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The Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

By

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Abstract

We survey recent results on a formulation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as an H-Hopf algebroid and its representation theory. We put emphasis on a connection of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and a constructive derivation of both finite and infinite-dimensional representations from those of $U_q(\widehat{\mathfrak{sl}}_2)$. Included is an announcement of a new result on a criterion for the finiteness of irreducible pseudo-highest weight representations stated in terms of an elliptic analogue of the Drinfeld polynomials. A derivation of the type I and II vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its implication in the algebraic analysis of elliptic solvable lattice models are also explained.

§ 1. Introduction

Theory of elliptic quantum groups has been developed in the two different approaches, the one based on H-Hopf algebroids [10] and the other on quasi-Hopf algebras [8].

The H-Hopf algebroid was introduced by Etingof and Varchenko [10], motivated by the work of Felder and Varchenko [12, 13]. There are some structures added by Koelink and Rosengren[19, 31]. See also a survey by van Norden [34]. A similar coalgebra structure was introduced by Lu [30] and Xu [35]. As an H-Hopf algebroid, Felder's elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ was formulated in terms of the L operator satisfying the RLL relation associated with the elliptic dynamical R matrices[12, 13, 10, 20].

The quasi-Hopf algebra formulation was carried out by Jimbo, Konno, Odake and Shiraishi [17] motivated by the works of Drinfeld[8], Babelon, Bernard and Billey [2] and Frønsdal [15]. There are two types of elliptic quantum groups (quasi-Hopf algebras),

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the vertex type $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, where $\mathfrak{g}$ is an affine Lie algebra. p is a complex parameter giving the nome of the related elliptic functions. λ denotes a Cartan subalgebra valued parameter which provides the elliptic nome and the dynamical parameters. Both $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ are isomorphic to the corresponding quantum affine algebras $U_q(\mathfrak{g})$ as associative algebras, but their coalgebra structures are deformed from $U_q(\mathfrak{g})$ by the twistors $E(r)$ and $F(\lambda)$, respectively[2, 15, 17]. Here r is related to p by $p = q^{2r}$. Felder's elliptic quantum group also has a formulation as a quasi-Hopf algebra[9].

The classification of the vertex and the face types is based on the fact that the vector representation of the universal dynamical R matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ yields Baxter's elliptic R matrix for the eight-vertex model[3], whereas the one of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ [17, 21, 22, 25] yields the face type elliptic Boltzmann weight of the SOS face model associated with $\mathfrak{g}$ [1, 16]. Since the latter Boltzmann weights are nothing but the elliptic dynamical R matrices used in Felder's elliptic quantum groups, we classify Felder's ones as the face type. See also [4] for a universal formulation of the vertex-face correspondence in the quasi-Hopf algebra scheme.

Each approach has advantages and disadvantages. An advantage of the quasi-Hopf algebra is that each of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ has an apparent connection to $U_q(\mathfrak{g})$ by the twist procedure. In particular, we can formulate both algebraic and representation theoretical objects of the quasi-Hopf algebra, such as the comultiplication, the universal dynamical R matrices and the vertex operators, from the corresponding objects of $U_q(\mathfrak{g})$[17]. However a disadvantage is a complication of the coalgebra structure due to the twist procedure mentioned above, so that it is not suitable for a practical calculation.

In contrast, the coalgebra structure of the known H-Hopf algebroid is simple enough for practical use. In fact it was already applied to a study of tensor product representations[13] and of co-representations[19, 20]. In particular, by studying the co-representation, Koelink, van Norden and Rosengren have succeeded to derive the terminating very-well-poised balanced elliptic hypergeometric series ${}_{12}V_{11}$ and their biorthogonality relations, which were introduced by Frenkel-Turaev[14] and developed by Spiridonov and Zhedanov[32, 33]. However a disadvantage is a lack of direct connection to $U_q(\mathfrak{g})$ even as an associative algebra. This defect seems to be an obstacle to extend the known H-Hopf algebroids associated with finite-dimensional simple Lie algebras to those associated with affine Lie algebras and to develop their representation theory in systematic way.

In this paper, we explain a new realization of the face type elliptic quantum group given by the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$[26, 27]. It is a realization by the Drinfeld generators of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and has an H-Hopf algebroid structure, so that it provides a complement to the above two approaches.

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was introduced in [24] as an elliptic analogue of the algebra of the Drinfeld currents for $U_q(\widehat{\mathfrak{sl}}_2)$. As an associative algebra, $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ and the Heisenberg algebra $\{P, e^Q\}$[18]. A similar algebra was studied in [35]. The generators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ are treated through the generating functions called the elliptic currents. In terms of the elliptic currents, we can construct the L operator and derive the RLL relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$[18]. It turns out that the resultant RLL relation is nothing but the one for Felder's elliptic quantum group with a central extension. We hence formulate the H-Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in a way similar to [10, 20], but with a modification due to the existence of the non-zero central element.

Due to a direct connection to $U_q(\widehat{\mathfrak{sl}}_2)$, we can derive all representations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ constructively from those of $U_q(\widehat{\mathfrak{sl}}_2)$. This yields quite a parallel structure to $U_q(\widehat{\mathfrak{sl}}_2)$ for both finite and infinite -dimensional representations. In particular, we can state a criterion for the finiteness of irreducible pseudo-highest weight representations in terms of an elliptic analogue of the Drinfeld polynomials. This provides an elliptic analogue of the works by Drinfeld[7] and by Chari and Pressley[5]. As an example of the application of infinite-dimensional representations, we also report on a formulation and derivation of the type I and II vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ studied in [26].

This paper is organized as follows. In the next section, we review the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The L operator and the RLL relation are also introduced. In Sect.3, we describe an H-Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following [26, 27]. In Sect.4 we summarize some basic facts on the dynamical representations. Theorem 4.2 is fundamental in a constructive derivation of the dynamical representations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Sect.5 is devoted to a study of finite-dimensional irreducible pseudo-highest weight representations and includes an announcement of new results. In Sect.6, we report a result on the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following [26].

In the forthcoming paper [27], we plan to provide proofs of the statements in Sect. 3, 4, 5 and also report on a structure of the finite-dimensional tensor product representations as well as an alternative derivation of the ${}_{12}V_{11}$.

§ 2. The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we review a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its RLL relation following [18, 26, 27].

§ 2.1. Definition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Let us fix a complex number q such that $q \neq 0, |q| < 1$.

Definition 2.1. [8] For a field $\mathbb{K}(\supset \mathbb{C})$, the quantum affine algebra $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the standard

Drinfeld generators a_n $(n \in \mathbb{Z}_{\neq 0})$, $x^{\pm}_n$ $(n \in \mathbb{Z})$, h, c, d. The defining relations are given as follows.

$$\begin{aligned}
&c: \text{ central },\\
&[h,d]=0, \quad [d,a_n]=na_n, \quad [d,x^{\pm}_n]=nx^{\pm}_n,\\
&[h,a_n]=0, \qquad [h,x^{\pm}(z)]=\pm 2x^{\pm}(z),\\
&[a_n,a_m]=\frac{[2n]_q[cn]_q}{n}q^{-c|n|}\delta_{n+m,0},\\
&[a_n,x^+(z)]=\frac{[2n]_q}{n}q^{-c|n|}z^n x^+(z),\\
&[a_n,x^-(z)]=-\frac{[2n]_q}{n}z^n x^-(z),\\
&(z-q^{\pm 2}w)x^{\pm}(z)x^{\pm}(w)=(q^{\pm 2}z-w)x^{\pm}(w)x^{\pm}(z),\\
&[x^+(z),x^-(w)]=\frac{1}{q-q^{-1}}\left(\delta\big(q^{-c}\frac{z}{w}\big)\psi(q^{c/2}w)-\delta\big(q^{c}\frac{z}{w}\big)\varphi(q^{-c/2}w)\right).
\end{aligned}$$

where we use $[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$, $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$ and the Drinfeld currents defined by

$$x^{\pm}(z)=\sum_{n\in\mathbb{Z}}x^{\pm}_n z^{-n},$$

$$\psi(q^{c/2}z)=q^h\exp\left((q-q^{-1})\sum_{n>0}a_n z^{-n}\right), \quad \varphi(q^{-c/2}z)=q^{-h}\exp\left(-(q-q^{-1})\sum_{n>0}a_{-n} z^{n}\right).$$

We also denote by $\mathbb{K}[U'_q(\widehat{\mathfrak{sl}}_2)]$ the subalgebra of $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ generated by the same generators as $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ except d.

Remark. We follows the conventions in [18]. In particular, we occasionally treat c as a complex number on the understanding that we make a specialization each time.

Let r be a generic complex number. We set $r^*=r-c$, $p=q^{2r}$ and $p^*=q^{2r^*}$. We define the Jacobi theta functions $[u]$ and $[u]^*$ by

$$\begin{aligned}
&[u]=\frac{q^{\frac{u^2}{r}-u}}{(p;p)^3_\infty}\Theta_p(q^{2u}), \qquad [u]^*=\frac{q^{\frac{u^2}{r^*}-u}}{(p^*;p^*)^3_\infty}\Theta_{p^*}(q^{2u}),\\
&\Theta_p(z)=(z;p)_\infty(p/z;p)_\infty(p;p)_\infty,
\end{aligned}$$

where

$$(z;p_1,p_2,\cdots,p_m)_\infty=\prod_{n_1,n_2,\cdots,n_m=0}^{\infty}(1-zp_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}).$$

Setting $p=e^{-\frac{2\pi i}{\tau}}$, $[u]$ satisfies the quasi-periodicity $[u+r]=-[u]$, $[u+r\tau]=-e^{-\pi i(2u/r+\tau)}[u]$.

Let $\{P, e^Q\}$ be a Heisenberg algebra commuting with $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ and satisfying

$$[P, e^Q] = -e^Q. \tag{2.1}$$

We take the realization $Q = \frac{\partial}{\partial P}$. We set $H = \mathbb{C}P \oplus \mathbb{C}r^*$ and $H^* = \mathbb{C}Q \oplus \mathbb{C}\frac{\partial}{\partial r^*}$ with the pairing $< \, , \, >$.

$$< Q, P >= 1 =< \frac{\partial}{\partial r^*}, r^* >,$$

the others are zero. We also consider the abelian group $\bar{H}^* = \mathbb{Z}Q$. We denote by $\mathbb{C}[\bar{H}^*]$ the group algebra over $\mathbb{C}$ of $\bar{H}^*$. We denote by e^α the element of $\mathbb{C}[\bar{H}^*]$ corresponding to $\alpha \in \bar{H}^*$. These e^α satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Let M_{H^*} be the field of meromorphic functions on H^*. We regard a meromorphic function $\widehat{f} = f(P, r^*)$ of P and r^* as an element of M_{H^*} by $\widehat{f}(\mu) = f(< \mu, P >, < \mu, r^* >) \;\; \mu \in H^*$.

Now we take the field $\mathbb{F} = M_{H^*}$ as $\mathbb{K}$ and consider the semi-direct product $\mathbb{C}$-algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2) = \mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$ of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$ and $\mathbb{C}[\bar{H}^*]$, whose multiplication is defined by

$$\begin{aligned}
&(f(P, r^*)a \otimes e^\alpha) \cdot (g(P, r^*)b \otimes e^\beta) = f(P, r^*)g(P+ < \alpha, P >, r^*)ab \otimes e^{\alpha+\beta},\\
&a, b \in \mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)], f(P, r^*), g(P, r^*) \in \mathbb{F}, \alpha, \beta \in \bar{H}^*.
\end{aligned}$$

The following automorphism ϕ_r of $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ is the key to our "elliptic deformation" [18].

$$\begin{aligned}
&c \mapsto c, \quad h \mapsto h, \quad d \mapsto d,\\
&x^+(z) \mapsto u^+(z, p)x^+(z), \quad x^-(z) \mapsto x^-(z)u^-(z, p),\\
&\psi(z) \mapsto u^+(q^{c/2}z, p)\psi(z)u^-(q^{-c/2}z, p),\\
&\varphi(z) \mapsto u^+(q^{-c/2}z, p)\varphi(z)u^-(q^{c/2}z, p).
\end{aligned}$$

Here we set

$$u^+(z, p) = \exp\left(\sum_{n>0} \frac{1}{[r^*n]_q} a_{-n}(q^r z)^n\right), \quad u^-(z, p) = \exp\left(-\sum_{n>0} \frac{1}{[rn]_q} a_n(q^{-r} z)^{-n}\right).$$

Definition 2.2. We define the elliptic currents $E(u), F(u), K(u) \in U_{q,p}(\widehat{\mathfrak{sl}}_2)[[u]]$

and $\hat{d}$ by

$$\begin{aligned} E(u) &= \phi_r(x^+(z))e^{2Q}z^{-\frac{P-1}{r^*}}, \\ F(u) &= \phi_r(x^-(z))z^{\frac{P+h-1}{r}}, \\ K(u) &= \exp\left(\sum_{n>0}\frac{[n]_q}{[2n]_q[r^*n]_q}a_{-n}(q^cz)^n\right)\exp\left(-\sum_{n>0}\frac{[n]_q}{[2n]_q[rn]_q}a_nz^{-n}\right) \\ &\qquad \times e^Q z^{-\frac{c}{4rr^*}(2P-1)+\frac{1}{2r}h}, \\ \hat{d} &= d - \frac{1}{4r^*}(P-1)(P+1) + \frac{1}{4r}(P+h-1)(P+h+1), \end{aligned}$$

where we set $z = q^{2u}$.

From Definition 2.1 and (2.1), we can derive the following relations.

Proposition 2.3.

$$\begin{aligned} &c: \ \text{central}, \\ &[h, a_n] = 0, \quad [h, E(u)] = 2E(u), \quad [h, F(u)] = -2F(u), \\ &[\hat{d}, h] = 0, \quad [\hat{d}, a_n] = na_n, \\ &[\hat{d}, E(u)] = \left(-z\frac{\partial}{\partial z} - \frac{1}{r^*}\right)E(u), \quad [\hat{d}, F(u)] = \left(-z\frac{\partial}{\partial z} - \frac{1}{r}\right)F(u), \\ &[a_n, a_m] = \frac{[2n]_q[cn]_q}{n}q^{-c|n|}\delta_{n+m,0}, \\ &[a_n, E(u)] = \frac{[2n]_q}{n}q^{-c|n|}z^nE(u), \\ &[a_n, F(u)] = -\frac{[2n]_q}{n}z^nF(u), \end{aligned}$$

$$\begin{aligned} E(u)E(v) &= \frac{[u-v+1]^*}{[u-v-1]^*}E(v)E(u), \\ F(u)F(v) &= \frac{[u-v-1]}{[u-v+1]}F(v)F(u), \\ [E(u), F(v)] &= \frac{1}{q-q^{-1}}\left(\delta\left(q^{-c}\frac{z}{w}\right)H^+(q^{c/2}w) - \delta\left(q^c\frac{z}{w}\right)H^-(q^{-c/2}w)\right), \end{aligned}$$

where $z = q^{2u}$, $w = q^{2v}$,

$$\begin{aligned} &H^\pm(z) = \kappa K\left(u \pm \frac{1}{2}(r-\frac{c}{2}) + \frac{1}{2}\right)K\left(u \pm \frac{1}{2}(r-\frac{c}{2}) - \frac{1}{2}\right), \\ &\kappa = \lim_{z\to q^{-2}}\frac{\xi(z;p^*,q)}{\xi(z;p,q)}, \qquad \xi(z;p,q) = \frac{(q^2z;p,q^4)_\infty(pq^2z;p,q^4)_\infty}{(q^4z;p,q^4)_\infty(pz;p,q^4)_\infty}. \end{aligned}$$

Definition 2.4. We call a pair ($\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$, ϕ_r) the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. We also denote by $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ the subalgebra $\mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

The following relations are crucial in the formulation of the H-Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. See Sect.3.2.

Proposition 2.5.

$$[K(u), P] = K(u), \quad [E(u), P] = 2E(u), \quad [F(u), P] = 0,$$
$$[K(u), P + h] = K(u), \quad [E(u), P + h] = 0, \quad [F(u), P + h] = 2F(u).$$

§ 2.2. The RLL-relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In order to define the L operator, we need the half currents defined by the following formulae.

Definition 2.6.

$$K^+(u) = K(u + \tfrac{r+1}{2}),$$
$$E^+(u) = a^* \oint_{C^*} E(u') \frac{[u - u' + c/2 - P + 1]^* \, [1]^*}{[u - u' + c/2]^* [P - 1]^*} \frac{dz'}{2\pi i z'},$$
$$F^+(u) = a \oint_{C} F(u') \frac{[u - u' + P + h - 1] \, [1]}{[u - u'][P + h - 1]} \frac{dz'}{2\pi i z'}.$$

Here $z' = q^{2u'}$ and $z = q^{2u}$. The contours C^* and C are chosen in such a way that C^* encircles $zq^c p^{*n}$ $(n \geq 1)$ but not $zq^c p^{*n}$ $(n \leq 0)$, C encircles zp^n $(n \geq 1)$ but not zp^n $(n \leq 0)$, respectively. The constants a, a^* are chosen to satisfy $\frac{a^* a [1]^* \kappa}{q - q^{-1}} = 1$.

Then we define the operator $\widehat{L}^+(u) \in \mathrm{End}_{\mathbb{C}} V \otimes U_{q,p}\big(\widehat{\mathfrak{sl}}_2\big)$ with $V = \mathbb{C}^2$ as follows.

Definition 2.7.

$$\widehat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u-1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}.$$

From the relations in Proposition 2.3, we obtain the RLL relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Proposition 2.8. *The operator $\widehat{L}^+(u)$ satisfies the following RLL relation.*

$$\begin{aligned} &R^{+(12)}(u_1 - u_2, P + h) \widehat{L}^{+(1)}(u_1) \widehat{L}^{+(2)}(u_2) \\ &\qquad = \widehat{L}^{+(2)}(u_2) \widehat{L}^{+(1)}(u_1) R^{+*(12)}(u_1 - u_2, P), \end{aligned} \tag{2.2}$$

where $R^+(u, P+h)$ *denotes the elliptic dynamical* R *matrices given by*

$$R^+(u,s) = \rho^+(u) \begin{pmatrix} 1 & & & \\ & b(u,s) & c(u,s) & \\ & \bar{c}(u,s) & \bar{b}(u,s) & \\ & & & 1 \end{pmatrix} \tag{2.3}$$

with

$$\rho^+(u) = z^{\frac{1}{2r}} \frac{\{pq^2z\}^2}{\{pz\}\{pq^4z\}} \frac{\{z^{-1}\}\{q^4z^{-1}\}}{\{q^2z^{-1}\}^2}, \quad \{z\} = (z;p,q^4)_\infty,$$
$$b(u,s) = \frac{[s+1][s-1]}{[s]^2}\frac{[u]}{[1+u]}, \quad c(u,s) = \frac{[1]}{[s]}\frac{[s+u]}{[1+u]},$$
$$\bar{c}(u,s) = \frac{[1]}{[s]}\frac{[s-u]}{[1+u]}, \quad \bar{b}(u,s) = \frac{[u]}{[1+u]},$$

and $R^{+*}(u,P)$ *denotes the* R *matrix obtained from* $R^+(u,P)$ *by the replacements* $r \to r^*, p \to p^*$ *and* $[\ \cdot\ ] \to [\ \cdot\ ]^*$.

It is also worth while noting the following proposition immediately obtained from Proposition 2.8, which indicates a connection between $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$.

Proposition 2.9. *[18] Let us set* $L^+(u,P) = \widehat{L}^+(u)e^{-h\otimes Q}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Then* $L^+(u,P)$ *is independent of* Q *and satisfies the following dynamical RLL relation.*

$$\begin{aligned} &R^{+(12)}(u_1-u_2, P+h)L^{+(1)}(u_1,P)L^{+(2)}(u_2,P+h^{(1)}) \\ &\qquad = L^{+(2)}(u_2,P)L^{+(1)}(u_1,P+h^{(2)})R^{+*(12)}(u_1-u_2,P). \end{aligned} \tag{2.4}$$

This is the same dynamical RLL relation that characterizes the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with the parametrization $\lambda = (r^*+2)\Lambda_0 + (P+1)\bar{\Lambda}_1$[18]. In fact, under this parametrization, the vector representation of the universal dynamical R matrix $\mathcal{R}(\lambda)$ of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ coincides with $R^{+*}(u,P)$ in (2.3). Recalling also that by definition $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with the usual parameter $\lambda \in \mathfrak{h}$ is isomorphic to $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ as an associative algebra, we have the isomorphism from $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with $\lambda = (r^*+2)\Lambda_0 + (P+1)\bar{\Lambda}_1$ to $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$. Combining these facts, we have the isomorphism $U_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2) \otimes \mathbb{C}[\bar{H}^*]$ with $\lambda = (r^*+2)\Lambda_0 + (P+1)\bar{\Lambda}_1$ as a semi-direct product algebra.

Note also that the $c=0$ case of (2.4) is the dynamical RLL relation studied by Felder [12, 9], whereas the $c=0$ case of (2.2) is the RLL relation studied in [10, 11, 19] for the trigonometric R, and in [20] for the elliptic R.

§ 3. H-Hopf Algebroid Structure

This section is an exposition of the H-Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$[26, 27]. Our H-Hopf algebroid is an extension of the one studied in [10, 11] and [19, 20] to the one with the central element c.

§ 3.1. Definition of the H-Hopf Algebroid[10, 11, 19]

Let A be a complex associative algebra, H be a finite dimensional commutative subalgebra of A, and M_{H^*} be the field of meromorphic functions on H^* the dual space of H.

Definition 3.1. An H-algebra is a complex associative algebra A with 1, which is bigraded over H^*, $A = \bigoplus_{\alpha,\beta\in H^*} A_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : M_{H^*} \to A_{00}$ (the left and right moment maps), such that

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \qquad a \in A_{\alpha\beta},\ \widehat{f} \in M_{H^*},$$

where T_α denotes the automorphism $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda+\alpha)$ of M_{H^*}.

Definition 3.2. An H-algebra homomorphism is an algebra homomorphism $\pi : A \to B$ between two H-algebras A and B preserving the bigrading and the moment maps, i.e. $\pi(A_{\alpha\beta}) \subseteq B_{\alpha\hat{\beta}}$ for all $\hat{\alpha}, \hat{\beta} \in H^*$ and $\pi(\mu_l^A(\widehat{f})) = \mu_l^B(\widehat{f}), \pi(\mu_r^A(\widehat{f})) = \mu_r^B(\widehat{f})$.

Let A and B be two H-algebras. The tensor product $A\widetilde{\otimes}B$ is the H^*-bigraded vector space with

$$(A\widetilde{\otimes}B)_{\alpha\beta} = \bigoplus_{\gamma\in H^*} (A_{\alpha\gamma} \otimes_{M_{H^*}} B_{\gamma\beta}),$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the following relation.

$$\mu_r^A(\widehat{f})a \otimes b = a \otimes \mu_l^B(\widehat{f})b, \qquad a \in A, b \in B, \widehat{f} \in M_{H^*}. \tag{3.1}$$

The tensor product $A\widetilde{\otimes}B$ is again an H-algebra with the multiplication $(a\otimes b)(c\otimes d) = ac \otimes bd$ and the moment maps

$$\mu_l^{A\widetilde{\otimes}B} = \mu_l^A \otimes 1, \qquad \mu_r^{A\widetilde{\otimes}B} = 1 \otimes \mu_r^B.$$

Let $\mathcal{D}$ be the algebra of automorphisms $M_{H^*} \to M_{H^*}$

$$\mathcal{D} = \{ \sum_i \widehat{f}_i T_{\beta_i} \mid \widehat{f}_i \in M_{H^*},\ \beta_i \in H^* \}.$$

Equipped with the bigrading $\mathcal{D}_{\alpha\alpha} = \{\ \widehat{f}T_{-\alpha} \mid \widehat{f} \in M_{H^*},\ \alpha \in H^*\ \}$, $\mathcal{D}_{\alpha\beta} = 0$ $(\alpha \neq \beta)$ and the moment maps $\mu_l^{\mathcal{D}}, \mu_r^{\mathcal{D}} : M_{H^*} \to \mathcal{D}_{00}$ defined by $\mu_l^{\mathcal{D}}(\widehat{f}) = \mu_r^{\mathcal{D}}(\widehat{f}) = \widehat{f}T_0$, $\mathcal{D}$ is an H-algebra. For any H-algebra A, we have the canonical isomorphism as an H-algebra

$$A \cong A\widetilde{\otimes}\mathcal{D} \cong \mathcal{D}\widetilde{\otimes}A \tag{3.2}$$

by $a \cong a\widetilde{\otimes}T_{-\beta} \cong T_{-\alpha}\widetilde{\otimes}a$ for all $a \in A_{\alpha\beta}$. Hence $\mathcal{D}$ plays the role of unit object in the category of H-algebras.

Definition 3.3. An H-bialgebroid is an H-algebra A equipped with two H-algebra homomorphisms $\Delta : A \to A\widetilde{\otimes}A$ (the comultiplication) and $\varepsilon : A \to \mathcal{D}$ (the counit) such that

$$(\Delta\widetilde{\otimes}\mathrm{id}) \circ \Delta = (\mathrm{id}\widetilde{\otimes}\Delta) \circ \Delta,$$
$$(\varepsilon\widetilde{\otimes}\mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id}\widetilde{\otimes}\varepsilon) \circ \Delta,$$

under the identification (3.2).

Definition 3.4. An H-Hopf algebroid is an H-bialgebroid A equipped with a $\mathbb{C}$-linear map $S : A \to A$ (the antipode), such that

$$S(\mu_r(\widehat{f})a) = S(a)\mu_l(\widehat{f}), \quad S(a\mu_l(\widehat{f})) = \mu_r(\widehat{f})S(a), \quad \forall a \in A, \widehat{f} \in M_{H^*},$$
$$m \circ (\mathrm{id}\widetilde{\otimes}S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \quad \forall a \in A,$$
$$m \circ (S\widetilde{\otimes}\mathrm{id}) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \quad \forall a \in A_{\alpha\beta},$$

where $m : A\widetilde{\otimes}A \to A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{H^*}$.

§ 3.2. H-Hopf Algebroid Structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Now let us consider the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The commutative subalgebra H is given in §2.1. Let $\bar{\mathfrak{h}} = \mathbb{C}h$ be the Cartan subalgebra, α_1 the simple root and $\bar{\Lambda}_1$ the fundamental weight of $\mathfrak{sl}(2,\mathbb{C})$. We set $\mathcal{Q} = \mathbb{Z}\alpha_1$ and $\bar{\mathfrak{h}}^* = \mathbb{C}\bar{\Lambda}_1$. Let $<,>$ be the standard paring of $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^*$. Using the isomorphism $\phi : \mathcal{Q} \to \bar{H}^*$ by $n\alpha_1 \mapsto nQ$, we define the $\bar{H}^*$-bigrading of $U_{q,p} = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by

$$U_{q,p} = \bigoplus_{\alpha,\beta \in \bar{H}^*} (U_{q,p})_{\alpha\beta},$$
$$(U_{q,p})_{\alpha\beta} = \left\{ x \in U_{q,p} \;\middle|\; \begin{array}{c} q^h x q^{-h} = q^{<\phi^{-1}(\alpha-\beta),h>}x, \\ q^P x q^{-P} = q^{<\beta,P>}x \end{array} \right\}. \tag{3.3}$$

Noting $<\phi^{-1}(\alpha), h> = <\alpha, P>$, we have $q^{P+h}xq^{-(P+h)} = q^{<\alpha,P>}x$ for $x \in (U_{q,p})_{\alpha\beta}$.

Let M_{H^*} be the field of meromorphic functions given in §2.1. We define two moment maps $\mu_l, \mu_r : M_{H^*} \to (U_{q,p})_{00}$ as follows

$$\mu_l(\widehat{f}) = f(P+h, r^*+c), \qquad \mu_r(\widehat{f}) = f(P, r^*). \tag{3.4}$$

From (3.3), one finds for $x \in (U_{q,p})_{\alpha\beta}$

$$\mu_l(\widehat{f})x = f(P+h, r^*+c)x = xf(P+h+<\alpha, P>, r^*+c) = x\mu_l(T_\alpha \widehat{f}),$$
$$\mu_r(\widehat{f})x = f(P, r^*)x = xf(P+<\beta, P>, r^*) = x\mu_r(T_\beta \widehat{f}),$$

where $T_\alpha = e^\alpha$ denotes a shift operator $M_{H^*} \to M_{H^*}$ defined by

$$(T_\alpha \widehat{f}) = e^\alpha f(P, r^*)e^{-\alpha} = f(P+<\alpha, P>, r^*).$$

Equipped with the bigrading structure (3.3) and the two moment maps (3.4), the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H-algebra.

We also have the H-algebra $\mathcal{D}$ of the shift operators on M_{H^*}

$$\mathcal{D} = \{ \sum_i \widehat{f}_i T_{\alpha_i} \mid \widehat{f}_i \in M_{H^*}, \alpha_i \in \bar{H}^* \}$$

whose bigrading structure and moment maps are given as in §3.1.

The tensor product among the H-algebras $U_{q,p}$ and $\mathcal{D}$ is defined as in §3.1. In particular, we have the H-algebra isomorphism $U_{q,p}\widetilde{\otimes}\mathcal{D} \cong U_{q,p} \cong \mathcal{D}\widetilde{\otimes}U_{q,p}$ by $x\widetilde{\otimes}T_{-\beta} = x = T_{-\alpha}\widetilde{\otimes}x$ for $x \in (U_{q,p})_{\alpha\beta}$.

Now let us consider the coalgebra structure of $U_{q,p}$. Let $\widehat{L}^+(u)$ be the L operator introduced in Sect.2.2. We write the entries of $\widehat{L}^+(u)$ as

$$\widehat{L}^+(u) = \begin{pmatrix} \widehat{L}^+_{++}(u) & \widehat{L}^+_{+-}(u) \\ \widehat{L}^+_{-+}(u) & \widehat{L}^+_{--}(u) \end{pmatrix}.$$

From Proposition 2.5 and Definition 2.7, one finds

$$\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) \in (U_{q,p})_{-\varepsilon_1 Q, -\varepsilon_2 Q}.$$

It is also easy to check the relations

$$f(P+h, r^*+c)\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) = \widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)f(P+h-\varepsilon_1, r^*+c),$$
$$f(P, r^*)\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) = \widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)f(P-\varepsilon_2, r^*).$$

Definition 3.5. We define H-algebra homomorphisms, $\varepsilon : U_{q,p} \to \mathcal{D}$ and Δ :

$U_{q,p} \to U_{q,p}\widetilde{\otimes}U_{q,p}$ by

$$\begin{aligned}
&\varepsilon(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)) = \delta_{\varepsilon_1,\varepsilon_2}T_{-\varepsilon_2 Q},\\
&\varepsilon(e^Q) = e^Q, \quad \varepsilon(\mu_l(\widehat{f})) = \varepsilon(\mu_r(\widehat{f})) = f(P,r^*)T_0,\\
&\Delta(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)) = \sum_{\varepsilon'} \widehat{L}^+_{\varepsilon_1\varepsilon'}(u)\widetilde{\otimes}\widehat{L}^+_{\varepsilon'\varepsilon_2}(u),\\
&\Delta(e^Q) = e^Q\widetilde{\otimes}e^Q,\\
&\Delta(\mu_l(\widehat{f})) = \mu_l(\widehat{f})\widetilde{\otimes}1, \quad \Delta(\mu_r(\widehat{f})) = 1\widetilde{\otimes}\mu_r(\widehat{f}).
\end{aligned}$$

We also define an H-algebra anti-homomorphism $S : U_{q,p} \to U_{q,p}$ by

$$\begin{aligned}
&S(\widehat{L}^+_{++}) = \widehat{L}^+_{--}(u-1), \quad S(\widehat{L}^+_{+-}(u)) = -\frac{[P+h+1]}{[P+h]}\widehat{L}^+_{+-}(u-1),\\
&S(\widehat{L}^+_{-+}(u)) = -\frac{[P]^*}{[P+1]^*}\widehat{L}^+_{-+}(u-1), \quad S(\widehat{L}^+_{--}(u)) = \frac{[P+h+1][P]^*}{[P+h][P+1]^*}\widehat{L}^+_{++}(u-1),\\
&S(e^Q) = e^{-Q}, \quad S(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), \quad S(\mu_l(\widehat{f})) = \mu_r(\widehat{f}).
\end{aligned}$$

One can check that Δ and S preserve the RLL relation (2.2). Furthermore one finds that ε, Δ and S are the counit, the comultiplication and the antipode satisfying the following relations.

Proposition 3.6. *[27] The maps ε, Δ and S satisfy*

$$\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,\\
&(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta.\\
&m \circ (\mathrm{id} \otimes S) \circ \Delta(x) = \mu_l(\varepsilon(x)1), \quad \forall x \in U_{q,p},\\
&m \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in (U_{q,p})_{\alpha\beta}.
\end{aligned}$$

Equipped with (Δ, ε, S), the H-algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H-Hopf algebroid.

Definition 3.7. We call the H-Hopf algebroid $(U_{q,p}(\widehat{\mathfrak{sl}}_2), H, M_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

§ 4. Representations

In this section, we summarize some basic facts on the dynamical representations of H-algebras [10, 19] and their application to $U'_{q.p}(\widehat{\mathfrak{sl}}_2)$.

Let us consider a vector space $\widehat{V}$ over $\mathbb{F}$, which is $\bar{\mathfrak{h}}$-diagonalizable,

$$\widehat{V} = \bigoplus_{\mu\in\bar{\mathfrak{h}}^*} \widehat{V}_\mu, \qquad \widehat{V}_\mu = \{v \in \widehat{V} \mid q^h v = q^\mu v \quad (h \in \bar{\mathfrak{h}})\}.$$

Let us define the H-algebra $\mathcal{D}_{H,\widehat{V}}$ of the $\mathbb{C}$-linear operators on $\widehat{V}$ by

$$\mathcal{D}_{H,\widehat{V}} = \bigoplus_{\alpha,\beta\in\bar{H}^*} (\mathcal{D}_{H,\widehat{V}})_{\alpha\beta},$$

$$(\mathcal{D}_{H,\widehat{V}})_{\alpha\beta} = \left\{ X \in \mathrm{End}_{\mathbb{C}}\widehat{V} \;\middle|\; \begin{array}{c} X(f(P,r^*)v) = f(P-<\beta,P>,r^*)X(v),\ v\in\widehat{V}, \\ f(P,r^*)\in\mathbb{F},\ X(\widehat{V}_\mu)\subseteq\widehat{V}_{\mu+\phi^{-1}(\alpha-\beta)} \end{array} \right\},$$

$$\mu_l^{\mathcal{D}_{H,\widehat{V}}}(\widehat{f})v = f(P+\mu,r^*+c)v, \quad \mu_r^{\mathcal{D}_{H,\widehat{V}}}(\widehat{f})v = f(P,r^*)v, \quad \widehat{f}\in M_{H^*},\ v\in\widehat{V}_\mu.$$

Note that the subspace $(\mathcal{D}_{H,\widehat{V}})_{\alpha\beta}$ consists of the $\mathbb{C}$-linear operators on $\widehat{V}$ of the form $x\widetilde{\otimes}e^{-\beta}$, where x is a $\mathbb{C}$-linear operator carrying the weight $\phi^{-1}(\alpha-\beta)$.

Definition 4.1. [10, 19] A dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\widehat{V}$ is an H-algebra homomorphism $\widehat{\pi}: U'_{q,p}(\widehat{\mathfrak{sl}}_2) \to \mathcal{D}_{H,\widehat{V}}$.

Let $(\widehat{\pi}_V,\widehat{V}),(\widehat{\pi}_W,\widehat{W})$ be two dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$. We define the tensor product $\widehat{V}\widetilde{\otimes}\widehat{W}$ by

$$\widehat{V}\widetilde{\otimes}\widehat{W} = \bigoplus_{\mu\in\bar{\mathfrak{h}}^*}(\widehat{V}\widetilde{\otimes}\widehat{W})_\mu, \quad (\widehat{V}\widetilde{\otimes}\widehat{W})_\mu = \bigoplus_{\nu\in\bar{\mathfrak{h}}^*}\widehat{V}_\nu\otimes_{M_{H^*}}\widehat{W}_{\mu-\nu},$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the relation

$$f(P,r^*)v\otimes w = v\otimes f(P+\nu,r^*+c)w \tag{4.1}$$

for $w\in\widehat{W}_\nu$. The action of scalars $f(P,r^*)\in\mathbb{F}$ on the tensor space $\widehat{V}\widetilde{\otimes}\widehat{W}$ is defined by

$$f(P,r^*).(v\widetilde{\otimes}w) = \Delta(\mu_r(\widehat{f}))(v\widetilde{\otimes}w) = v\widetilde{\otimes}f(P,r^*)w.$$

Then one finds a natural H-algebra embedding $\theta_{\widehat{V}\widehat{W}}: \mathcal{D}_{H,\widehat{V}}\widetilde{\otimes}\mathcal{D}_{H,\widehat{W}} \to \mathcal{D}_{H,\widehat{V}\widetilde{\otimes}\widehat{W}}$ by $X_{\widehat{V}}\widetilde{\otimes}X_{\widehat{W}} \in (\mathcal{D}_{H,\widehat{V}})_{\alpha\gamma}\widetilde{\otimes}(\mathcal{D}_{H,\widehat{W}})_{\gamma\beta} \mapsto X_{\widehat{V}}\widetilde{\otimes}X_{\widehat{W}} \in (\mathcal{D}_{H,\widehat{V}\widetilde{\otimes}\widehat{W}})_{\alpha\beta}$. Hence the map $\theta_{\widehat{V}\widehat{W}}\circ(\widehat{\pi}_V\otimes\widehat{\pi}_W)\circ\Delta: U'_{q,p}(\widehat{\mathfrak{sl}}_2)\to\mathcal{D}_{H,\widehat{V}\widetilde{\otimes}\widehat{W}}$ gives a dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\widehat{V}\widetilde{\otimes}\widehat{W}$.

Now let us consider a construction of dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$. Let V be an $\bar{\mathfrak{h}}^*$-diagonalizable vector space over $\mathbb{F}$. Let V_Q be a vector space over $\mathbb{C}$, and assume that an action of e^Q on V_Q is defined appropriately. Two important examples of V_Q are $V_Q=\mathbb{C}1$ and $V_Q=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}e^{nQ}$, where 1 denotes the vacuum state satisfying $e^Q.1=1$. We then consider the vector space $\widehat{V}=V\otimes_{\mathbb{C}}V_Q$ over $\mathbb{F}$, where the actions of $f(P,r^*)\in\mathbb{F}$ and e^Q on $\widehat{V}$ are defined by

$$f(P,r^*)(v\otimes\xi) = f(P,r^*)v\otimes\xi,$$
$$e^Q(f(P,r^*)v\otimes\xi) = f(P+1,r^*)v\otimes e^Q\xi$$

for $v \otimes \xi \in V \otimes V_Q$.

Then the following theorem is fundamental in our construction of dynamical representations.

Theorem 4.2. *[27] Let $\widehat{V}$ be as above and $\pi_V : \mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \to \mathrm{End}_{\mathbb{F}} V$ be an algebra homomorphism. Define a map $\widehat{\pi}_V = \pi_V \otimes \mathrm{id} : U'_{q,p}(\widehat{\mathfrak{sl}}_2) = \mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*] \to \mathrm{End}_{\mathbb{C}} \widehat{V}$ by*

$$\widehat{\pi}_V(E(u)) = \pi_V(\phi_r(x^+(z)))e^{2Q}z^{-\frac{P-1}{r^*}},$$
$$\widehat{\pi}_V(F(u)) = \pi_V(\phi_r(x^-(z)))z^{\frac{P+\pi_V(h)-1}{r}},$$
$$\widehat{\pi}_V(K(u)) = \exp\left(\sum_{n>0} \frac{[n]_q}{[2n]_q[r^*n]_q}\pi_V(a_{-n})(q^c z)^n\right) \exp\left(-\sum_{n>0} \frac{[n]_q}{[2n]_q[rn]_q}\pi_V(a_n)z^{-n}\right) \times e^Q z^{-\frac{c}{4rr^*}(2P-1)+\frac{1}{2r}\pi_V(h)}.$$

Then $(\widehat{\pi}_{\widehat{V}}, \widehat{V})$ is a dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\widehat{V}$.

Through this paper we consider dynamical representations obtained in this way.

Let us also state the Poincaré-Birkhoff-Witt theorem for $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Definition 4.3. Let $\mathcal{H}$ (resp. $\mathcal{N}_\pm$) be the subalgebras of $\mathbb{F}[U'_{q,p}(\widehat{\mathfrak{sl}}_2)]$ generated by c, h and a_k ($k \in \mathbb{Z}_{\neq 0}$) (resp. by $x^\pm_n$ ($n \in \mathbb{Z}$)).

Theorem 4.4. *[27]*

$$U'_{q,p}(\widehat{\mathfrak{sl}}_2) = (\mathcal{N}_- \otimes \mathcal{H} \otimes \mathcal{N}_+) \otimes \mathbb{C}[\bar{H}^*].$$

Here the last $\otimes$ should be understood as a semi-direct product.

§ 5. Finite-Dimensional Representations

This section is an announcement of some new results on the finite-dimensional dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$. Detailed discussion will be published elsewhere [27].

§ 5.1. Pseudo-highest Weight Representations

We begin by stating a characteristic feature of the finite-dimensional irreducible dynamical representations.

Theorem 5.1. *Every finite-dimensional irreducible dynamical representation $(\widehat{\pi}_V, \widehat{V} = V \otimes_{\mathbb{C}} V_Q)$ of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ contains a non-zero vector of the form $\widehat{\Omega} = \Omega \otimes 1, \Omega \in V$ such* *that*

$$
\begin{aligned}
&1)\ x^+_n.\widehat{\Omega}=0 \qquad \forall n\in\mathbb{Z},\\
&2)\ \psi_n.\widehat{\Omega}=d^+_n\widehat{\Omega}, \qquad \phi_{-n}.\widehat{\Omega}=d^-_{-n}\widehat{\Omega} \quad \forall n\in\mathbb{Z}_{\geq 0},\\
&3)\ e^Q.\widehat{\Omega}=\widehat{\Omega},\\
&4)\ \widehat{V}=U'_{q,p}.\widehat{\Omega}.
\end{aligned}
$$

with some complex numbers $d^{\pm}_{\pm n}$, $d^+_0 d^-_0=1$. *Furthermore* q^c *acts as* 1 *or* -1 *on* $\widehat{V}$.

Definition 5.2. We define a pseudo-highest weight representation, a pseudo-highest weight vector and a pseudo-highest weight to be a dynamical representation (not necessarily irreducible) $(\widehat{\pi}_V,\widehat{V})$, a vector $\widehat{\Omega}\in\widehat{V}$ and a set of complex numbers $\boldsymbol{d}=\{d^{\pm}_{\pm n}\}_{n\in\mathbb{Z}_{\geq 0}}$ satisfying the conditions 1) $-$ 4) in Theorem 5.1, respectively.

The following theorem is useful.

Theorem 5.3. *For a vector* $\widehat{\Omega}\in\widehat{V}$ *satisfying* $e^Q.\widehat{\Omega}=\widehat{\Omega}$, *the conditions* 1) *and* 2) *in Definition 5.2 are equivalent to the following.*

$$
\begin{aligned}
&i)\ \widehat{L}^+_{-+}(u).\widehat{\Omega}=0 \quad \forall u,\\
&ii)\ q^h.\widehat{\Omega}=q^{\lambda}\widehat{\Omega} \quad \exists\lambda\in\mathbb{C},\\
&\qquad \widehat{L}^+_{++}(u).\widehat{\Omega}=A(u)\widehat{\Omega}, \qquad \widehat{L}^+_{--}(u).\widehat{\Omega}=D(u)\widehat{\Omega}
\end{aligned}
$$

with some meromorphic functions $A(u)$ *and* $D(u)$ *satisfying* $D(u-1)^{-1}=A(u)$ *and*

$$
A(u)=z^{\frac{\lambda}{2r}}\sum_{m\in\mathbb{Z},n\in\mathbb{Z}_{\geq 0}} A_{m,n}z^m p^n \qquad A_{m,n}\in\mathbb{C}, z=q^{2u}, p=q^{2r}. \tag{5.1}
$$

$U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ admits the universal pseudo-highest weight representation defined as follows.

Definition 5.4. Let $\boldsymbol{d}=\{d^{\pm}_{\pm n}\}_{n\in\mathbb{Z}_{\geq 0}}$ be any sequence of complex numbers. The Verma module $M(\boldsymbol{d})$ is the quotient of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the two sided ideal generated by q^c-1 and the left ideal generated by $\{x^+_k\ (k\in\mathbb{Z}),\ \psi_n-d^+_n\cdot 1, \phi_{-n}-d^-_{-n}\cdot 1\ (n\in\mathbb{Z}_{\geq 0}),\ e^Q-1\}$.

Proposition 5.5. *The Verma module* $M(\boldsymbol{d})$ *is pseudo-highest weight with pseudo-highest weight* $\boldsymbol{d}$. *Every pseudo-highest weight representation with pseudo-highest weight* $\boldsymbol{d}$ *is isomorphic to a quotient of* $M(\boldsymbol{d})$. *Moreover* $M(\boldsymbol{d})$ *has a unique irreducible pseudo-highest weight module.*

§ 5.2. Elliptic Analogue of the Drinfeld Polynomials

We state a necessary and sufficient condition for an irreducible dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ to be finite-dimensional. We introduce a natural elliptic analogue of the Drinfeld polynomials.

Theorem 5.6. *The irreducible pseudo-highest weight representation* $(\widehat{\pi}_V, \widehat{V})$ *of* $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ *is finite-dimensional if and only if there exists an entire and quasi-periodic function* $P_V(u)$ *such that*

$$\begin{aligned} &H^{\pm}(u).\widehat{\Omega} = c_V \frac{P_V(u+1)}{P_V(u)}\widehat{\Omega} \\ &P_V(u+r) = (-)^{\mathrm{deg}P} P_V(u), \\ &P_V(u+r\tau) = (-)^{\mathrm{deg}P} e^{-\pi i \sum_{j=1}^{\mathrm{deg}P}(\frac{2(u-\alpha_j)}{r}+\tau)} P_V(u). \end{aligned}$$

Here $\widehat{\Omega}$ *denotes the pseudo-highest weight vector in* $\widehat{V}$*, and* $\tau = -\frac{2\pi i}{\log p}$*. The symbol* c_V *denotes a constant given by*

$$c_V = q^{\frac{r-1}{r}\mathrm{deg}P} \prod_{j=1}^{\mathrm{deg}P} a_j^{\frac{1}{r}},$$

where $\mathrm{deg}P$ *is the number of zeros of* $P_V(u)$ *in the period parallelogram* $(1, \tau)$ *(= the degree of the Drinfeld polynomial* $P(z) = \lim_{r\to\infty} P_V(u),\ z = q^{2u}$*), and* $a_j = q^{2\alpha_j}$ *with* α_j *being a zero of* $P_V(u)$ *in the period parallelogram. The function* $P_V(u)$ *is unique up to a scalar multiple.*

Remark. We can take $c_V = 1$ by the gauge transformation given by (2.11) in [18]. An example is given in Theorem5.8.

The following Proposition is a direct consequence of the comultiplication formula for $\widehat{L}^+(u)$ and Definition 2.7.

Proposition 5.7. *Let* $\widehat{V}$ *and* $\widehat{W}$ *be finite dimensional dynamical representations of* $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ *and assume that the tensor product* $\widehat{V}\widetilde{\otimes}\widehat{W}$ *is irreducible. Let* $P_V(u), P_W(u)$ *and* $P_{V\widetilde{\otimes}W}(u)$ *be the entire quasi-periodic function associated to* $\widehat{V}, \widehat{W}$ *and* $\widehat{V}\widetilde{\otimes}\widehat{W}$ *in Theorem 5.6. Then*

$$P_{V\widetilde{\otimes}W}(u) = P_V(u)P_W(u).$$

§ 5.3. Evaluation Representations

An important example of finite-dimensional irreducible dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ is the evaluation representation. We here give a summary on the $l+1$-dimensional evaluation representation obtained from the one of $\mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)]$.

Let $V^{(l)} = \oplus_{m=0}^{l}\mathbb{F}v_m^l,\ V_w^{(l)} = V^{(l)}\otimes\mathbb{C}[w,w^{-1}]$, and consider the operators $h, S^{\pm}$ on $V^{(l)}$ defined by

$$hv_m^l = (l-2m)v_m^l, \qquad S^{\pm}v_m^l = v_{m\mp1}^l, \qquad v_m^l = 0 \quad \text{for } m<0,\quad m>l.$$

In terms of the Drinfeld generators, the $l+1$-dimensional evaluation representation $(\pi_{l,w}, V_w^{(l)})$ of $\mathbb{F}[U_q'(\widehat{\mathfrak{sl}}_2)]$ is given by

$$
\begin{aligned}
&\pi_{l,w}(q^c) = 1,\\
&\pi_{l,w}(a_n) = \frac{w^n}{n}\frac{1}{q-q^{-1}}((q^n+q^{-n})q^{nh} - (q^{(l+1)n}+q^{-(l+1)n})),\\
&\pi_{l,w}(x^\pm(z)) = S^\pm \left[\frac{\pm h+l+2}{2}\right]_q \delta\left(q^{h\pm 1}\frac{w}{z}\right).
\end{aligned}
\tag{5.2}
$$

Now let us consider the vector space $\widehat{V}^{(l)}(w) = V^{(l)}(w)\otimes \mathbb{C}1$ and the map $\widehat{\pi}_{l,w} = \pi_{l,w}\otimes \mathrm{id}$ on $U'_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathbb{F}[U_q'(\widehat{\mathfrak{sl}}_2)]\otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$. Applying (5.2) to Theorem 4.2 and noting Definitions 2.2 and 2.6, we can derive the evaluation representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ as follows.

Theorem 5.8. *$(\widehat{\pi}_{l,w}, \widehat{V}^{(l)}(w))$ is the $l+1$-dimensional irreducible dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the pseudo-highest weight vector $v_0^l\otimes 1$. In particular, the images of the matrix elements of $\widehat{L}^+(u)$ by the map $\widehat{\pi}_{l,w}$ are given, up to fractional powers of z, w and q, by*

$$
\begin{aligned}
\widehat{\pi}_{l,w}(\widehat{L}^+_{++}(u)) &= -\frac{[u-v+\frac{h+1}{2}][P-\frac{l-h}{2}][P+\frac{l+h+2}{2}]}{\varphi_l(u-v)[P][P+h+1]}e^Q,\\
\widehat{\pi}_{l,w}(\widehat{L}^+_{+-}(u)) &= -S^-\frac{[u-v+\frac{h-1}{2}+P][\frac{l-h+2}{2}]}{\varphi_l(u-v)[P+h-1]}e^{-Q},\\
\widehat{\pi}_{l,w}(\widehat{L}^+_{-+}(u)) &= S^+\frac{[u-v-\frac{h+1}{2}-P][\frac{l+h+2}{2}]}{\varphi_l(u-v)[P]}e^{Q},\\
\widehat{\pi}_{l,w}(\widehat{L}^+_{--}(u)) &= -\frac{[u-v-\frac{h-1}{2}]}{\varphi_l(u-v)}e^{-Q},
\end{aligned}
$$

where $z=q^{2u}$, $w=q^{2v}$, and

$$
\begin{aligned}
\varphi_l(u) &= -z^{-\frac{l}{2r}}\rho^+_{1l}(z,p)^{-1}[u+\frac{l+1}{2}],\\
\rho^+_{kl}(z,p) &= q^{\frac{kl}{2}}\frac{\{pq^{k-l+2}z\}\{pq^{-k+l+2}z\}}{\{pq^{k+l+2}z\}\{pq^{-k-l+2}z\}}\frac{\{q^{k+l+2}/z\}\{q^{-k-l+2}/z\}}{\{q^{k-l+2}/z\}\{q^{-k+l+2}/z\}}.
\end{aligned}
$$

Corollary 5.9. *The elliptic analogue of the Drinfeld polynomial associated to $\widehat{V}^{(l)}(q^{2v})$ is given by*

$$
P_{l,v}(u) = [u-v-\frac{l-1}{2}][u-v-\frac{l-1}{2}+1]\cdots[u-v+\frac{l-1}{2}].
$$

Obviously the zeros of $P_{l,v}(u)$ modulo $\mathbb{Z}r+\mathbb{Z}r\tau$ coincide with those of the Drinfeld polynomial corresponding to the evaluation representation $V^{(l)}(q^{2v})$ of $U_q'(\widehat{\mathfrak{sl}}_2)$.

The following Proposition indicates a consistency of our construction of $\widehat{\pi}_{l,w}$ and the standard fusion construction of the dynamical R matrices (=face type Boltzmann weights).

Proposition 5.10. *Let us define the matrix elements of* $\widehat{\pi}_{l,w}(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u))$ *by*

$$\widehat{\pi}_{l,w}(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u))v^l_m = \sum_{m'=0}^{l} (\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u))_{\mu_{m'}\mu_m} v^l_{m'},$$

where $\mu_m = l - 2m$. *Then we have*

$$(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u))_{\mu_{m'}\mu_m} = R^+_{1l}(u-v,P)^{\varepsilon_2\mu_m}_{\varepsilon_1\mu_{m'}}.$$

Here $R^+_{1l}(u-v,P)$ *is the R matrix from (C.17) in [18]. In the case* $l=1$, $R^+_{11}(u-v,P)$ *coincides with the image* $(\pi_{1,z}\otimes\pi_{1,w})$ *of the universal R matrix* $\mathcal{R}^+(\lambda)$ *[17] given in* (2.3). *In the case* $l>1$, $R^+_{1l}(u-v,P)$ *coincides with the R matrix obtained by the standard fusion procedure from* $R^+_{11}(u-v,P)$. *In particular the matrix element* $R^+_{1l}(u-v,P)^{\varepsilon'\mu'}_{\varepsilon\mu}$ *is gauge equivalent to the fusion face weight* $W_{l1}(P+\varepsilon', P+\varepsilon'+\mu', P+\mu, P|u-v)$ *from (4) in [6].*

§ 6. Infinite-dimensional Representations and Vertex Operators

Theorem 4.2 is valid also for infinite-dimensional representations. Let $(\pi, V(\lambda_l))$ be the level-k ($c=k$) irreducible highest weight representation of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$ with the highest weight $\lambda_l = (k-l)\Lambda_0 + l\Lambda_1$ ($0 \le l \le k$). Here Λ_i ($i=0,1$) denote the fundamental weights of $\widehat{\mathfrak{sl}}(2,\mathbb{C})$. Then ($\widehat{\pi} = \pi\otimes \mathrm{id}$, $\widehat{V}(\lambda_l) = \bigoplus_{m\in\mathbb{Z}} V(\lambda_l)\otimes\mathbb{C}e^{-mQ}$) is the level-$k$ highest weight irreducible dynamical representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

A realization of $\widehat{V}(\lambda_l)$ in terms of the Drinfeld generators a_n ($n\in\mathbb{Z}_{\neq 0}$) and the q-deformed $\mathbb{Z}_k$-parafermion algebra was given in [23, 26]. In [24, 18], we also studied the Wakimoto representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ labeled by an integer J ($0\le J\le k$), which is nothing but the dynamical representation $\widehat{V}(\lambda_J)$.

The H-Hopf algebroid structure allows us to define the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as follows.

Definition 6.1. The type I and II vertex operators of spin $n/2$ are the intertwiners of the $U_{q,p}$-modules of the form

$$\widehat{\Phi}(u)\ :\ \widehat{V}(\lambda)\to \widehat{V}^{(n)}_z\widetilde{\otimes}\widehat{V}(\nu),\quad \widehat{\Psi}^*(u)\ :\ \widehat{V}(\lambda)\widetilde{\otimes}\widehat{V}^{(n)}_z\to\widehat{V}(\nu),$$

where $z=q^{2u}$, and $\widehat{V}(\lambda)$ and $\widehat{V}(\nu)$ denote the level-k highest weight $U_{q,p}$-modules of highest weights λ and ν, respectively. They satisfy the intertwining relations with respect to the comultiplication Δ in Definition 3.5.

$$\Delta(x)\widehat{\Phi}(u) = \widehat{\Phi}(u)x,\quad x\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\Delta(x)\qquad \forall x\in U_{q,p}. \tag{6.1}$$

The physically interesting cases are $n = k, \lambda = \lambda_l, \nu = \lambda_{k-l}$ for the type I and $n = 1, \lambda = \lambda_l, \nu = \lambda_{l\pm1}$ for the type II vertex operators. See for example [23].

Let us define the components of the vertex operators as follows.

$$\widehat{\Phi}(v - \frac{1}{2}) = \sum_{m=0}^{n} v_m^n \widetilde{\otimes} \Phi_m(v), \quad \widehat{\Psi}^*(v - \frac{c+1}{2})(\,\cdot\,\widetilde{\otimes} v_m^n) = \Psi_m^*(v). \tag{6.2}$$

By using the comultiplication formula for $\widehat{L}^+(u)$ in Definition 3.5 and Proposition 5.10, we obtain the following theorem.

Theorem 6.2. *[26] The vertex operators satisfy the following linear equations.*

$$\widehat{\Phi}(u)\widehat{L}^+(v) = R_{1n}^{+(12)}(v-u, P+h)\widehat{L}^+(v)\widehat{\Phi}(u), \tag{6.3}$$

$$\widehat{L}^+(v)\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\widehat{L}^+(v)R_{1n}^{+*(13)}(v-u, P-h^{(1)}-h^{(3)}). \tag{6.4}$$

The relation (6.3) *should be understood on* $\widehat{V}_w^{(1)}\widetilde{\otimes}\widehat{V}(\lambda)$, *whereas* (6.4) *on* $\widehat{V}_w^{(1)}\widetilde{\otimes}\widehat{V}(\lambda)\widetilde{\otimes}\widehat{V}_z^{(n)}$.

Equations (6.3) and (6.4) coincide with (5.3) and (5.4) in [18], respectively. In [18], those equations were derived by using the quasi-Hopf algebra structure on $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and the isomorphism $U_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2) \otimes \mathbb{C}[\bar{H}^*]$ as a semi-direct product algebra. Under certain analyticity conditions, these equations determine the vertex operators uniquely up to normalization. Combining the results obtained here and those in [29, 24, 18, 28, 23], we have established the algebraic analysis scheme for the fusion RSOS models as well as for the fusion eight-vertex models on the basis of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

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