_L(\mu) = \left\{ \left(J_i^{(k)}\right)_{1 \leq i \leq m_k, k \in H} \in \mathbb{Z}^{m_{k_1}} \times \cdots \times \mathbb{Z}^{m_{k_s}} \;\middle|\; 0 \leq J_1^{(k)} \leq \cdots \leq J_{m_k}^{(k)} \leq p_k \right\}. \tag{5.4}$$

We sometimes omit L of $\mathrm{Rig}_L(\mu)$ such as $\mathrm{Rig}(\mu)$. Here integer p_k is called the vacancy number defined by

$$p_k = L - 2\sum_{i=1}^{N} \min(k, \mu_i). \tag{5.5}$$

In the present setting, we have $0 \leq p_{k_s} < p_{k_{s-1}} < \cdots < p_{k_1}$ (positivity $0 \leq p_{k_i}$ follows from the fact that b_+ is highest weight, and other inequalities $<$ follow from the shape of the quantum space (1^L)).

5.4.2. Extension of riggings First we give motivations for extension of riggings. From Proposition 6.2, the rigged configuration corresponding to $b_+^{\otimes n}$ have $n \times m_k$ rows of length k, and the associated riggings takes the form

$$\begin{aligned}
&J_1^{(k)} \leq J_2^{(k)} \leq \cdots \leq J_{m_k}^{(k)} \\
&\leq J_1^{(k)} + p_k \leq J_2^{(k)} + p_k \leq \cdots \leq J_{m_k}^{(k)} + p_k \\
&\leq \cdots \\
&\leq J_1^{(k)} + (n-1)p_k \leq J_2^{(k)} + (n-1)p_k \leq \cdots \leq J_{m_k}^{(k)} + (n-1)p_k.
\end{aligned} \tag{5.6}$$

In view of this observation, we define extension of riggings

$$\iota : \left(J_i^{(k)}\right)_{1\leq i\leq m_k} \longmapsto \left(J_i^{(k)}\right)_{i\in\mathbb{Z}} \tag{5.7}$$

by the relation

$$J_{i+m_k}^{(k)} = J_i^{(k)} + p_k. \tag{5.8}$$

This $\iota(J)$ can be considered as an element of the following set

$$\bar{\mathcal{J}}_k = \left\{ \left(J_i^{(k)}\right)_{i\in\mathbb{Z}} \middle| J_i^{(k)} \in \mathbb{Z},\ J_i^{(k)} \leq J_{i+1}^{(k)},\ J_{i+m_k}^{(k)} = J_i^{(k)} + p_k,\ \forall i \right\}. \tag{5.9}$$

We also define

$$\bar{\mathcal{J}} = \bar{\mathcal{J}}(\mu) = \bar{\mathcal{J}}_{k_1} \times \bar{\mathcal{J}}_{k_2} \times \cdots \times \bar{\mathcal{J}}_{k_s}. \tag{5.10}$$

5.4.3. Slide σ_l and equivalence relation On the extended riggings, we define the following important operations.

Definition 5.7. *For $l \in \mathbb{Z}_{\geq 1}$, we define $\sigma_l : \bar{\mathcal{J}}_k \mapsto \bar{\mathcal{J}}_k$ by*

$$\sigma_l : \left(J_i^{(k)}\right)_{i\in\mathbb{Z}} \longmapsto \left(J_{i+\delta_{l,k}}^{(k)} + 2\min(l,k)\right)_{i\in\mathbb{Z}}. \tag{5.11}$$

We define abelian group $\mathcal{A}$ by

$$\mathcal{A} = \left\{ \sigma_{k_1}^{n_1}\sigma_{k_2}^{n_2}\cdots\sigma_{k_s}^{n_s} \middle| n_1, n_2, \ldots, n_s \in \mathbb{Z} \right\}. \tag{5.12}$$

We call an element of $\mathcal{A}$ slide. □

We naturally define σ_l on $\bar{\mathcal{J}}$ by $\sigma_l(\bar{\mathcal{J}}) = \sigma_l(\bar{\mathcal{J}}_{k_1}) \times \sigma_l(\bar{\mathcal{J}}_{k_2}) \times \cdots \times \sigma_l(\bar{\mathcal{J}}_{k_s})$.

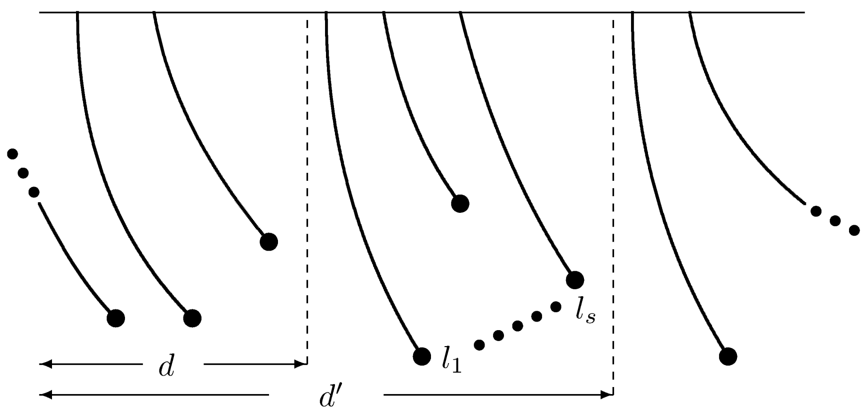
Definition 5.8. *We define equivalence relation $\simeq$ between $J, K \in \bar{\mathcal{J}}$ by the following condition: $J \simeq K$ if $\exists \sigma \in \mathcal{A}$ such that $J = \sigma(K)$.* □

We have the following standard form with respect to the above $\simeq$.

Proposition 5.9. *For any $\bar{J} \in \bar{\mathcal{J}}(\mu)$, there exist $d \in \mathbb{Z}$ and $J \in \mathrm{Rig}(\mu)$ such that $\bar{J} \simeq \iota(J) + d$.*

Sketch of proof. There is a general algorithm to derive the standard form. Basis of the algorithm is the relation $p_{k_i} > 0$ $(2 \leq i \leq s)$. See Lemma 3.9 of [KTT] for more details. □

5.4.4. Interpretation of σ_l By using Theorem 4.1, we can give interpretation of slide σ_l in terms of the bijection ϕ. Suppose we have two expressions $p = \bar{T}_1^d(p_+)$ and $p = \bar{T}_1^{d'}(p'_+)$ with highest paths p_+ and p'_+. Let the rigged configuration corresponding to p_+ (resp. p'_+) be (μ, J) (resp. (μ, J')). Draw local energy distribution of p.



If the difference between p_+ and p'_+ is solitons of lengths $l_1, \dots, l_s$, then we have

$$\iota(J') + d' = \prod_{i=1}^{s} \sigma_{l_i} \iota(J) + d. \tag{5.13}$$

Hence $\iota(J') + d' \simeq \iota(J) + d$. This follows from direct calculation using Eq.(4.1).

Example 5.10. *Consider the paths*

$$p = 221221112221111 \qquad \textit{and} \qquad p' = 111222111122122.$$

Here we have omitted $\otimes$ and frames of tableaux, and $p = \bar{T}_1^5(p')$. Then the local energy distribution takes the following form (all letters 0 are suppressed):

	2	2	1	2	2	1	1	1	2	2	2	1	1	1	1
$\bar{E}_{1,j} - \bar{E}_{0,j}$	1			1					1						
$\bar{E}_{2,j} - \bar{E}_{1,j}$		1								1					
$\bar{E}_{3,j} - \bar{E}_{2,j}$					1						1				

We are setting $d = 0$, $d' = 5$, $l_1 = 3$ and $l_2 = 1$ (vacancy numbers are $p_3 = 1$ and $p_1 = 9$). The rigged configurations are

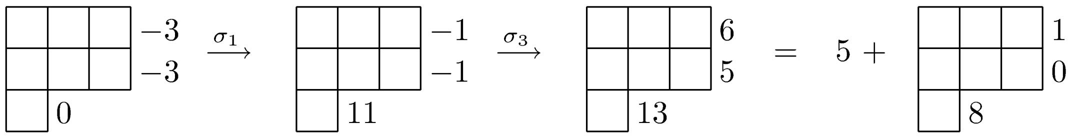
p

□	□	□	−3
□	□	□	−3
□	0		

p'

□	□	□	1
□	□	□	0
□	8		

Calculation goes as follows:



This coincides with Eq.(5.13). □

5.4.5. Angle variables and inverse scattering formalism

Definition 5.11. *Given arbitrary element $b \in B_1^{\otimes L}$ where number of $\boxed{2}$ in b is equal to or less than that of $\boxed{1}$. Then the angle variable $[\iota(J)+d] \in \mathcal{J}(\mu)$ corresponding to b is defined by the following procedure.*

1. *Find integer d and highest path b_+ such that $b = \bar{T}_1^d(b_+)$.*
2. *Apply the bijection ϕ and obtain $\phi(b_+) = \big((1^L),(\mu,J)\big)$.*
3. *Extend the rigging to obtain $\iota(J)+d \in \bar{\mathcal{J}}(\mu)$.*
4. *Take equivalent class with respect to $\simeq$ and obtain $[\iota(J)+d] \in \mathcal{J}(\mu)$.* □

Remark. The procedure in Definition 5.11 uniquely determines the angle variable, despite the non-uniqueness of (d, b_+) in Step 1. □

The following is the main theorem of [KTT].

Theorem 5.12. *Let the angle variable corresponding to b be $\left(J_i^{(k)}\right)_{i\in\mathbb{Z},k\in H}$. Then the angle variable corresponding to $\bar{T}_l(b)$ is $\left(J_i^{(k)} + \min(k,l)\right)_{i\in\mathbb{Z},k\in H}$.* □

§ 5.5. Ultradiscrete Riemann theta function

In this section, we assume that all solitons have distinct lengths $\mu_1 < \mu_2 < \cdots < \mu_g$. More general case including solitons with same length can be treated similarly (see [KS2]). Define ultradiscrete Riemann theta function as follows:

$$\begin{aligned}\Theta(\mathbf{z}) &= \lim_{\epsilon\to+0} \epsilon \log\left(\sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\Big(-\frac{{}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}}{\epsilon}\Big)\right) \\ &= -\min_{\mathbf{n}\in\mathbb{Z}^g}\{{}^t\mathbf{n}A\mathbf{n}/2 + {}^t\mathbf{n}\mathbf{z}\}.\end{aligned} \tag{5.14}$$

Here A is the symmetric positive definite $g \times g$ integer matrix appearing in the string center equation [KN]:

$$(A)_{i,j} = \delta_{i,j} p_{\mu_i} + 2\min(\mu_i, \mu_j). \tag{5.15}$$

Here p_i is the vacancy number.

We introduce the vectors

$$\mathbf{h}_l = (\min(\mu_i, l))_{i=1}^g \in \mathbb{Z}^g, \tag{5.16}$$

$$\mathbf{p} = (p_{\mu_i})_{i=1}^g \in \mathbb{Z}^g. \tag{5.17}$$

Again, consider the highest path b_+ obtained by $b = \bar{T}_1^d(b_+)$. Let the rigging corresponding to b_+ be $\mathbf{J} = (J_i)_{i=1}^g$. Then we define $\mathbf{I} = (J_i + d)_{i=1}^g$.

Definition 5.13. *For $1 \le k \le L$ and $r = 0, 1$, we define the ultradiscrete tau function as follows:*

$$\tau_r(k) = \Theta\left(\mathbf{I} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1 + r\mathbf{h}_\infty\right). \tag{5.18}$$

□

Theorem 5.14. *Under the above settings, the state p is expressed as $p = (1 - x(1), x(1)) \otimes \cdots \otimes (1 - x(L), x(L))$, where*

$$x(k) = \tau_0(k) - \tau_0(k-1) - \tau_1(k) + \tau_1(k-1). \tag{5.19}$$

□

Proof of this assertion uses Proposition 6.2 and the main result of [KSY]. See [KS1]. Note that this result itself is independent to Theorem 5.12.

Combining this and Theorem 5.12, we solve the initial value problem of the pBBS.

§ 6. Appendix: Tensor product of highest paths

In the appendix, we clarify special property of tensor product of highest paths which gives the basis for the inverse scattering formalism of pBBS. To begin with, we recall famous characterization of highest paths.

Lemma 6.1. *The highest elements b ($\tilde{e}_1 b = 0$) of the form $b_1 \otimes \cdots \otimes b_L \in B_1^{\otimes L}$ are characterized by the following Yamanouchi condition:*

$$\#\left\{1 \le i \le k \,\middle|\, b_i = \boxed{1}\right\} \ge \#\left\{1 \le i \le k \,\middle|\, b_i = \boxed{2}\right\} \tag{6.1}$$

for all $1 \le k \le L$. □

In order to prove the following assertion, we need to look at the original combinatorial description of the map ϕ in addition to Theorem 4.1. For description of the combinatorial algorithm of ϕ, see, e.g., Appendix A of [KTT] or Appendix C of [KSY].

Here we summarize basic definitions which will be used in the proof. Consider the rigged configuration $\left((\lambda_i)_{i=1}^L, (l_j, I_j)_{j=1}^N\right)$. We call λ quantum space and (l, I) configuration. Then the vacancy number p_k for $k > 0$ is defined by

$$p_k := \sum_{i=1}^{L} \min(k, \lambda_i) - 2\sum_{j=1}^{N} \min(k, l_j). \tag{6.2}$$

The row l_j is called singular if the corresponding rigging I_j is equal to the vacancy number p_{l_j} for the row l_j, i.e., $p_{l_j} = I_j$. Finally, we call quantity $p_{l_j} - I_j$ corigging. It is known that $p_{l_j} \geq I_j$ for all (l_j, I_j).

In the following, we have to consider paths of the form $q \otimes r$ where q is arbitrary highest path and r is highest path of the form $r \in B_1^{\otimes M}$. The basic points of the combinatorial procedure of ϕ in this setting are the following. First of all, recall that the combinatorial procedure proceeds recursively from the left of path to the right. So we assume that we have done the procedure on q and we exclusively consider the combinatorial procedure on r. To be more precise, suppose that we have constructed the rigged configuration corresponding to $q \otimes r_{[k]}$ where $r_{[k]}$ is the first k components of r. Then we are going to construct the rigged configuration corresponding to $q \otimes r_{[k+1]}$ according to $\boxed{1}$ or $\boxed{2}$ of $(k+1)$th factor of r. In both cases, we add one row of length one to the quantum space of the rigged configuration corresponding to $q \otimes r_{[k]}$. For $\boxed{2}$, we add one box to the longest singular row of configuration or, if there is no singular row, we add one row of length one to the configuration. The riggings for $q \otimes r_{[k+1]}$ are the same as those for $q \otimes r_{[k]}$ except the rigging of row of configuration that is different from $q \otimes r_{[k]}$. We set the latter rigging equal to the vacancy number (computed with the data of the rigged configuration for $q \otimes r_{[k+1]}$) for the corresponding row.

Proposition 6.2. *Given two highest paths q and r as follows:*

$$q \in B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_L}, \tag{6.3}$$

$$r \in B_{\mu_1} \otimes B_{\mu_2} \otimes \cdots \otimes B_{\mu_M}. \tag{6.4}$$

Suppose that their rigged configurations are $\phi(q) = (\lambda, (l, I))$ and $\phi(r) = (\mu, (m, K))$. Then the rigged configuration of the highest path $q \otimes r$ is given by $\phi(q \otimes r) = (\lambda \cup \mu, (l \cup m, I \cup K'))$, where $K' = \left(K_i^{'(j)}\right)$ is given by

$$K_i^{'(j)} = K_i^{(j)} + \tilde{p}_j, \qquad \tilde{p}_j := \sum_{k_1} \min(j, \lambda_{k_1}) - 2\sum_{k_2} \min(j, l_{k_2}), \tag{6.5}$$

and $(l \cup m, I \cup K')$ means the union of (l, I) and (m, K') as multi-sets of rows assigned with rigging. $\tilde{p}_j$ is the vacancy number for $(\lambda, (l, I))$.

Proof. Special version ($\lambda_i = 1$ and $\mu_i = 1$ for all i) of this claim is proved in Lemma C.1 of [KTT], omitting some of details. We include here an alternative proof intended to clarify why the highest weight condition is necessary for this result.

Consider the path $q \otimes r \otimes \boxed{1}^{\otimes \Lambda}$ with $\Lambda \gg |\mu|$. Recall the property of the energy function $H(b \otimes u_l) = 0$ for arbitrary $b \in B_k$. This means that entries of the local energy distribution under $\boxed{1}^{\otimes \Lambda}$ are all 0. Therefore the rigged configuration obtained by Theorem 4.1 is the same as that for the path $q \otimes r$ except the extra (1^Λ) of the quantum space. So we can always think about paths of the form $q \otimes r \otimes \boxed{1}^{\otimes \Lambda}$ by putting the tail $\boxed{1}^{\otimes \Lambda}$ on the right of a given path. Consider the isomorphism $q \otimes r \otimes \boxed{1}^{\otimes \Lambda} \simeq q \otimes p^* \otimes (\bigotimes_i u_{\mu_i})$, where $p^* \in B_1^{\otimes \Lambda}$ and p^* is highest. From Lemma 8.5 of [KSS], these two isomorphic paths correspond to the same rigged configuration. Therefore we can assume $r \in B_1^{\otimes M}$ without loss of generality.

Recall the Yamanouchi condition on r and consider the combinatorial procedure of ϕ. Assume that we have finished ϕ on q part and we are going to apply ϕ to r. Since we are assuming $r \in B_1^{\otimes M}$, all letters 1 contained in r correspond to length 1 row of the quantum space. Fix a row of length l that was constructed from q, and consider the change of corrigings induced by r. During the procedure ϕ, if the chosen row does not obtain new box, then the corresponding rigging does not change. In such situation, we only need to keep track of change of the vacancy numbers by using Eq.(6.2). Then letters $\boxed{1}$ of r increase its corigging by 1, on the other hand, letters $\boxed{2}$ of r decrease the coriggings at most 1. Therefore the Yamanouchi condition means that, after creating rows corresponding to q, those rows never become singular during r.

Hence we can assume that rows corresponding to r are independent to that of q, thus we have $(l \cup m)$ as a configuration. In terms of the local energy distribution, this means that no solitons cross the boundary between q and r. As to the riggings, those corresponding to m are larger than K by $\tilde{p}_j$. This follows from direct calculation using Eq.(4.1). □

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