A q-analogue of Catalan Hankel determinants

By

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Abstract

In this article we shall survey the various methods of evaluating Hankel determinants and as an illustration we evaluate some Hankel determinants of a q-analogue of Catalan numbers. Here we consider $\frac{(aq;q)_n}{(abq^2;q)_n}$ as a q-analogue of Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is known as the moments of the little q-Jacobi polynomials. We also give several proofs of this q-analogue, in which we use lattice paths, the orthogonal polynomials, or the basic hypergeometric series. We also consider a q-analogue of Schröder Hankel determinants, and give a new proof of Moztkin Hankel determinants using an addition formula for $_2F_1$.

§1. Introduction

Given a sequence a_0, a_1, a_2,..., we set the Hankel matrix of the sequence to be

$$
(1.1) \qquad A_n^{(t)} = (a_{i+j+t})_{0 \le i,j \le n-1} = \begin{pmatrix} a_t & a_{t+1} & \cdots & a_{t+n-1} \\ a_{t+1} & a_{t+2} & \cdots & a_{t+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t+n-1} & a_{t+n} & \cdots & a_{t+2n-2} \end{pmatrix}.
$$

For $n = 0, 1, 2, \ldots$, let

$$
(1.2) \qquad C_n = \frac{1}{n+1}\binom{2n}{n},
$$

which are called the *Catalan numbers*. The generating function for the Catalan numbers is given by

$$
\sum_{n \ge 0} C_n t^n = \frac{1 - \sqrt{1-4t}}{2t}.
$$

If we put $a_n = C_n$ in (1.1), then the following identity is well-known and several proofs are known [5, 6, 14, 17, 19]:

$$(1.3) \qquad \det A_n^{(t)} = \det\,(C_{i+j+t})_{0\le i,j\le n-1} = \prod_{1\le i\le j\le t-1} \frac{i+j+2n}{i+j}.$$

If we put $B_n = \binom{2n+1}{n}$ and $D_n = \binom{2n}{n}$, then the following variations are also known [17]:

$$(1.4) \qquad \det\,(B_{i+j+t})_{0\le i,j\le n-1} = \prod_{1\le i\le j\le t-1} \frac{i+j-1+2n}{i+j-1},$$

$$(1.5) \qquad \det\,(D_{i+j+t})_{0\le i,j\le n-1} = 2^n \prod_{1\le i<j\le t-1} \frac{i+j+2n}{i+j}.$$

As a generalization of (1.3), Krattenthaler [12] has obtained

$$(1.6) \qquad \det\,(C_{k_i+j})_{0\le i,j\le n-1} = \prod_{0\le i<j\le n-1} (k_j-k_i) \prod_{i=0}^{n-1} \frac{(i+n)!(2k_i)!}{(2i)!k_i!(k_i+n)!}$$

for a positive integer n and non-negative integers k_0, k_1,…, k_{n-1}.

In this article we shall survey the various methods of evaluating Hankel determinants and as an illustration we give a q-analogue of the above results. We first recall some terminology in q-series (see Gasper-Rahman's book [9]) before stating the main theorem. Next some terminology is defined before stating the main theorem. We use the notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

for a nonnegative integer $n \ge 0$. Usually $(a;q)_n$ is called the *q-shifted factorial*, and we frequently use the compact notation:

$$(a_1,a_2,\ldots,a_r;q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty,$$
$$(a_1,a_2,\ldots,a_r;q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n.$$

If we put $a = q^\alpha$ and $q \to 1$, then we have

$$\lim_{q\to 1} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n,$$

where $(\alpha)_n = \prod_{k=0}^{n-1}(\alpha + k)$ is called the *raising factorial*. We shall define the $_{r+1}\phi_r$ *basic hypergeometric series* by

$$_{r+1}\phi_r \left[\begin{array}{c} a_1,a_2,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{array} ; q,z \right] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{r+1};q)_n}{(q,b_1,\ldots,b_r;q)_n} z^n.$$

If we put $a_i = q^{\alpha_i}$ and $b_i = q^{\beta_i}$ in the above series and let $q \to 1$, then we obtain the $_{r+1}F_r$ *hypergeometric series*

$$_{r+1}F_r \left[\begin{array}{c} \alpha_1,\alpha_2,\ldots,\alpha_{r+1} \\ \beta_1,\ldots,\beta_r \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_{r+1})_n}{n!(\beta_1)_n \ldots (\beta_r)_n} z^n.$$

The Motzkin number M_n is defined to be

$$M_n = {}_2F_1\left[\begin{array}{c}(1-n)/2, -n/2\\2\end{array}; 4\right].$$

The generating function for the Motzkin numbers is given by

$$\sum_{n=0}^{\infty} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

It is known [1] that

(1.7) $$\det(M_{i+j})_{0 \le i,j \le n-1} = 1$$

for $n \ge 1$, and

(1.8) $$\det(M_{i+j+1})_{0 \le i,j \le n-1} = 1, 0, -1$$

for $n \equiv 0, 1 \pmod 6$, $n \equiv 2, 5 \pmod 6$, $n \equiv 3, 4 \pmod 6$, respectively.

The large Schröder number S_n is defined to be

$$S_n = 2 {}_2F_1\left[\begin{array}{c}-n+1, n+2\\2\end{array}; -1\right]$$

for $n \ge 1$ ($S_0 = 1$). The generating function for the large Schröder numbers is

(1.9) $$\sum_{n=0}^{\infty} S_n x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

Eu and Fu [8] have proved

(1.10) $$\det(S_{i+j})_{0 \le i,j \le n-1} = 2^{\binom{n}{2}}, \qquad \det(S_{i+j+1})_{0 \le i,j \le n-1} = 2^{\binom{n+1}{2}}$$

for $n \ge 1$ (see [4, 8, 16]). We can also prove that

(1.11) $$\det(S_{i+j+2})_{0 \le i,j \le n-1} = 2^{\binom{n+1}{2}}(2^{n+1} - 1)$$

holds for $n \ge 1$.

In this article, as a generalization of (1.2), we choose

(1.12) $$\mu_n = \frac{(aq; q)_n}{(abq^2; q)_n}$$

for a nonnegative integer n. The aim of this article is to give three different proofs of the following theorem:

Theorem 1.1. *Let n be a positive integer. Then we have*

(1.13) $$\det(\mu_{i+j})_{0 \le i,j \le n-1} = a^{\frac{1}{2}n(n-1)} q^{\frac{1}{6}n(n-1)(2n-1)} \prod_{k=1}^{n} \frac{(q, aq, bq; q)_{n-k}}{(abq^{n-k+1}; q)_{n-k}(abq^2; q)_{2(n-k)}}.$$

As a corollary of this theorem we can get the following more general identity.

Corollary 1.2.   *Let n be a positive integer, and t a nonnegative integer. Then we have*

$$\det\left(\mu_{i+j+t}\right)_{0\leq i,j\leq n-1} = a^{\frac{1}{2}n(n-1)}q^{\frac{1}{6}n(n-1)(2n-1)+\frac{1}{2}n(n-1)t}\left\{\frac{(aq;q)_t}{(abq^2;q)_t}\right\}^n$$

$$(1.14)\qquad\qquad\qquad \times\prod_{k=1}^{n}\frac{(q,aq^{t+1},bq;q)_{n-k}}{(abq^{n-k+t+1};q)_{n-k}(abq^{t+2};q)_{2(n-k)}}.$$

Proof.   If we use

$$\mu_{n+t} = \frac{(aq;q)_{n+t}}{(abq^2;q)_{n+t}} = \frac{(aq;q)_t}{(abq^2;q)_t}\cdot\frac{(aq^{t+1};q)_n}{(abq^{t+2};q)_n},$$

then we have

$$\det\left(\mu_{i+j+t}\right)_{0\leq i,j\leq n-1} = \det\left(\frac{(aq;q)_t}{(abq^2;q)_t}\cdot\frac{(aq^{t+1};q)_{i+j}}{(abq^{t+2};q)_{i+j}}\right)_{0\leq i,j\leq n-1}$$

$$= \left\{\frac{(aq;q)_t}{(abq^2;q)_t}\right\}^n\det\left(\frac{(a'q;q)_{i+j}}{(a'bq^2;q)_{i+j}}\right)_{0\leq i,j\leq n-1},$$

where $a' = aq^t$. If we use (1.13), then we obtain (1.14) by a straightforward computation.   □

We can prove (1.3), (1.4) and (1.5) as a corollary of Corollary 1.2.

Proof of (1.3), (1.4) *and* (1.5)*.* If we substitute $a = q^\alpha$ and $b = q^\beta$ into ν_n, and we put $q\to 1$, then we obtain $\mu_n \to \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}$, which we write ν_n. Thus (1.14), leads to

$$\det\left(\nu_{i+j+t}\right)_{0\leq i,j\leq n-1} = \nu_t^n\prod_{k=1}^{n}\frac{(n-k)!(\alpha+t+1)_{n-k}(\beta+1)_{n-k}}{(\alpha+\beta+t+n-k+1)_{n-k}(\alpha+\beta+t+2)_{2(n-k)}}.$$

Note that

$$\nu_n = \begin{cases} C_n/2^{2n} & \text{if } \alpha = -\frac{1}{2} \text{ and } \beta = \frac{1}{2}, \\ B_n/2^{2n} & \text{if } \alpha = \frac{1}{2} \text{ and } \beta = -\frac{1}{2}, \\ D_n/2^{2n} & \text{if } \alpha = -\frac{1}{2} \text{ and } \beta = -\frac{1}{2}. \end{cases}$$

Hence we obtain

$$2^{2n(n+t-1)}\det\left(\nu_{i+j+t}\right)_{0\leq i,j\leq n-1} = \begin{cases} \det\left(C_{i+j+t}\right)_{0\leq i,j\leq n-1} & \text{if } \alpha = -\frac{1}{2} \text{ and } \beta = \frac{1}{2}, \\ \det\left(B_{i+j+t}\right)_{0\leq i,j\leq n-1} & \text{if } \alpha = \frac{1}{2} \text{ and } \beta = -\frac{1}{2}, \\ \det\left(D_{i+j+t}\right)_{0\leq i,j\leq n-1} & \text{if } \alpha = -\frac{1}{2} \text{ and } \beta = -\frac{1}{2}. \end{cases}$$

Thus we can prove (1.3), (1.4) and (1.5) by direct computations from the above identity.   □

In fact we can also obtain the following generalization of (1.6).

Theorem 1.3.   *Let n be a positive integer, and k_0, ..., k_{n-1} nonnegative integers. Then we have*

$$(1.15)\ \det\left(\mu_{k_i+j}\right)_{0\leq i,j\leq n-1} = a^{\binom{n}{2}}q^{\binom{n+1}{3}}\prod_{i=0}^{n-1}\frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}}\prod_{0\leq i<j\leq n-1}(q^{k_i}-q^{k_j})\prod_{i=0}^{n-1}(bq;q)_i.$$

§2. Non-intersecting lattice paths

In this section we give our first proof of Theorem 1.1 using non-intersecting lattice paths.

Let m and n be nonnegative integers. A *Dyck path* is, by definition, a lattice path in the plane lattice $\mathbb{Z}^2$ consisting of two types of steps: rise vector $(1,1)$ and fall vector $(1,-1)$, which never passes below the x-axis. We say a rise vector (resp. fall vector) whose origin is (x,y) and ends at $(x+1, y+1)$ (resp. $(x+1, y-1)$) has *height* y. For example, Figure 1 presents
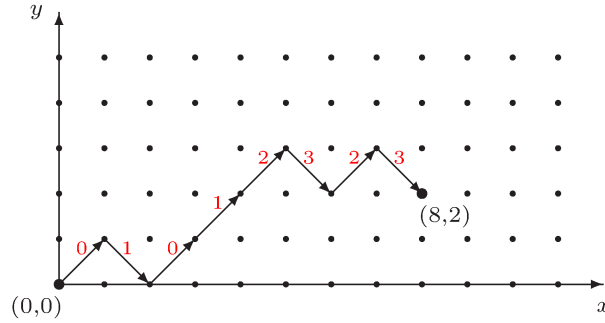

Figure 1. A Dyck Path starting from $(0,0)$ and ending at $(8,2)$

a Dyck path starting from $(0,0)$ and ending at $(8,2)$, in which each red number stands for the height of the step. Let $\mathscr{D}_{m,n}$ denote the set of Dyck paths starting from $(0,0)$ and ending at (m,n). Especially, the cardinality of $\mathscr{D}_{2n,0}$ is known to be the Catalan number C_n.

A *Motzkin path* is, by definition, a lattice path in $\mathbb{Z}^2$ consisting of three types of steps: rise vectors $(1,1)$, fall vectors $(1,-1)$, and (short) level vectors $(1,0)$ which never passes below the x-axis. We say a rise vector, fall vector and level vector whose origin is (x,y) and ends at $(x+1, y+1)$, $(x+1, y-1)$ and $(x+1, y)$ has *height* y, respectively. Figure 2 presents a
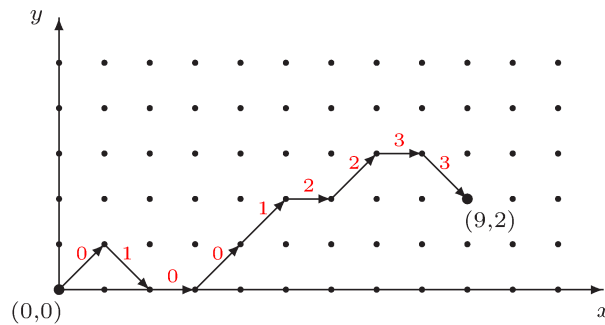
Figure 2. A Moztkin path starting from $(0,0)$ and ending at $(9,2)$

Motzkin path starting from $(0,0)$ and ending at $(9,2)$, in which each red number stands for the height of the step. Let $\mathscr{M}_{m,n}$ denote the set of Motzkin paths starting from $(0,0)$ and ending at (m,n). Note that the cardinality of $\mathscr{M}_{n,0}$ is known to be the Motzkin number M_n. We define the height of each step similarly as before.
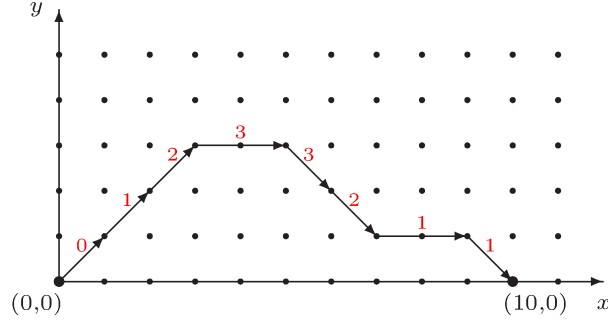
Figure 3. A Schröder path starting from $(0,0)$ and ending at $(10,0)$

A *Schröder path* is, by definition, a lattice path in $\mathbb{Z}^2$ consisting of three types of steps: rise vectors $(1,1)$, fall vectors $(1,-1)$, and long level vectors $(2,0)$ which never passes below the x-axis. Figure 3 presents a Schröder path starting from $(0,0)$ and ending at $(10,0)$, in which each red number stands for the height of the step. Let $\mathscr{S}_{m,n}$ denote the set of Schröder paths starting from $(0,0)$ and ending at (m,n). Note that the cardinality of $\mathscr{S}_{2n,0}$ is known to be the large Schröder number S_n.

Assign the weight a_h, b_h, c_h to each rise vector, fall vector, (short or long) level vector of height h, respectively. Set the weight of a path P to be the product of the weights of its edges and denote it by $w(P)$. Given any family $\mathscr{F}$ of paths, we write the generating function of $\mathscr{F}$ as

$$\mathrm{GF}\left[\mathscr{F}\right] = \sum_{P \in \mathscr{F}} w(P).$$

Proposition 2.1.   *(Flajolet [7]) The generating function for the Dyck paths is given by the following Stieltjes type continued fraction:*

$$\sum_{n \geq 0} \mathrm{GF}\left[\mathscr{D}_{(2n,0)}\right] t^{2n} = \cfrac{1}{1 - \cfrac{a_0 b_1 t^2}{1 - \cfrac{a_1 b_2 t^2}{1 - \cfrac{a_2 b_3 t^2}{\ddots}}}}.$$

Meanwhile, the generating function for the Motzkin paths is given by the following Jacobi type continued fraction:

$$\sum_{n \geq 0} \mathrm{GF}\left[\mathscr{M}_{(n,0)}\right] t^n = \cfrac{1}{1 - c_0 t - \cfrac{a_0 b_1 t^2}{1 - c_1 t - \cfrac{a_1 b_2 t^2}{1 - c_2 t - \cfrac{a_2 b_3 t^2}{\ddots}}}}.$$

It is also easy to see the following proposition holds.

Proposition 2.2.   *Let n be a positive integer. Then the generating function for Schröder*

paths is given by the following continued fraction:

$$\sum_{n \geq 0} \mathrm{GF}\left[\mathscr{S}_{(2n,0)}\right] t^{2n} = \cfrac{1}{1 - c_0 t^2 - \cfrac{a_0 b_1 t^2}{1 - c_1 t^2 - \cfrac{a_1 b_2 t^2}{1 - c_2 t^2 - \cfrac{a_2 b_3 t^2}{\ddots}}}}.$$

Next we recall notation and definitions used for the lattice path method due to Gessel and Viennot [10]. Let $D = (V, E)$ be an acyclic digraph without multiple edges. If u and v are any pair of vertices, let $\mathscr{P}(u, v)$ denote the set of all directed paths from u to v. For a fixed positive integer n, an *n-vertex* is an n-tuple of vertices of D. If $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ are n-vertices, an *n-path* from $\boldsymbol{u}$ to $\boldsymbol{v}$ is an n-tuple $\boldsymbol{P} = (P_1, \ldots, P_n)$ such that $P_i \in \mathscr{P}(u_i, v_i)$, $i = 1, \ldots, n$. The n-path $\boldsymbol{P} = (P_1, \ldots, P_n)$ is said to be *non-intersecting* if any two different paths P_i and P_j have no vertex in common. We will write $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$ for the set of all n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$, and write $\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})$ for the subset of $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$ consisting of non-intersecting n-paths. If $\boldsymbol{u} = (u_1, \ldots, u_m)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ are linearly ordered sets of vertices of D, then $\boldsymbol{u}$ is said to be *D-compatible* with $\boldsymbol{v}$ if every path $P \in \mathscr{P}(u_i, v_l)$ intersects with every path $Q \in \mathscr{P}(u_j, v_k)$ whenever $i < j$ and $k < l$. Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. Then for $\pi \in S_n$, by $\boldsymbol{v}^\pi$ we mean the n vertex $(v_{\pi(1)}, \ldots, v_{\pi(n)})$.

The weight $w(\boldsymbol{P})$ of an n-path $\boldsymbol{P}$ is defined to be the product of the weights of its components. Thus, if $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ are n-vertices, we define the generating functions $F(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{GF}\left[\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})\right] = \sum_{\boldsymbol{P} \in \mathscr{P}(\boldsymbol{u}, \boldsymbol{v})} w(\boldsymbol{P})$ and $F_0(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{GF}\left[\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})\right] = \sum_{\boldsymbol{P} \in \mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})} w(\boldsymbol{P})$. In particular, if u and v are any pair of vertices, we write

$$h(u, v) = \mathrm{GF}\left[\mathscr{P}(u, v)\right] = \sum_{P \in \mathscr{P}(u, v)} w(P).$$

The following lemma is called the Gessel-Viennot formula for counting lattice paths in terms of determinants. (See [10].)

Lemma 2.3. *(Lidström-Gessel-Viennot)*
Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ be two n-vertices in an acyclic digraph D. Then

$$(2.1) \qquad \sum_{\pi \in S_n} \mathrm{sgn}\,\pi \; F_0(\boldsymbol{u}^\pi, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}.$$

In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then

$$(2.2) \qquad F_0(\boldsymbol{u}, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}.$$

If we apply Lemma 2.3 to Dyck paths, then we obtain the following proposition:

Proposition 2.4. *Let $G_m = \mathrm{GF}\left[\mathscr{D}_{2m,0}\right]$ for non-negative integer m.*
(i) If $t = 0$, then we have

$$(2.3) \qquad \det\left(G_{i+j}\right)_{0 \leq i, j \leq n-1} = \prod_{i=1}^{n} \left(a_{2i-2} b_{2i-1} a_{2i-1} b_{2i}\right)^{n-i}.$$

(ii) If $t = 1$, then we have

$$(2.4) \qquad \det\left(G_{i+j+1}\right)_{0 \leq i, j \leq n-1} = \prod_{i=1}^{n} \left(a_{2i-2} b_{2i-1}\right)^{n-i+1} \left(a_{2i-1} b_{2i}\right)^{n-i}.$$

(iii) If $t = 2$, then we have $\det{(G_{i+j+2})}_{0 \le i,j \le n-1}$ equals

$$(2.5) \qquad \sum_{k=0}^{n} \prod_{i=1}^{k} \left(a_0 a_1 \cdots a_{2i-3} a_{2i-2}^2 b_1 b_2 \cdots b_{2i-1} b_{2i-1}^2 \right) \cdot \prod_{i=1}^{k} \left(a_0 a_1 \cdots a_{2i-1} b_1 b_2 \cdots b_{2i} \right).$$

(iv) If $t = 3$, then we have $\det{(G_{i+j+3})}_{0 \le i,j \le n-1}$ equals

$$\sum_{k=0}^{n} \Big\{ \sum_{l=0}^{k} \prod_{i=1}^{l} \left(a_0 a_1 \cdots a_{2i-3} a_{2i-2}^2 b_{2i-1} \right) \prod_{i=l+1}^{k} \left(a_0 a_1 \cdots a_{2i-3} a_{2i-2} a_{2i-1} b_{2i} \right) \Big\}$$

$$\times \Big\{ \sum_{l=0}^{k} \prod_{i=1}^{l} \left(b_1 b_2 \cdots b_{2i-2} b_{2i-1}^2 a_{2i-2} \right) \prod_{i=l+1}^{k} \left(b_1 b_2 \cdots b_{2i-2} b_{2i-1} b_{2i} a_{2i-1} \right) \Big\}$$

$$(2.6) \qquad \times \prod_{i=k+1}^{n} \left(a_0 a_1 \cdots a_{2i-1} a_{2i} b_1 b_2 \cdots b_{2i} b_{2i+1} \right).$$

((i) and (ii) of this proposition are originally appeared in [18, Ch. 4, §3].)

Proof. We consider the digraph (V, E), in which V is the plane lattice $\mathbb{Z}^2$ and E the set of rise vectors and fall vectors in the above half plane. Let $u_i = (x_0 - 2(i-1), 0)$ and $v_j = (x_0 + 2(j + t - 1), 0)$ for $i, j = 1, 2, \ldots, n$, $t = 0, 1, 2, 3$ and a fixed integer x_0. It is easy to see that the n-vertex $\boldsymbol{u} = (u_1, \ldots, u_n)$ is D-compatible with the n-vertex $\boldsymbol{v} = (v_1, \ldots, v_n)$. If $t = 0$, then there is always a unique n-path $\boldsymbol{P} = (P_1, \ldots, P_n)$ that connect $\boldsymbol{u}$ to $\boldsymbol{v}$ as in
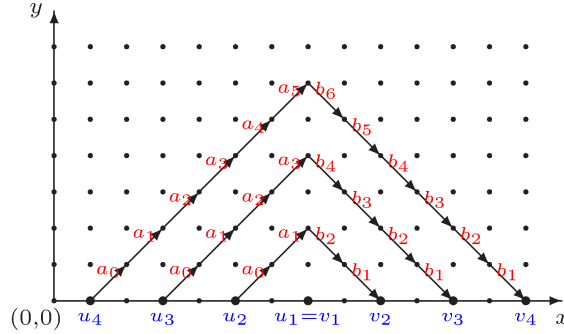

Figure 4. $t = 0$ and $n = 4$

Figure 4. By multiplying the weights of all edges in $\boldsymbol{P}$, we obtain the right-hand side of (2.3). On the other hand, applying Lemma 2.3, we obtain the left-hand side of (2.3).

The other identities can be proven similarly. For example, if $t = 1$, there is only one n-path $\boldsymbol{P} = (P_1, \ldots, P_n)$ that connect $\boldsymbol{u}$ to $\boldsymbol{v}$ as in Figure 5. As the product of the weights of all edges in $\boldsymbol{P}$ we obtain (2.4). If $t = 2$, there are $(n+1)$ ways to connect $\boldsymbol{u}$ to $\boldsymbol{v}$ with n-path $\boldsymbol{P} = (P_1, \ldots, P_n)$. As an example, we show one way in Figure 6. A similar reasoning leads to (2.5). One can also derive (2.6) by a similar argument. □

We assign the following weight to each step: the weight of a rise vector is 1, while the weight of a fall vector of height h is

$$(2.7) \qquad \lambda_h = \begin{cases} \frac{q^k (1 - a q^{k+1})(1 - a b q^{k+1})}{(1 - a b q^{2k+1})(1 - a b q^{2k+2})} & \text{if } h = 2k+1 \text{ is odd,} \\ \frac{a q^k (1 - q^k)(1 - b q^k)}{(1 - a b q^{2k})(1 - a b q^{2k+1})} & \text{if } h = 2k \text{ is even.} \end{cases}$$
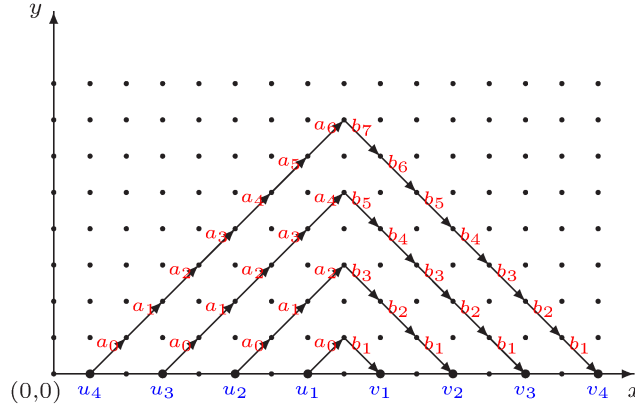
Figure 5. $t = 1$ and $n = 4$

For example, we have $\lambda_1 = \frac{1-aq}{1-abq^2}$, $\lambda_2 = \frac{aq(1-q)(1-bq)}{(1-abq^2)(1-abq^3)}$, $\lambda_3 = \frac{q(1-aq^2)(1-abq^2)}{(1-abq^3)(1-abq^4)}$, and an example of the weight of a path is Figure 7.

Lemma 2.5. *Let m and n be a non-negative integers such that $m \equiv n$ (mod 2). Then the generating function of $\mathscr{D}_{m,n}$ is given by*

$$(2.8) \qquad \mathrm{GF}\left(\mathscr{D}_{m,n}\right) = \begin{bmatrix} \lfloor \frac{m}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(aq^{1+\lceil \frac{n}{2} \rceil}; q)_{\frac{m-n}{2}}}{(abq^{2+n}; q)_{\frac{m-n}{2}}}.$$

Here $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) stands for the greatest integer that does not exceed x (resp. the smallest integer that is not smaller than x). Especially, we have

$$(2.9) \qquad \mathrm{GF}\left(\mathscr{D}_{2n,0}\right) = \frac{(aq; q)_n}{(abq^2; q)_n}.$$

Proof. We prove (2.8) by induction on m. If $m = 0$, then it is obvious that $\mathrm{GF}\left(\mathscr{D}_{0,n}\right)$ equals 1 if $n = 0$, and 0 otherwise. Assume that (2.8) holds up to $m - 1$. Then we have

$$\mathrm{GF}\left[\mathscr{D}_{m,n}\right] = \mathrm{GF}\left[\mathscr{D}_{m-1,n-1}\right] + \lambda_{n+1}\mathrm{GF}\left[\mathscr{D}_{m-1,n+1}\right].$$

If $m = 2r$ and $n = 2s$, then, by induction hyperthesis and the above recursion, we obtain $\mathrm{GF}\left[\mathscr{D}_{2r,2s}\right]$ equals

$$\begin{bmatrix} r-1 \\ s-1 \end{bmatrix}_q \frac{\left(aq^{s+1}; q\right)_{r-s}}{(abq^{2s+1}; q)_{r-s}} + \frac{q^s(1-aq^{s+1})(1-abq^{s+1})}{(1-abq^{2s+1})(1-abq^{2s+2})} \begin{bmatrix} r-1 \\ s \end{bmatrix}_q \frac{\left(aq^{s+2}; q\right)_{r-s-1}}{(abq^{2s+3}; q)_{r-s-1}}$$

$$= \frac{(q;q)_{r-1}\left(aq^{s+1}; q\right)_{r-s}}{(q;q)_s (q;q)_{r-s}(abq^{2s+1}; q)_{r-s+1}} \left\{ \left(1-q^s\right)\left(1-abq^{r+s+1}\right) + q^s\left(1-q^{r-s}\right)\left(1-abq^{s+1}\right) \right\}$$

$$= \begin{bmatrix} r \\ s \end{bmatrix}_q \frac{\left(aq^{s+1}; q\right)_{r-s}}{(abq^{2s+2}; q)_{r-s}}.$$

This equals the right-hand side of (2.8) with $m = 2r$ and $n = 2s$. Hence (2.8) holds when $m = 2r$. One can prove (2.8) similarly when $m = 2r + 1$ and $n = 2s + 1$. $\qquad\square$

For example, if $m = 4$ and $n = 0$, then $\mathscr{D}_{4,0}$ has the two Dyck paths shown in Figure 9. Thus, the generating function of $\mathscr{D}_{4,0}$ equals

$$\mathrm{GF}(\mathscr{D}_{4,0}) = \lambda_1^2 + \lambda_1\lambda_2 = \frac{(1-aq)(1-aq^2)}{(1-abq^2)(1-abq^3)}.$$

Proof of Theorem 1.1. If we use (2.3), (2.7) and (2.9), then we conclude that $\det(\mu_{i+j})_{0 \le i,j \le n-1}$ equals

$$\prod_{i=1}^{n} (\lambda_{2i-1}\lambda_{2i})^{n-i} = \prod_{i=1}^{n} \left\{ \frac{aq^{2i-1}(1-q^i)(1-aq^i)(1-bq^i)(1-abq^i)}{(1-abq^{2i-1})(1-abq^{2i})^2(1-abq^{2i+1})} \right\}^{n-i}.$$

An easy computation leads to (1.13). □

Remark. One can also prove Theorem 1.1 by using Motzkin paths and giving the weight λ_{2h+1} to rise vector of hight h, λ_{2h} to fall vector of hight h and $\lambda_{2h} + \lambda_{2h+1}$ to level vector of hight h. Then one can prove

$$(2.10) \qquad \mathrm{GF}(\mathscr{M}_{m,n}) = q^{\binom{m}{n}} \begin{bmatrix} m \\ n \end{bmatrix}_q \frac{(aq;q)_m(1-abq^{2n+1})}{(abq^{n+1};q)_{m+1}}.$$

for nonnegative integers m and n.

§ 3.   Orthogonal Polynomials

In this section we give our second proof of Theorem 1.1 using the little q-Jacobi polynomials. We use the notation $S(t; \lambda_1, \lambda_2, \ldots)$ for the Stieltjes-type continued fraction

$$(3.1) \qquad \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{1 - \cdots}}},$$

and $J(t; b_0, b_1, b_2, \ldots; \lambda_1, \lambda_2, \ldots)$ for the Jacobi-type continued fraction

$$(3.2) \qquad \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{\cfrac{\ddots}{1 - b_n x - \cfrac{\lambda_n x^2}{\ddots}}}}}.$$

Given a moment sequence $\{\mu_n\}$, we define the linear functional $\mathscr{L} : x^n \mapsto \mu_n$ on the vector space of polynomials $\mathbb{C}[x]$. Then the monic polynomials $p_n(x)$ orthogonal with respect to $\mathscr{L}$ and of $\deg p_n(x) = n$ satisfy a three term recurrence relation (Favard's theorem), say

$$(3.3) \qquad p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x),$$

where $p_{-1}(x) = 0$ and $p_0(x) = 1$. The moment sequence $\{\mu_n\}$ is related to the coefficients b_n and λ_n by the identity:

$$(3.4) \qquad 1 + \sum_{n \geq 1} \mu_n x^n = J(t; b_0, b_1, b_2, \ldots; \lambda_1, \lambda_2, \ldots).$$

Hereafter we assume $\lambda_0 = \mu_0 = 1$ for simplicity of arguments.

Define Δ_n and $D_n(x)$ by

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}, \qquad D_n(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

Then $p_n(x) = (\Delta_{n-1})^{-1} D_n(x)$ is the monic OPS for $\mathscr{L}$.

It is easy to see that

$$(3.5) \qquad \mathscr{L}(x^n p_n(x)) = \frac{\Delta_n}{\Delta_{n-1}} = \lambda_n \lambda_{n-1} \ldots \lambda_1 \mu_0,$$

$$(3.6) \qquad \mathscr{L}(x^{n+1} p_n(x)) = \frac{\chi_n}{\Delta_{n-1}} = \lambda_n \lambda_{n-1} \ldots \lambda_1 \mu_0 (b_0 + \cdots + b_n),$$

where

$$\chi_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu^{n-1} & \mu^n & \cdots & \mu_{2n-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{vmatrix}.$$

Therefore

$$(3.7) \qquad \lambda_n = \frac{\mathscr{L}[p_n^2(x)]}{\mathscr{L}[p_{n-1}^2(x)]} = \frac{\Delta_{n-2} \Delta_n}{\Delta_{n-1}^2},$$

and

$$(3.8) \qquad b_n = \frac{\mathscr{L}[x p_n^2(x)]}{\mathscr{L}[p_n^2(x)]} = \frac{\chi_n}{\Delta_n} - \frac{\chi_{n-1}}{\Delta_{n-1}}.$$

Theorem 3.1 (The Stieltjes-Rogers addition formula). *The formal power series $f(x) = \sum_{i \geq 0} a_i x^i / i!$ ($a_0 = 1$) has the property that*

$$f(x + y) = \sum_{m \geq 0} \alpha_m f_m(x) f_m(y),$$

where α_m is independent of x and y and

$$f_m(x) = \frac{x^m}{m!} + \beta_m \frac{x^{m+1}}{(m+1)!} + O(x^{m+2}),$$

if and only if the formal power series $\hat{f}(x) = \sum_{i \geq 0} a_i x^i$ has the J-continued fraction expansion $J(x; b_0, b_1, b_2, \ldots; \lambda_1, \lambda_2, \ldots)$ with the parameters

$$b_m = \beta_{m+1} - \beta_m \quad and \quad \lambda_m = \frac{\alpha_{m+1}}{\alpha_m}, \quad m \geq 0.$$

From (3.5), one can compute the Hankel determinants

$$(3.9) \qquad \det{(\mu_{i+j})}_{0 \le i, j \le n-1} = \Delta_{n-1} = \mu_0^n \lambda_1^{n-1} \lambda_2^{n-2} \cdots \lambda_{n-2}^2 \lambda_{n-1},$$

of (1.13) by taking appropriate orthogonal polynomials $p_n(x)$. Recall the definition of Heine's q-hypergeometric series

$$_2\phi_1(a, b; c; q; x) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n} \frac{x^n}{(q; q)_n}.$$

The following is one of Heine's three-term contiguous relations for $_2\phi_1$:

$$_2\phi_1(a, b; c; q; x) = {}_2\phi_1(a, bq; cq; q; x) + \frac{(1-a)(c-b)}{(1-c)(1-cq)} x \, _2\phi_1(aq, bq; cq^2; q; x).$$

It follows that

$$\frac{_2\phi_1(a, bq; cq; q; x)}{_2\phi_1(a, b; c; q; x)}$$
$$= S\left(x; \frac{(1-a)(b-c)}{(1-c)(1-cq)}, \frac{(1-bq)(a-cq)}{(1-cq)(1-cq^2)}, \frac{(1-aq)(bq-cq^2)}{(1-cq^2)(1-cq^3)}, \cdots \right).$$

Hence, by induction, we can prove that

$$\frac{_2\phi_1(a, bq; cq; q; x)}{_2\phi_1(a, b; c; q; x)} = S(x; \lambda_1, \lambda_2, \dots),$$

where

$$\lambda_{2n+1} = \frac{(1-aq^n)(b-cq^n)q^n}{(1-cq^{2n})(1-cq^{2n+1})}, \quad \lambda_{2n} = \frac{(1-bq^n)(a-cq^n)q^{n-1}}{(1-cq^{2n-1})(1-cq^{2n})}.$$

Making the substitution $b \leftarrow 1$, $a \leftarrow aq$ and $c \leftarrow abq$ into the above equation, we obtain

$$\sum_{n \ge 0} \frac{(aq; q)_n}{(abq^2; q)_n} x^n = S\left(x; \lambda_1, \lambda_2, \dots \right),$$

where

$$\lambda_{2n+1} = \frac{(1-aq^{n+1})(1-abq^{n+1})q^n}{(1-aq^{2n+1})(1-abq^{2n+2})}, \qquad \lambda_{2n} = \frac{(1-q^n)(1-bq^n)aq^n}{(1-abq^{2n})(1-abq^{2n+1})}.$$

This corresponds to the little q-Jacobi polynomials. Indeed, the little q-Jacobi polynomials

$$(3.10) \qquad p_n(x; a, b; q) = \frac{(aq; q)_n}{(abq^{n+1}; q)_n} (-1)^n q^{\binom{n}{2}} \, _2\phi_1 \left[\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \; ; q, xq \right]$$

are introduced in [2]. The polynomials satisfy the recurrence equation

$$(3.11) \qquad x p_n(x) = p_{n+1}(x) + (A_n + C_n) p_n(x) + A_{n-1} C_{n-1} p_{n-1}(x)$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and

$$(3.12) \qquad A_n = \frac{q^n(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})}, \qquad C_n = \frac{aq^n(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

They are orthogonal with respect to the moment sequence $\{\mu_n\}_{n\geq 0}$ where

$$(3.13) \qquad \mu_n = \frac{(aq;q)_n}{(abq^2;q)_n}.$$

For the passage from the Stieltjes-type continued fraction to the Jacobi-type continued fraction we use the following contraction formula:

$$S(x, \lambda_1, \lambda_2, \dots) = \cfrac{1}{1 - \lambda_1 t - \cfrac{\lambda_1\lambda_2 t^2}{1 - (\lambda_2 + \lambda_3)t - \cfrac{\lambda_3\lambda_4 t^2}{1 - \cdots}}}.$$

Thus, by the same computation as in the former section, we conclude that the determinant (3.13) is equal to (1.13). This proof gives us an insight to the determinant (1.13) from the point of view of the classical orthogonal polynomial theory.

§ 4. q-Dougall's formula

In this section we give our third proof of Theorem 1.1 using q-Dougall's formula and LU-decomposition of the Hankel matrix.

First the following formula is known as q-Dougall's formula: We have

$$(4.1) \qquad {}_6\phi_5\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d \end{matrix}; q, \frac{aq}{bcd}\right] = \frac{(qa, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}.$$

provided $|aq/bcd| < 1$ (see [9, (2.7.1)]). If we perform the substitution $a \leftarrow abq$, $b \leftarrow bq$, $c \leftarrow q^{-i}$ and $d \leftarrow q^{-j}$ in (4.1), then we obtain

$$(4.2) \qquad {}_6\phi_5\left[\begin{matrix} abq, a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{3}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{3}{2}}, bq, q^{-i}, q^{-j} \\ a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{2}}, aq, abq^{i+2}, abq^{j+2} \end{matrix}; q, aq^{i+j+1}\right] = \frac{\mu_{i+j}}{\mu_i\mu_j},$$

where $\mu_n = \frac{(aq;q)_n}{(abq^2;q)_n}$ as before. If we use

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}$$

then this identity can be rewritten as

$$(4.3) \qquad \sum_{k=0}^{\infty} a^k q^{k^2} \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \frac{(q, abq, a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{3}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{3}{2}}, bq; q)_k}{(a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{2}}, aq, abq^{i+2}, abq^{j+2}; q)_k} = \frac{\mu_{i+j}}{\mu_i\mu_j}.$$

If we put

$$(4.4) \qquad l_{ij} = \frac{\mu_i}{\mu_j} \frac{(abq^{j+2};q)_j}{(abq^{i+2};q)_j} \begin{bmatrix} i \\ j \end{bmatrix}_q,$$

$$(4.5) \qquad u_{ij} = a^i q^{i^2} \mu_i \mu_j \begin{bmatrix} j \\ i \end{bmatrix}_q \frac{(q, abq, a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{3}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{3}{2}}, bq; q)_i}{(a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{2}}, -a^{\frac{1}{2}}b^{\frac{1}{2}}q^{\frac{1}{2}}, aq, abq^{i+2}, abq^{j+2}; q)_i},$$

then (4.3) implies

$$(4.6) \qquad \sum_{k=0}^{\infty} l_{ik} u_{kj} = \mu_{i+j}$$

Note that $L_n = (l_{ij})_{0 \le i,j \le n-1}$ is a lower triangular matrix such that all main-diagonal entries are 1, and $U_n = (u_{ij})_{0 \le i,j \le n-1}$ is an upper-triangular matrix with diagonal entries

$$u_{ii} = a^i q^{i^2} \mu_i^2 \frac{(q, abq, a^{\frac{1}{2}} b^{\frac{1}{2}} q^{\frac{3}{2}}, -a^{\frac{1}{2}} b^{\frac{1}{2}} q^{\frac{3}{2}}, bq; q)_i}{(a^{\frac{1}{2}} b^{\frac{1}{2}} q^{\frac{1}{2}}, -a^{\frac{1}{2}} b^{\frac{1}{2}} q^{\frac{1}{2}}, aq, abq^{i+2}, abq^{i+2}; q)_i}$$

$$(4.7) \qquad = a^i q^{i^2} \frac{1 - abq^{2i+1}}{1 - abq} \frac{(q, aq, bq, abq; q)_i}{(abq^2; q)_{2i}^2}.$$

Since $(\mu_{i+j})_{0 \le i,j \le n-1} = L_n U_n$, $\det (\mu_{i+j})_{0 \le i,j \le n-1}$ is the product of the diagonal entries, i.e.,

$$\det (\mu_{i+j})_{0 \le i,j \le n-1} = \prod_{i=0}^{n-1} u_{ii}.$$

Using (4.7), one can easily prove (1.13) by a direct computation.

    Remark.   We should note that Corollary 1.2 can be proven by induction using the following Desnanot-Jacobi adjoint matrix theorem: If M is an $n \times n$ matrix, then we have

$$(4.8) \qquad \det M \det M_{1,n}^{1,n} = \det M_1^1 \det M_n^n - \det M_n^1 \det M_1^n,$$

where $M_{j_1,\ldots,j_r}^{i_1,\ldots,i_r}$ denotes the $(n-r) \times (n-r)$ submatrix obtained by removing rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ from M.

    Corollary 1.2 can be also proven as a special case of Theorem 1.3, which will be proven in Section 5.1. In fact, if one puts $k_i = i + t$ in (1.15), then he obtains (1.14).

§ 5.   Miscellany

§ 5.1.   A proof of Theorem 1.3

    In this subsection we give a proof of Theorem 1.3. Before we prove the formula, we need to cite a lemma from [11, 12].

    Lemma 5.1 (Krattenthaler [11]).   *Let $X_0, \ldots, X_n, A_1, \ldots, A_{n-1}$, and $B_1, \ldots, B_{n-1}$ be indeterminates. Then there holds*

$$(5.1)$$

$$\det \left[\prod_{l=1}^{j} (X_i + B_l) \prod_{l=j+1}^{n-1} (X_i + A_l) \right]_{0 \le i,j \le n-1} = \prod_{0 \le i < j \le n-1} (X_i - X_j) \prod_{1 \le i \le j \le n-1} (B_i - A_j).$$

Proof of Theorem 1.3. Using

$$\mu_n = \frac{(aq; q)_n}{(abq^2; q)_n},$$

we can write

$$\det\left(\mu_{k_i+j}\right)_{0\le i,j\le n-1} = \prod_{i=0}^{n-1} \frac{(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \det\left(\prod_{l=1}^{j}\left(1-aq^{k_i+l}\right)\prod_{l=j+1}^{n-1}\left(1-abq^{k_i+l+1}\right)\right)_{0\le i,j\le n-1}$$

$$= \prod_{i=0}^{n-1} \frac{q^{(n-1)k_i}(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \det\left(\prod_{l=1}^{j}\left(q^{-k_i}-aq^{l}\right)\prod_{l=j+1}^{n-1}\left(q^{-k_i}-abq^{l+1}\right)\right)_{0\le i,j\le n-1}.$$

If we substitute $X_i = q^{-k_i}$, $B_l = -aq^l$ and $A_l = -abq^{l+1}$ into (5.1), then we see that

$$\det\left(\mu_{k_i+j}\right)_{0\le i,j\le n-1} = \prod_{i=0}^{n-1} \frac{q^{(n-1)k_i}(aq;q)_{k_i}}{(abq^2;q)_{k_i+n-1}} \prod_{0\le i<j\le n-1}\left(q^{-k_i}-q^{-k_j}\right)\prod_{1\le i\le j\le n-1}\left(abq^{j+1}-aq^i\right)$$

One can derive (1.15) easily by a direct computation. $\qquad\square$

§5.2. An addition formula for $_2F_1$

In this subsection we give a new proof of (1.7) using an addition formula for $_2F_1$ and LU-decomposition of Motzkin Hankel matrices. First, we shall prove the following identity.

Lemma 5.2. *If i and j are nonnegative integers, then we have*

$$(5.2) \qquad \sum_{k\ge 0}\binom{i}{k}\binom{j}{k}{}_2F_1\left[\begin{matrix}\frac{k-i+1}{2},\frac{k-i}{2}\\k+2\end{matrix};4\right]{}_2F_1\left[\begin{matrix}\frac{k-j+1}{2},\frac{k-j}{2}\\k+2\end{matrix};4\right] = {}_2F_1\left[\begin{matrix}\frac{1-i-j}{2},\frac{-i-j}{2}\\2\end{matrix};4\right].$$

Proof. Recall the quadratic transformation formula (see [9, (3.1.5)]):

$$(5.3) \qquad (1-z)^a{}_2F_1(a,b;2b;2z) = {}_2F_1\left(\frac{a}{2},\frac{a+1}{2};b+\frac{1}{2};\frac{z^2}{(1-z)^2}\right).$$

Applying (5.3) with $a = k-i$, $b = k+3/2$ and $z = 2$ we obtain

$${}_2F_1\left[\begin{matrix}\frac{k-i+1}{2},\frac{k-i}{2}\\k+2\end{matrix};4\right] = (-1)^{k-i}{}_2F_1\left[\begin{matrix}k-i,k+3/2\\2k+3\end{matrix};4\right].$$

Substituting i by j yields

$${}_2F_1\left[\begin{matrix}\frac{k-j+1}{2},\frac{k-j}{2}\\k+2\end{matrix};4\right] = (-1)^{k-j}{}_2F_1\left[\begin{matrix}k-j,k+3/2\\2k+3\end{matrix};4\right].$$

Now, applying (5.3) with $a = -i-j$, $b = 3/2$ and $z = 2$ we obtain

$${}_2F_1\left[\begin{matrix}\frac{1-i-j}{2},\frac{-i-j}{2}\\2\end{matrix};4\right] = (-1)^{i+j}{}_2F_1\left[\begin{matrix}-i-j,\frac{3}{2}\\3\end{matrix};4\right].$$

Therefore we can rewrite (5.2) as follows:

$$(5.4) \qquad \sum_{k\ge 0}\binom{i}{k}\binom{j}{k}{}_2F_1\left[\begin{matrix}k-i,k+3/2\\2k+3\end{matrix};4\right]{}_2F_1\left[\begin{matrix}k-j,k+3/2\\2k+3\end{matrix};4\right] = {}_2F_1\left[\begin{matrix}-i-j,\frac{3}{2}\\3\end{matrix};4\right].$$

Now we recall a formula of Burchnall and Chaundy [3, (43)]:

$$
{}_2F_1 \left[\begin{matrix} c-a, c-b \\ c \end{matrix} ; x \right] = \sum_{k \geq 0} \frac{(c-a)_k (a)_k (d)_k (c-b-d)_k}{k!(c+k-1)_k (c)_{2k}} x^{2k}
$$

$$
(5.5) \qquad \times {}_2F_1 \left[\begin{matrix} c-a+k, c-b-d+k \\ c+2k \end{matrix} ; x \right] {}_2F_1 \left[\begin{matrix} c-a+k, d+k \\ c+2k \end{matrix} ; x \right].
$$

It is then easy to check that the specialization of (5.5) with

$$
a = \frac{3}{2}, \quad b = 3+i+j, \quad c = 3, \quad d = -j, \quad x = 4
$$

yields (5.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proof of (1.7). Define l_{ij} and u_{ij} by

$$
(5.6) \qquad\qquad l_{ij} = \binom{i}{j} {}_2F_1 \left[\begin{matrix} \frac{j-i+1}{2}, \frac{j-i}{2} \\ j+2 \end{matrix} ; 4 \right],
$$

$$
(5.7) \qquad\qquad u_{ij} = \binom{j}{i} {}_2F_1 \left[\begin{matrix} \frac{i-j+1}{2}, \frac{i-j}{2} \\ i+2 \end{matrix} ; 4 \right].
$$

Then $L_n = (l_{ij})_{0 \leq i,j \leq n-1}$ is a lower triangular matrix with all diagonal entries 1, and $U_n = (l_{ij})_{0 \leq i,j \leq n-1}$ is an upper triangular matrix with all diagonal entries 1. The formula (5.2) gives the LU-decomposition of Motzkin Hankel matrix:

$$
(M_{ij}) = L_n U_n.
$$

Hence we conclude that $\det (M_{ij}) = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

§ 5.3.   A q-analogue of Schröder numbers

We define $S_n(q)$ $(n \geq 0)$ by the following recurrence:

$$
S_0(q) = 1, \ S_n(q) = q^{2n-1} S_{n-1}(q) + \sum_{k=0}^{n-1} q^{2(k+1)(n-1-k)} S_{n-1-k}(q) S_k(q).
$$

In fact one can show that

$$
S_n(q) = \sum_{P \in \mathscr{S}_{2n,0}} \omega(P),
$$

where $\omega(P)$ is the number of triangles below the path P (see Figure 10), and the sum runs over all Schröder paths from the origin to $(2n, 0)$. As a q-analogue of (1.10) and (1.11) we consider the matrix

$$
(5.8) \qquad\qquad S_n^{(t)}(q) = \left(q^{(i-j)(i-j-1)} S_{i+j+t}(q) \right)_{0 \leq i,j \leq n-1}.
$$

Note that this matrix is not a Hankel matrix, but as a q-analogue of (1.10) and (1.11), the following theorem holds:

Theorem 5.3. *Let n be a positive integer.*
(i) If $t = 0$ or 1, then we have

$$\det S_{n-1}^{(1)}(q) = \det S_n^{(0)}(q) = \prod_{k=1}^{n-1}(q^{2k-1}+1)^{n-k}. \tag{5.9}$$

(ii) If $t = 2$, then we have

$$\det S_n^{(2)}(q) = q^{-1}\prod_{k=1}^{n}(q^{2k-1}+1)^{n+1-k}(\prod_{k=1}^{n+1}(q^{2k-1}+1)-1). \tag{5.10}$$

To prove this theorem, we define the matrices

$$\widehat{S}_n^{(t)}(q) = \left(q^{2(n-i)(t+i+j-2)}S_{t+i+j-2}(q)\right)_{1\le i,j\le n},$$

$$\widetilde{S}_n^{(t)}(q) = \left(q^{-(t+i+j)(t+i+j-1)}S_{t+i+j-2}(q)\right)_{1\le i,j\le n},$$

then the following lemma can be easily proven by direct computations:

Lemma 5.4. *Let n be a positive integer. Then*

$$\det \widehat{S}_n^{(t)}(q) = q^{\frac{n(n-1)(2n+3t-4)}{3}}\det S_n^{(t)}(q), \tag{5.11}$$

$$\det \widetilde{S}_n(q) = q^{-\frac{n(4n^2+3(2t+1)n+3t^2+3t-1)}{3}}\det S_n^{(t)}(q). \tag{5.12}$$

Lemma 5.5. *Let n be a positive integer.*
(i) If $n \ge 2$, then we have

$$\det \widehat{S}_n^{(0)}(q) = q^{(n-1)(n-2)}\det \widehat{S}_{n-1}^{(1)}(q). \tag{5.13}$$

(ii) If $n \ge 2$, then we have

$$\det \widehat{S}_n^{(1)}(q) = q^{n^2-n+1}\det \widehat{S}_{n-1}^{(2)}(q) + q^{2(n-1)^2}\det \widehat{S}_{n-1}^{(1)}(q). \tag{5.14}$$

(iii) If $n \ge 3$, then we have

$$\det \widetilde{S}_n^{(0)}(q)\det \widetilde{S}_{n-2}^{(2)}(q) = \det \widetilde{S}_{n-1}^{(0)}(q)\det \widetilde{S}_{n-1}^{(2)}(q) - \left\{\det \widetilde{S}_{n-1}^{(1)}(q)\right\}^2. \tag{5.15}$$

Proof. We consider the digraph (V, E), in which V is the plane lattice $\mathbb{Z}^2$ and E the set of rise vectors, fall vectors and long level vectors in the above half plane. Let $u_i = (x_0 - 2(i-1), 0)$ and $v_j^{(t)} = (x_0 + 2(j+t-1), 0)$ for $i, j = 1, 2, \ldots, n$, $t = 0, 1, 2$ and a fixed integer x_0. It is easy to see that the n-vertex $\boldsymbol{u} = (u_1, \ldots, u_n)$ is D-*compatible* with the n-vertex $\boldsymbol{v}^{(t)} = (v_1^{(t)}, \ldots, v_n^{(t)})$. We assign the weight of each edge as a rise vector, a fall vector and a long level vector whose origin is (x, y) and ends at $(x+1, y+1)$, $(x+1, y-1)$ and $(x+2, y)$ has weight $q^{x-y-x_0+2(n-1)}$, 1 and q^{x-y-x_0+2n-1}, respectively, which is visualized in Figure 11. Then, by applying Lemma 2.3, we can obtain

$$\mathrm{GF}\left[\mathscr{P}_0\left(\boldsymbol{u}, \boldsymbol{v}^{(t)}\right)\right] = \det \widehat{S}_n^{(t)}(q). \tag{5.16}$$

This is important to prove the following.

(i) Assume $t = 0$ and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be as above. Put $\widetilde{u}_i = (x_0 - 2i + 3, 1)$ and $\widetilde{v}_j^{(1)} = (x_0 + 2j - 3, 1)$ for $i, j = 2, \ldots, n$, and let $\widetilde{\boldsymbol{u}} = (\widetilde{u}_2, \ldots, \widetilde{u}_n)$ and $\widetilde{\boldsymbol{v}}^{(1)} = (\widetilde{v}_2^{(1)}, \ldots, \widetilde{v}_n^{(1)})$. Then each n-path $\boldsymbol{P} = (P_1, P_2, \ldots, P_n)$ from $\boldsymbol{u}$ to $\boldsymbol{v}^{(0)}$ corresponds to an $(n-1)$-path $\widetilde{\boldsymbol{P}} = (\widetilde{P}_2, \ldots, \widetilde{P}_n)$ from $\widetilde{\boldsymbol{u}}$ to $\widetilde{\boldsymbol{v}}^{(1)}$ by regarding $\widetilde{\boldsymbol{P}}$ as the subpath of $\boldsymbol{P}$. In fact, note that P_1 is always the path composed of a single vertex $u_1 = v_1$, each P_i always starts from the rise vector $u_i \to \widetilde{u}_i$ and ends at the fall vector $\widetilde{v}_i^{(1)} \to v_i^{(0)}$ for $i = 2, \ldots, n$. Hence this gives a bijection, and the product of the weight of the rise vectors $u_i \to \widetilde{u}_i$ and the fall vectors $\widetilde{v}_i^{(1)} \to v_i^{(0)}$ for $i = 2, \ldots, n$ is $q^{(n-1)(n-2)}$. This proves (5.13).

(ii) Assume $t = 1$ and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be as above, i.e., $u_i = (x_0 - 2(i-1), 0)$ and $v_j^{(1)} = (x_0 + 2j, 0)$ for $1 \leq i, j \leq n$ (see Figure 12). Put $\widetilde{u}_i = (x_0 - 2i + 3, 1)$ $(2 \leq i \leq n)$ and $\widetilde{v}_j^{(2)} = (x_0 + 2j - 1, 1)$ $(2 \leq j \leq n)$, and let $\widetilde{\boldsymbol{u}} = (\widetilde{u}_2, \ldots, \widetilde{u}_n)$ and $\widetilde{\boldsymbol{v}}^{(2)} = (\widetilde{v}_2^{(2)}, \ldots, \widetilde{v}_n^{(2)})$. Further, put $\check{u}_i = (x_0 - 2i + 4, 2)$ $(2 \leq i \leq n)$ and $\check{v}_j^{(1)} = (x_0 + 2j - 2, 2)$ $(2 \leq j \leq n)$, and let $\check{\boldsymbol{u}} = (\check{u}_2, \ldots, \check{u}_n)$ and $\check{\boldsymbol{v}}^{(1)} = (\check{v}_2^{(1)}, \ldots, \check{v}_n^{(1)})$. Let $\boldsymbol{P} = (P_1, P_2, \ldots, P_n)$ be any non-intersecting n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}^{(1)}$. Then, it is easy to see that $\boldsymbol{P}$ must satisfy one of the following two conditions:

(1) P_1 is the long level vector whose origin is u_1 and ends at $v_1^{(1)}$, and P_i goes through the vertices $\widetilde{u}_i$ and $\widetilde{v}_i^{(2)}$ for $i = 2, 3, \ldots, n$.

(2) P_1 is a path which goes through only three vertices u_1, $u_1 + (1,1)(= v_1^{(1)} - (1,-1))$ and $v_1^{(1)}$, and P_i goes through the vertices $\widetilde{u}_i$, $\check{u}_i$, $\check{v}_i^{(1)}$ and $\widetilde{v}_i^{(2)}$ for $i = 2, 3, \ldots, n$.

By a similar argument as in the proof of (i), we can deduce that

$$\mathrm{GF}\left[\mathscr{P}_0(\boldsymbol{u}_n, \boldsymbol{v}_n^{(1)})\right] = q^{n^2 - n + 1} \mathrm{GF}\left[\mathscr{P}_0(\boldsymbol{u}_{n-1}, \boldsymbol{v}_{n-1}^{(2)})\right] + q^{2(n-1)^2} \mathrm{GF}\left[\mathscr{P}_0(\boldsymbol{u}_{n-1}, \boldsymbol{v}_{n-1}^{(1)})\right]$$

holds. By the equality (5.16), we obtain the identity (5.14).

(iii) This identity can be proven by applying the Desnanot-Jacobi adjoint matrix theorem (4.8) to $\widetilde{S}_n^{(t)}(q)$. $\square$

Proof of Theorem 5.3. (i) The first equality of (5.9) is easily obtained from (5.11) and (5.13). By applying the equalities (5.11) and (5.12) to (5.14) and (5.15), we have

$$(5.17) \qquad\qquad \det S_n^{(1)}(q) = q \det S_{n-1}^{(2)}(q) + \det S_{n-1}^{(1)}(q)$$

for $n \geq 2$, and we have

$$(5.18) \qquad \det S_n^{(0)}(q) \det S_{n-2}^{(2)}(q) = \det S_{n-1}^{(0)}(q) \det S_{n-1}^{(2)}(q) - q^{2(n-1)}\{\det S_{n-1}^{(1)}(q)\}^2$$

for $n \geq 3$. By the equalities (5.17) and (5.18), for $n \geq 3$, the following identity holds:

$$\det S_n^{(0)}(q)(\det S_{n-1}^{(1)}(q) - \det S_{n-2}^{(1)}(q))$$
$$(5.19) \qquad = \det S_{n-1}^{(0)}(q)(\det S_n^{(1)}(q) - \det S_{n-1}^{(1)}(q)) - q^{2n-1}\{\det S_{n-1}^{(1)}(q)\}^2.$$

Moreover, by applying the first equality of (5.9) to (5.19) and replacing n with $n-1$, we obtain

$$(5.20) \qquad\qquad (1 + q^{2n-3})\{\det S_{n-1}^{(0)}(q)\}^2 = \det S_{n-2}^{(0)}(q) \det S_n^{(0)}(q)$$

for $n \geq 4$. We prove the second equality of (5.9) by induction on n. If $n = 1, 2, 3$, then it is easily obtained by direct computations. Assume that (5.9) holds up to $n-1$. Then, by (5.20) and induction hypothesis, we can obtain the second equality (5.9).

(ii) It follows from our result of (i) and the equality (5.17) that (5.10) holds. $\square$

By applying the Desnanot-Jacobi adjoint matrix theorem (4.8) to $S_{n+1}^{(1)}(1)$, then we have

$$(5.21) \qquad \det S_{n+1}^{(1)}(1) \det S_{n-1}^{(3)}(1) = \det S_n^{(1)}(1) \det S_n^{(3)}(1) - \{\det S_n^{(2)}(1)\}^2$$

for $n \geq 2$. Therefore the following identity is easily obtained by induction on n and the formula (5.21):

Remark. For positive integer n, we have

$$(5.22) \qquad \det S_n^{(3)}(1) = 2^{\binom{n+3}{2}} - (2n+3)2^{\binom{n+2}{2}} - 2^{\binom{n+1}{2}}.$$

Note that there is a relation between domino tilings of the Aztec diamonds and Schröder paths (see [8, 15]). It might be an interesting problem to consider what this weight means.

§ 5.4. Delannoy numbers

The *Delannoy numbers* $D(a, b)$ are the number of lattice paths from $(0, 0)$ to (b, a) in which only east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$ steps are allowed. They are given by the recurrence relation

$$
\begin{aligned}
& D(a, 0) = D(0, b) = 1, \\
(5.23) \qquad & D(a, b) = D(a-1, b) + D(a-1, b-1) + D(a, b-1).
\end{aligned}
$$

The first few terms of $D(n, n)$ $(n = 0, 1, 2, \dots)$ are given by 1, 3, 13, 63, 321, ... By a similar argument we can derive the following result. We may give a proof in another occasion.

Proposition 5.6. *Let n be a positive integers. Then the following identities would hold:*

$$(5.24) \qquad \det \left(D(i+j, i+j) \right)_{0 \leq i,j \leq n-1} = 2^{\binom{n+1}{2}-1},$$

$$(5.25) \qquad \det \left(D(i+j+1, i+j+1) \right)_{0 \leq i,j \leq n-1} = 2^{\binom{n+2}{2}-2} + 2^{\binom{n+1}{2}-1},$$

$$(5.26) \qquad \det \left(D(i+j+2, i+j+2) \right)_{0 \leq i,j \leq n-1} = 2^{\binom{n+3}{2}-3} + (2n+1)2^{\binom{n+2}{2}-2} - 2^{\binom{n+1}{2}-1}.$$

§ 6. Concluding remarks

Since a hyperpfaffian version of (1.3) is obtained in [13], we believe it will be interesting problem to consider a hyperpfaffian version of Theorem 1.1 and Theorem 1.3. We shall argue on it in another chance. The authors also would like to express their gratitude to the anonymous referee for his (her) constructive comments.

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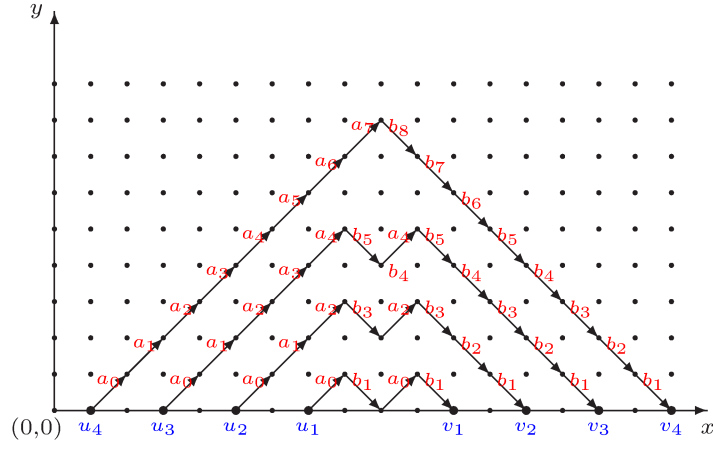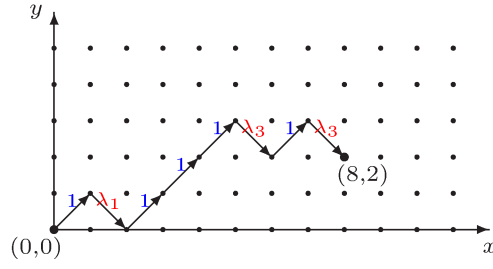
Figure 6. $t = 1$ and $n = 4$

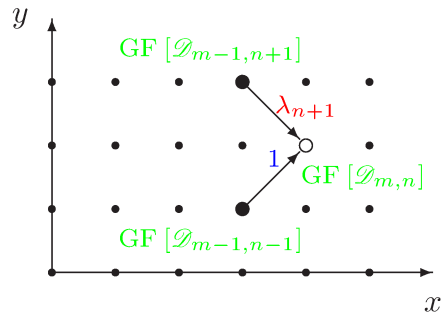Figure 7. A Dyck Path of weight $\lambda_1 \lambda_3^2$

Figure 8. $\mathrm{GF}\left[\mathscr{D}_{m,n}\right] = \mathrm{GF}\left[\mathscr{D}_{m-1,n-1}\right] + \lambda_{n+1}\mathrm{GF}\left[\mathscr{D}_{m-1,n+1}\right]$
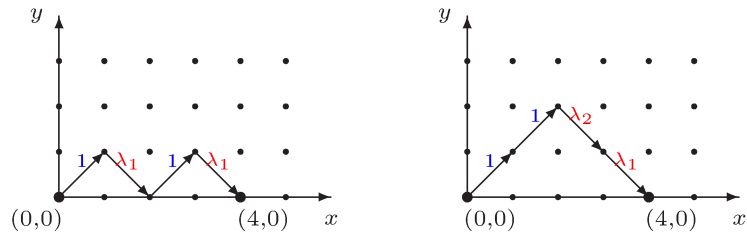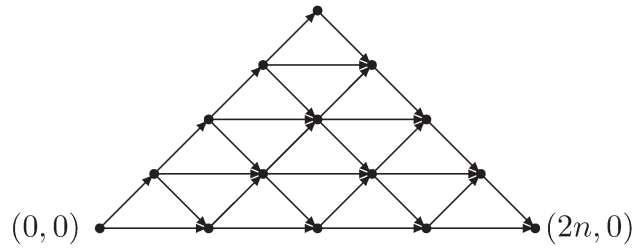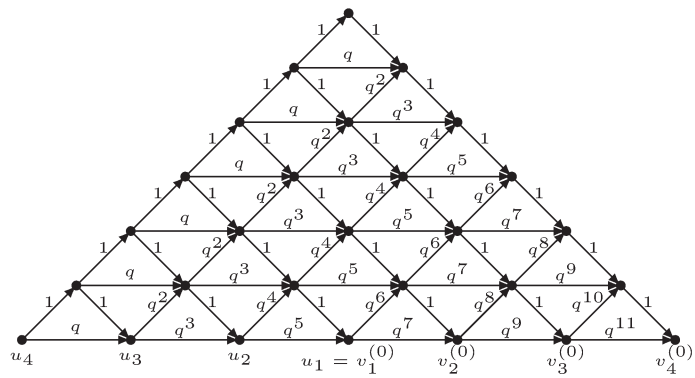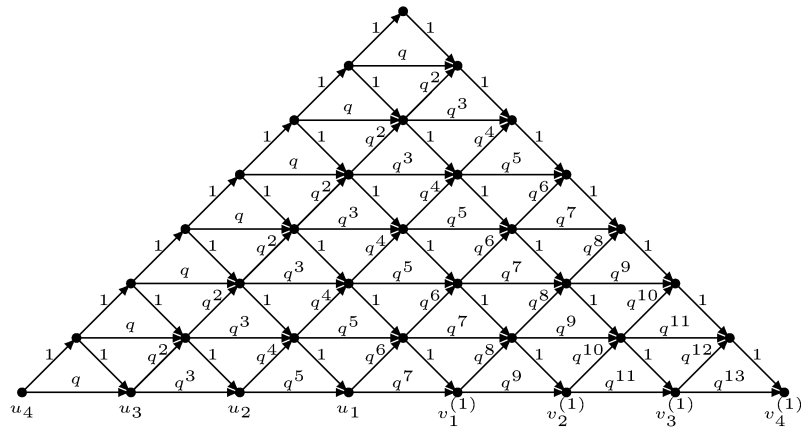
Figure 9. Dyck Paths in $\mathscr{D}_{4,0}$

Figure 10. Weight of Schröder paths in $\mathscr{S}_{4,0}$

Figure 11. Weight of each edge

Figure 12. $t = 1$ case