RIMS Kôkyûroku Bessatsu
B11 (2009), 197–222

Iwahori-Hecke type algebras associated with the Lie superalgebras $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$

By

Hiroyuki Yamane*

Abstract

In this paper we give Iwahori-Hecke type algebras $H_q(\mathbf{g})$ associated with the Lie superalgebras $\mathbf{g} = A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$. We classify the irreducible representations of $H_q(\mathbf{g})$ for generic q.

Introduction

Recently, motivated by a question posed by V. Serganova [S] and study of the Weyl groupoids [H1][H2] associated with Nichols algebras [AS1][AS2] including generalizations of quantum groups, I. Heckenberger and the author [HY] introduced a notion of 'Coxeter groupoids' (in fact they can be defined as semigroups), and showed that a Matsumoto-type theorem holds for the groupoids, so they have the solvable word problem. We mention that the Coxeter groupoid associated with the affine Lie superalgebra $D^{(1)}(2,1;x)$ was used in the study [HSTY], where Drinfeld second realizations of $U_q(D^{(1)}(2,1;x))$ was analized by physical motivation in recent study of AdS/CFT correspondence.

It would be able to be said that one of the main purposes at present of the representation theory is to study the Kazhdan-Lusztig polynomials (cf. [Hu, 7.9]) and their versions. The polynomials are defined by using the standard and canonical bases of the Iwahori-Hecke algebras. The existence of those bases is closely related to the Matsumoto theorem of the Coxeter groups. So it would be natural to ask what to be

Received April 14, 2008. Accepted January 14, 2009.
2000 Mathematics Subject Classification(s): 17B20, 33D80, 20H15
Key Words: Lie superalgebra, Hecke algebra, Weyl groupoid
*Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, 560-0043, Japan.
e-mail: `yamane@ist.osaka-u.ac.jp`

the Iwahori-Hecke algebras of the Coxeter groupoids. In this paper, we give a tentative answer to this question for the Coxeter groupoids W associated with the Lie superalgebras $\mathbf{g} = A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$. We introduce the Iwahori-Hecke type algebra $H_q(\mathbf{g})$ (in the text, it is also denoted by $H_q(W)$) as q-analogue of the semigroup algebra $\mathbb{C}W/\mathbb{C}0$, where 0 is the zero element of W. We also show that if q is nonzero and not any root of unity, $H_q(\mathbf{g})$ is semisimple and there exists a natural one-to-one correspondence between the equivalence classes of the irreducible representations of $H_q(\mathbf{g})$ and those of the Iwahori-Hecke algebra $H_q(W_0)$ associated with the Weyl group W_0 of the Lie algebra $\mathbf{g}(0)$ obtained as the even part of $\mathbf{g} = \mathbf{g}(0) \oplus \mathbf{g}(1)$.

Until now, no relation has been achieved between the groupoids treated in [SV] and this paper.

This paper is composed of the two sections. Main results and their proofs are given in Section 2. Results of [HY] used in Section 2 are introduced in Section 1.

The author thanks to the referee for careful reading and valuable comments, which encourage him so much to make future study.

§ 1. Preliminary—Matsumoto-type theorem of Coxeter groupoids

This section is preliminary. Here we collect the results which have already been given in [HY] and will be used in the next section.

§ 1.1. Semigroups and Monoids

Let K be a non-empty set. Assume that K has a product map $K \times K \to K$, $(x,y) \mapsto xy$. We call K a *semigroup* if $(xy)z = x(yz)$ for $\forall\ x,y,z \in K$. We call K a *monoid* if K is a semigroup and there exists a unit $1 \in K$, that is, $1x = x1 = x$ for all $x \in K$.

§ 1.2. Free semigroup $F_{-1}(N)$ and Free monoid $F_0(N)$

Let N be a non-empty set. Let $F_{-1}(N)$ be the set of all the finite sequences of elements of N, that is

$$F_{-1}(N) := \coprod_{n=1}^{\infty} N^n = \{(h_1,\dots,h_n)|n \in \mathbb{N}, h_i \in N\}.$$

We regard $F_{-1}(N)$ as the semigroup by

$$(h_1,\dots,h_m)(h_{m+1},\dots,h_{m+n}) = (h_1,\dots,h_m,h_{m+1},\dots,h_{m+n}).$$

Then we call $F_{-1}(N)$ a *free semigroup*. Let $F_0(N)$ be the semigroup obtained by adding the unit 1, that is, $F_0(N) := \{1\} \cup F_{-1}(N)$, $1 \notin F_{-1}(N)$, and $1x = x1 = x$ for all $x \in F_0(N)$.

§ 1.3. Semigroup generated by the generators and and defined by the relations

Let $Q = \{(x_j, y_j) | j \in J\}$ be a subset of $F_{-1}(N) \times F_{-1}(N)$, where J is an index set. For g_1, $g_2 \in F_{-1}(N)$, we write $g_1 \sim_1 g_2$ if there exist $(x, y) \in Q$ and $(f_1, f_2) \in F_0(N) \times F_0(N)$ such that either of the following (i), (ii), (iii) holds.

(i) $g_1 = f_1 x f_2 \neq g_2 = f_1 y f_2$.
(ii) $g_1 = f_1 y f_2 \neq g_2 = f_1 x f_2$.
(iii) $g_1 = g_2 = f_1 x f_2 = f_1 y f_2$.

For g, $g' \in F_{-1}(N)$, we write $g \sim g'$ if $g = g'$ or there exists $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in F_{-1}(N)$ such that $g_1 = g$, $g_r = g'$, and $g_i \sim_1 g_{i+1}$ for $1 \leq i \leq r-1$. Then $F_{-1}(N)/\sim$ can be regarded as a semigroup by the product $[g][g'] = [gg']$, where for $g \in F_{-1}(N)$, we denote $[g] := \{g' | g' \sim g\} \in F_{-1}(N)/\sim$. We call $F_{-1}(N)/\sim$ *the semigroup generated by N and defined by the relations $x_j = y_j$ ($j \in J$).* When there is no fear of misunderstanding, we also denote $[g]$ by its representative g by abuse of notation.

§ 1.4. Free group $F_1(N)$ and Involutive free group $F_2(N)$

Let N be a set. Let N^{-1} be a copy of N so that the bijective map $N \to N^{-1}$, $x \mapsto x^{-1}$, is given. Let $F_1(N)$ be the semigroup generated by

$$\{e\} \cup N \cup N^{-1} \quad \text{(disjoint union)}$$

and defined by the relations

$$ee = e, \quad ex = xe = x, \quad ex^{-1} = x^{-1}e = x^{-1}, \quad xx^{-1} = x^{-1}x = e \quad \text{for } \forall x \in N.$$

We call $F_1(N)$ the *free group* over N.

Let $F_2(N)$ be the semigroup generated by

$$\{e\} \cup N \quad \text{(disjoint union)}$$

and defined by the relations

$$ee = e, \quad ex = xe = x, \quad x^2 = e \quad \text{for } \forall x \in N.$$

We call $F_2(N)$ the *involutive free group* over N. Note that $F_2(N)$ can be identified with the quotient group of $F_1(N)$ in the natural sense:

$$F_2(N) = F_1(N)/\{g_1 y_1^2 g_1^{-1} \cdots g_r y_r^2 g_r^{-1} | r \in \mathbb{N}, y_i \in N \cup N^{-1}, g_i \in F_1(N)\}.$$

§ 1.5. Action $\triangleright$ of $F_2(N)$ on A

Let N and A be non-empty sets. An action $\triangleright$ of $F_2(N)$ on A is a map

$$\triangleright : F_2(N) \times A \to A$$

such that

$$e \triangleright a = a, \quad g \triangleright (h \triangleright a) = (gh) \triangleright a \quad \text{for } \forall g, \forall h \in F_2(N), \forall a \in A.$$

Note that $n \triangleright (n \triangleright a) = a$ for all $n \in N, a \in A$.

For n, $n' \in N$ and $a \in A$, define

$$\Theta(n, n'; a) := \{(nn')^m \triangleright a,\ (n'n)^m \triangleright a \mid m \in \mathbb{N} \cup \{0\}\}.$$

Let

$$\theta(n, n'; a) := |\Theta(n, n'; a)|.$$

This is the cardinality of $\Theta(n, n'; a)$, which is either in $\mathbb{N}$ or is ∞. One obviously has $\Theta(n, n'; a) = \Theta(n', n; a)$ and $\Theta(n, n'; n \triangleright a) = n \triangleright \Theta(n', n; a)$.

Let $a_0 := a$, $b_0 := a$, and define recursively $a_{m+1} := n \triangleright b_m$, $b_{m+1} := n' \triangleright a_m$ for all $m \in \mathbb{N} \cup \{0\}$. That is:

$$\begin{aligned} &b_0 := a, \quad a_1 := n \triangleright a, \quad b_2 := n' \triangleright n \triangleright a, \quad a_3 := n \triangleright n' \triangleright n \triangleright a, \ldots \\ &a_0 := a, \quad b_1 := n' \triangleright a, \quad a_2 := n \triangleright n' \triangleright a, \quad b_3 := n' \triangleright n \triangleright n' \triangleright a, \ldots \end{aligned}$$

Then we have

$$\theta(n, n'; a) = \begin{cases} \infty & \text{if } a_m \neq b_m \text{ for all } m \in \mathbb{N}, \\ \min\{m \in \mathbb{N} \mid a_m = b_m\} & \text{otherwise.} \end{cases}$$

§ 1.6. Coxeter groupoids

Definition 1.1. [HY, Definition 1] Let N and A be non-empty sets. Let $\triangleright$ be a transitive action of $F_2(N)$ on A. For each $a \in A$ and $i, j \in N$ with $i \neq j$ let

$$m_{i,j;a} = m_{j,i;a} \in (\mathbb{N} + 1) \cup \{\infty\}$$

be such that $\theta(i, j; a) \in \mathbb{N} \Longrightarrow \frac{m_{i,j;a}}{\theta(i,j;a)} \in \mathbb{N} \cup \{\infty\}$ or $\theta(i, j; a) = \infty \Longrightarrow m_{i,j;a} = \infty$. Set

$$\mathbf{m} := (m_{i,j;a} \mid i, j \in N, i \neq j, a \in A).$$

Let

$$W = (W, N, A, \triangleright, \mathbf{m}) \tag{1.1}$$

be the semigroup generated by the set

$$\{0, e_a, s_{i,a} \,|\, a \in A, i \in N\}$$

and defined by the relations

$$00 = e_a 0 = 0 e_a = s_{i,a} 0 = 0 s_{i,a} = 0, \tag{1.2}$$

$$\begin{aligned} e_a^2 =& e_a, & e_a e_b =& 0 \text{ for } a \neq b, \\ e_{i \triangleright a} s_{i,a} =& s_{i,a} e_a = s_{i,a}, \quad & s_{i,i \triangleright a} s_{i,a} =& e_a, \end{aligned} \tag{1.3}$$

$$\begin{aligned} s_i s_j \cdots s_j s_{i,a} &= s_j s_i \cdots s_i s_{j,a} \ (m_{i,j;a} \text{ factors}) \qquad \text{if } m_{i,j;a} \text{ is finite and odd,} \\ s_j s_i \cdots s_j s_{i,a} &= s_i s_j \cdots s_i s_{j,a} \ (m_{i,j;a} \text{ factors}) \qquad \text{if } m_{i,j;a} \text{ is finite and even,} \end{aligned} \tag{1.4}$$

where we use the convention:

$$s_j s_{i,a} := s_{j, i \triangleright a} s_{i,a}, \quad s_i s_j s_{i,a} := s_{i, j \triangleright i \triangleright a} s_j s_{i,a}, \ldots \tag{1.5}$$

See also (1.7) below.

§ 1.7. Sign representation

Let $\mathbb{Z}A$ be the free $\mathbb{Z}$-module generated by A, that is,

$$\mathbb{Z}A = \oplus_{a \in A} \mathbb{Z}a.$$

Then there exists a unique semigroup homomorphism

$$\widetilde{\mathrm{sgn}} : W \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}A)$$

such that

$$\widetilde{\mathrm{sgn}}(0)(b) = 0, \qquad \widetilde{\mathrm{sgn}}(e_a)(b) = \delta_{ab} b, \qquad \widetilde{\mathrm{sgn}}(s_{i,a})(b) = (-1)\delta_{ab} i \triangleright a \tag{1.6}$$

for $a, b \in A$ and $i \in N$, where δ means Kronecker's symbol. Hence for $w \in W$ one has

$$w \neq 0$$

if and only if $w = e_a$ for some $a \in A$ or there exist $m \in \mathbb{N}$ and $i_j \in N$, $b_j \in A$ with $1 \leq j \leq m$ such that $b_j = i_{j+1} \triangleright b_{j+1}$ and $w = s_{i_1,b_1} \cdots s_{i_{m-1},b_{m-1}} s_{i_m,b_m}$. If this is the case, we use the convention

$$s_{i_1} \cdots s_{i_{m-1}} s_{i_m,b_m} := w, \tag{1.7}$$

and, if $m = 0$, $s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a}$ means e_a. We note again

Lemma 1.2. (1) *$s_{i_1}\cdots s_{i_{m-1}}s_{i_m,a}\neq 0$ for all $a\in A$ and $m\in\mathbb{N}\cup\{0\}$.*

(2) *If $s_{i_1}\cdots s_{i_{m-1}}s_{i_m,a}=s_{j_1}\cdots s_{j_{r-1}}s_{j_r,b}$, then $a=b$, $i_1\cdots i_m\triangleright a=j_1\cdots j_r\triangleright b$ and $(-1)^m=(-1)^r$.*

§ 1.8. Generalization of Root systems

Definition 1.3. [HY, Definition 2] We call a quadruple $(R,N,A,\triangleright)$ a *multi-domains root system* if the following conditions hold.

1. N and A are non-empty sets and $\triangleright$ is a transitive action of $F_2(N)$ on A.
2. Let V_0 be the $|N|$-dimensional $\mathbb{R}$-linear space. Then

$$R=\{(R_a,\pi_a,S_a)\,|a\in A\},$$

   where $\pi_a=\{\alpha_{n,a}\,|\,n\in N\}\subset R_a\subset V_0$, and π_a is a basis of V_0 for all $a\in A$.
3. $R_a=R_a^+\cup -R_a^+$ for all $a\in A$, where $R_a^+=(\mathbb{N}\cup\{0\})\pi_a\cap R_a$.
4. For any $i\in N$ and $a\in A$ one has $\mathbb{R}\alpha_{i,a}\cap R_a=\{\alpha_{i,a},-\alpha_{i,a}\}$.
5. $S_a=\{\sigma_{i,a}\,|\,i\in N\}$, and for each $a\in A$ and $i\in N$ one has $\sigma_{i,a}\in\mathrm{GL}(V_0)$,

$$\sigma_{i,a}(R_a)=R_{i\triangleright a},\quad \sigma_{i,a}(\alpha_{i,a})=-\,\alpha_{i,i\triangleright a},\quad \sigma_{i,a}(\alpha_{j,a})\in\alpha_{j,i\triangleright a}+(\mathbb{N}\cup\{0\})\alpha_{i,i\triangleright a}$$

   for all $j\in N\setminus\{i\}$.
6. $\sigma_{i,i\triangleright a}\sigma_{i,a}=\mathrm{id}$ for $a\in A$ and $i\in N$.
7. Let $a\in A$, $i,j\in N$, $i\neq j$, and $d=|((\mathbb{N}\cup\{0\})\alpha_{i,a}+(\mathbb{N}\cup\{0\})\alpha_{j,a})\cap R_a|$. If d is finite then $\theta(i,j;a)$ is finite and it divides d.

Convention. We write

$$(R,N,A,\triangleright)\in\mathcal{R}$$

if $(R,N,A,\triangleright)$ is a multi-domains root system, that is, $\mathcal{R}=\{(R,N,A,\triangleright)\}$ denotes the family of all the multi-domains root systems.

Definition 1.4. [HY, Definition 4] Let $(R,N,A,\triangleright)\in\mathcal{R}$. Let $\mathbf{m}:=(m_{i,j;a}\,|\,i,j\in N, i\neq j, a\in A)$ be such that $m_{i,j;a}:=|((\mathbb{N}\cup\{0\})\alpha_{i,a}+(\mathbb{N}\cup\{0\})\alpha_{j,a})\cap R_a|$. Then we call $(W,N,A,\triangleright,\mathbf{m})$ the *Coxeter groupoid associated with* $(R,N,A,\triangleright)$.

Theorem 1.5. [HY, Theorem 1] *Let $(R,N,A,\triangleright)\in\mathcal{R}$. Set $V=\bigoplus_{a\in A}V_a$, where $V_a=V_0$. Let $P_a:V\to V_a$ and $\iota_a:V_a\to V$ be the canonical projection and the canonical inclusion map respectively. Then the assignment $\rho:0\mapsto 0\cdot\mathrm{id}_V$, $e_a\mapsto\iota_aP_a$, $s_{i,a}\mapsto\iota_{i\triangleright a}\sigma_{i,a}P_a$, gives a faithful representation (ρ,V) of the Coxeter groupoid $(W,N,A,\triangleright,\mathbf{m})$ associated with $(R,N,A,\triangleright)$.*

§ 1.9. Matsumoto-type theorem

Define $\ell : W \to \mathbb{N} \cup \{0\} \cup \{-\infty\}$ to be the map such that $\ell(0) = -\infty$, $\ell(e_a) = 0$ for all $a \in A$, and

$$\ell(w) = \min\{m \in \mathbb{N} \,|\, w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a} \text{ for some } i_1, \dots, i_m \in N,\ a \in A\}$$

for all $w \in W \setminus (\{0\} \cup \{e_a | a \in A\})$; we also refer to Lemma 1.2 (1) for this definition of ℓ. One has

$$\ell(w) = \ell(w^{-1}) \tag{1.8}$$

for $w \in W \setminus \{0\}$, and

$$\ell(ww') \leq \ell(w) + \ell(w') \tag{1.9}$$

for $w, w' \in W$. We say that a product $w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a} \in W$ is *reduced* if $m = \ell(w)$.

Definition 1.6. [HY, Definition 5] Let $W = (W, N, A, \triangleright, \mathbf{m})$ be a Coxeter groupoid. Let $\widetilde{W} = (\widetilde{W}, N, A, \triangleright, \mathbf{m})$ denote the semigroup generated by the set

$$\{0, \tilde{e}_a, \tilde{s}_{i,a} \,|\, a \in A, i \in N\}$$

and defined by the relations

$$00 = 0, \quad 0\tilde{e}_a = \tilde{e}_a 0 = 0\tilde{s}_{i,a} = \tilde{s}_{i,a} 0 = 0, \tag{1.10}$$

$$\tilde{e}_a^2 = \tilde{e}_a, \quad \tilde{e}_a \tilde{e}_b = 0 \text{ for } a \neq b, \quad \tilde{e}_{i \triangleright a} \tilde{s}_{i,a} = \tilde{s}_{i,a} \tilde{e}_a = \tilde{s}_{i,a}, \tag{1.11}$$

$$\begin{array}{ll} \tilde{s}_i \tilde{s}_j \cdots \tilde{s}_j \tilde{s}_{i,a} = \tilde{s}_j \tilde{s}_i \cdots \tilde{s}_i \tilde{s}_{j,a} \ (m_{i,j;a} \text{ factors}) & \text{if } m_{i,j;a} \text{ is finite and odd,} \\ \tilde{s}_j \tilde{s}_i \cdots \tilde{s}_j \tilde{s}_{i,a} = \tilde{s}_i \tilde{s}_j \cdots \tilde{s}_i \tilde{s}_{j,a} \ (m_{i,j;a} \text{ factors}) & \text{if } m_{i,j;a} \text{ is finite and even.} \end{array} \tag{1.12}$$

Theorem 1.7. [HY, Theorem 5] (Matsumoto-type theorem of the Coxeter groupoids) *Let*

$$W = (W, N, A, \triangleright, \mathbf{m})$$

be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ (see Definition 1.4). Suppose that $m \in \mathbb{N} \cup \{0\}$, $a \in A$, and $(i_1, \dots, i_m), (j_1, \dots, j_m) \in N^m$ such that

$$\ell(s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a}) = m$$

and equation

$$s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a} = s_{j_1} \cdots s_{j_{m-1}} s_{j_m,a}$$

holds in W. Then in the semigroup $(\widetilde{W}, N, A, \triangleright, \mathbf{m})$ one has

$$\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m,a} = \tilde{s}_{j_1} \cdots \tilde{s}_{j_{m-1}} \tilde{s}_{j_m,a}.$$

Corollary 1.8. [HY, Corollary 6] *Let*

$$W = (W, N, A, \triangleright, \mathbf{m})$$

be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ *(see Definition 1.4). Suppose that* $m \in \mathbb{N} \cup \{0\}$*,* $a \in A$*, and* $(i_1, \ldots, i_m) \in N^m$ *such that* $\ell(s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a}) < m$ *holds in* $(W, N, A, \triangleright, \mathbf{m})$*. Then there exist* $j_1, \ldots, j_m \in N$ *and* $t \in \{1, \ldots, m-1\}$ *such that* $j_t = j_{t+1}$ *and in the semigroup* $(\widetilde{W}, N, A, \triangleright, \mathbf{m})$ *one has the equation*

$$\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m,a} = \tilde{s}_{j_1} \cdots \tilde{s}_{j_t} \tilde{s}_{j_{t+1}} \cdots \tilde{s}_{j_{m-1}} \tilde{s}_{j_m,a}.$$

In the next section, we also need

Proposition 1.9. [HY, Corollary 3] *Let* $m \in \mathbb{N}$*,* $(i_1, \ldots, i_m, j) \in N^{m+1}$*, and* $a \in A$*, and suppose that* $\ell(s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a}) = m$*. Then:*

(1) $m = |\sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m,a}(R^+_a) \cap -R^+_{i_1 \cdots i_m \triangleright a}|$.

(2) $\ell(s_{i_1} \cdots s_{i_m} s_{j,j \triangleright a}) = m - 1 \Longleftrightarrow \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m,a}(\alpha_{j,a}) \in -R^+_{i_1 \cdots i_m \triangleright a}$.

(3) $\ell(s_{i_1} \cdots s_{i_m} s_{j,j \triangleright a}) = m + 1 \Longleftrightarrow \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m,a}(\alpha_{j,a}) \in R^+_{i_1 \cdots i_m \triangleright a}$.

Example 1.10. Here we treat the finite dimensional simple Lie superalgebra $D(2,1;x)$, where $x \notin \{0, -1\}$. Note that it has 14 (positive and negative) roots. One has $m_{i,j;a(k)} = 2 + \delta_{k0} + (1 - \delta_{k0})(\delta_{ik} + \delta_{jk})$ and $\sigma_{i,a(k)}(\alpha_{j,a(k)}) = \alpha_{j,i \triangleright a(k)} + \delta_{3,m_{i,j;a(k)}} \alpha_{i,i \triangleright a(k)}$ for $i \neq j$. Moreover

$$\begin{aligned} R^+_{a(k)} = \pi_{a(k)} &\cup \{\alpha_{i,a(k)} + \alpha_{j,a(k)} | m_{i,j;a(k)} = 3\} \\ &\cup \{\alpha_{1,a(k)} + \alpha_{2,a(k)} + \alpha_{3,a(k)}\} \\ &\cup \{\alpha_{i,a(k)} + 2\alpha_{k,a(k)} + \alpha_{j,a(k)} | m_{i,j;a(k)} = 2\}. \end{aligned}$$

Note that $D(2,1;1) = D(2,1) = \mathfrak{osp}(4|2)$ (see also Section 2.2).

Let $w_{a(2)} := s_{3,a(2)} s_{2,a(0)} s_{3,a(3)} s_{1,a(3)} s_{3,a(0)} s_{2,a(2)} s_{1,a(2)}$. Then $\rho(w_{a(2)}) = -\mathrm{id}_{V_{a(2)}}$. Indeed:

$$\begin{aligned} &\alpha_{1,a(2)} \mapsto -\alpha_{1,a(2)} \mapsto -\alpha_{1,a(0)} - \alpha_{2,a(0)} \mapsto -\alpha_{1,a(3)} - \alpha_{2,a(3)} - 2\alpha_{3,a(3)} \\ &\quad \mapsto -\alpha_{1,a(3)} - \alpha_{2,a(3)} - 2\alpha_{3,a(3)} \mapsto -\alpha_{1,a(0)} - \alpha_{2,a(0)} \mapsto -\alpha_{1,a(2)} \mapsto -\alpha_{1,a(2)}, \\ &\alpha_{2,a(2)} \mapsto \alpha_{1,a(2)} + \alpha_{2,a(2)} \mapsto \alpha_{1,a(0)} \mapsto \alpha_{1,a(3)} + \alpha_{3,a(3)} \mapsto \alpha_{3,a(3)} \mapsto -\alpha_{3,a(0)} \\ &\quad \mapsto -\alpha_{2,a(2)} - \alpha_{3,a(2)} \mapsto -\alpha_{2,a(2)}, \\ &\alpha_{3,a(2)} \mapsto \alpha_{3,a(2)} \mapsto \alpha_{2,a(0)} + \alpha_{3,a(0)} \mapsto \alpha_{2,a(3)} \mapsto \alpha_{2,a(3)} \mapsto \alpha_{2,a(0)} + \alpha_{3,a(0)} \\ &\quad \mapsto \alpha_{3,a(2)} \mapsto -\alpha_{3,a(2)}. \end{aligned}$$

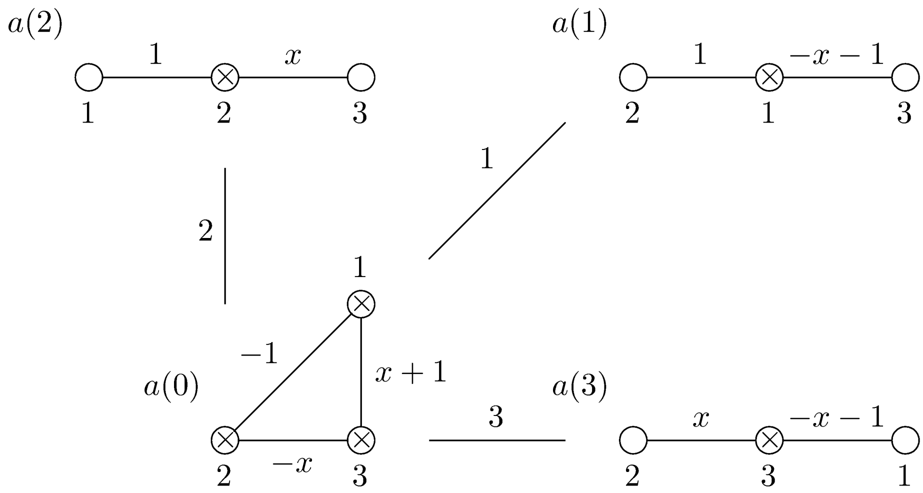


Figure 1. Dynkin diagrams of the Lie superalgebra $D(2,1;x)$

By Proposition 1.9(1), we have $\ell(w_{a(2)}) = |R^+_{a(2)}| = 7$, $w_{a(2)}$ is the longest word. Let $w' := (s_{3,a(2)}s_{2,a(0)})^{-1}w_{a(2)}$. Then $\ell(w') = 5$. By Theorem 1.7, w' has the following four reduced expressions:

$$\begin{aligned} w' &= \underline{s_{3,a(3)}s_{1,a(3)}s_{3,a(0)}}s_{2,a(2)}s_{1,a(2)} = s_{1,a(1)}s_{3,a(1)}\underline{s_{1,a(0)}s_{2,a(2)}s_{1,a(2)}} \\ &= s_{1,a(1)}\underline{s_{3,a(1)}s_{2,a(1)}}s_{1,a(0)}s_{2,a(2)} = s_{1,a(1)}s_{2,a(1)}s_{3,a(1)}s_{1,a(0)}s_{2,a(2)}. \end{aligned}$$

§ 2. Main theorems—Irreducible representations of the Iwahori-Hecke type algeras $H_q(A(m,n))$, $H_q(B(m,n))$, $H_q(C(n))$ and $H_q(D(m,n))$ associated with the Lie superalgebras $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$

§ 2.1. Definition of Lie superalgebras

As for the terminology concerning Lie superalgebras, we refer to [K].

Let $\mathfrak{v} = \mathfrak{v}(0) \oplus \mathfrak{v}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear space. If $i \in \{0,1\}$ and $j \in \mathbb{Z}$ such that $j - i \in 2\mathbb{Z}$ then let $\mathfrak{v}(j) = \mathfrak{v}(i)$. If $X \in \mathfrak{v}(0)$ (resp. $X \in \mathfrak{v}(1)$) then we write

$$\deg(X) = 0 \text{ (resp. } \deg(X) = 1) \tag{2.1}$$

and we say that X is an *even* (resp. *odd*) element. If $X \in \mathfrak{v}(0) \cup \mathfrak{v}(1)$, then we say that X is a *homogeneous* element and that $\deg(X)$ is the *parity* (or *degree*) of X. If $\mathfrak{w} \subset \mathfrak{v}$ is a subspace and $\mathfrak{w} = (\mathfrak{w} \cap \mathfrak{v}(0)) \oplus (\mathfrak{w} \cap \mathfrak{v}(1))$ (resp. $\mathfrak{w} \subset \mathfrak{v}(0)$, resp. $\mathfrak{w} \subset \mathfrak{v}(1)$), then we say that $\mathfrak{w}$ is a *graded* (resp. *even*, resp. *odd*) subspace.

Let $\mathbf{g} = \mathbf{g}(0) \oplus \mathbf{g}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear space equipped with a bilinear map $[\,,\,] : \mathbf{g} \times \mathbf{g} \to \mathbf{g}$ such that $[\mathbf{g}(i), \mathbf{g}(j)] \subset \mathbf{g}(i+j)$ $(i,\ j \in \mathbb{Z})$; we recall from the above paragraph that

$$\mathbf{g}(i) = \{X \in \mathbf{g} \,|\, \deg(X) = i\}. \tag{2.2}$$

We say that $\mathbf{g} = (\mathbf{g}, [\,,\,])$ is a ($\mathbb{C}$-)*Lie superalgebra* if for all homogeneous elements X, Y, Z of $\mathbf{g}$ the following equations hold.

$$\begin{aligned} [Y, X] &= -(-1)^{\deg(X)\deg(Y)}[X, Y], && \text{(skew-symmetry)} \\ [X, [Y, Z]] &= [[X, Y], Z] + (-1)^{\deg(X)\deg(Y)}[Y, [X, Z]]. && \text{(Jacobi identity)} \end{aligned}$$

We call the Lie algebra $\mathbf{g}(0)$ the *even part* of $\mathbf{g}$.

§ 2.2. Lie superalgebras $\mathfrak{gl}(m+1|n+1)$ and $\mathfrak{osp}(m|n)$

Let $m,\ n \in \mathbb{N} \cup \{0\}$. Let:

$$\mathcal{D}_{m+1|n+1} := \{(p_1, \ldots, p_{m+n+2}) \in \mathbb{Z}^{m+n+2} \,|\, p_i \in \{0,1\}, \sum_{i=1}^{m+n+2} p_i = n+1\}.$$

For $i, j \in \{1, \ldots, m+n+2\}$, let $\mathbf{E}_{i,j}$ denote the $(m+n+2) \times (m+n+2)$ matrix having 1 in (i,j) position and 0 otherwise, that is, the (i,j)-matrix unit. Let $\mathbf{E}_{m+n+2}$ denote the $(m+n+2) \times (m+n+2)$ unit matrix, that is, $\mathbf{E}_{m+n+2} = \sum_{i=1}^{m+n+2} \mathbf{E}_{i,i}$. Denote by $\mathrm{M}_{m+n+2}(\mathbb{C})$ the $\mathbb{C}$-linear space of the $(m+n+2) \times (m+n+2)$-matrices, i.e., $\mathrm{M}_{m+n+2}(\mathbb{C}) = \oplus_{i,j=1}^{m+n+2} \mathbb{C}\mathbf{E}_{i,j}$.

Let $d = (p_1, \ldots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$. The Lie superalgebra $\mathfrak{gl}(m+1|n+1) = \mathfrak{gl}(d)$ is defined by $\mathfrak{gl}(d) = \mathrm{M}_{m+n+2}(\mathbb{C})$ (as a $\mathbb{C}$-linear space),

$$\mathfrak{gl}(d)(0) = \oplus_{1 \le p_i = p_j \le m+n+2} \mathbb{C}\mathbf{E}_{i,j}, \quad \mathfrak{gl}(d)(1) = \oplus_{1 \le p_i \neq p_j \le m+n+2} \mathbb{C}\mathbf{E}_{i,j}, \tag{2.3}$$

and $[X, Y] = XY - (-1)^{r_1 r_2} YX$ for $X \in \mathfrak{gl}(d)(r_1)$ and $Y \in \mathfrak{gl}(d)(r_2)$,

where XY and YX mean the matrix product, that is, $\mathbf{E}_{i,j}\mathbf{E}_{k,l} = \delta_{j,k}\mathbf{E}_{i,l}$. Define the $\mathbb{C}$-linear map $\mathrm{str} : \mathfrak{gl}(d) \to \mathbb{C}$ by $\mathrm{str}(\mathbf{E}_{i,j}) = \delta_{i,j}(-1)^{p_i}$. The Lie subsuperalgebra $\{X \in \mathfrak{gl}(d) \,|\, \mathrm{str}(X) = 0\}$ of $\mathfrak{gl}(d)$ is denoted as $\mathfrak{sl}(m+1|n+1) = \mathfrak{sl}(d)$. The finite dimensional simple Lie superalgebra $A(m,n)$ is defined as follows, where $m+n \ge 1$. Let $\mathfrak{z}$ be the one dimensional ideal $\mathbb{C}\mathbf{E}_{m+n+2}$ of $\mathfrak{gl}(d)$. If $m \neq n$, then $A(m,n)$ means $\mathfrak{sl}(d)$. On the other hand, $A(n,n)$ means $\mathfrak{sl}(d)/\mathfrak{z}$, and is also denoted as $\mathfrak{psl}(n+1|n+1)$.

Let $d = (p_1, \ldots, p_{m+2n}) \in \mathcal{D}_{m|2n}$. Define the map $\theta : \{1, \ldots, m+2n\} \to \{1, \ldots, m+2n\}$ by $\theta(i) = m+2n+1-i$. Assume that $p_{\theta(i)} = p_i$. Let $g_i \in \{1, -1\}$ be such that $g_i = -1$ if $p_i = 1$ and $i < \theta(i)$ and $g_i = 1$ otherwise. We have an automorphism Ω of

$\mathfrak{gl}(d)$ defined by $\Omega(\mathbf{E}_{i,j}) = -(-1)^{p_ip_j+p_j} g_ig_j\mathbf{E}_{\theta(j),\theta(i)}$. The Lie superalgebra $\mathfrak{osp}(m|2n)$ means $\{X \in \mathfrak{gl}(d)|\Omega(X) = X\}$. We also denote $\mathfrak{osp}(m|2n)$ as follows:

$$\begin{aligned} B(m-1,n) &= \mathfrak{osp}(2m-1|2n) && \text{if } m,\, n \in \mathbb{N}, \\ D(m+1,n) &= \mathfrak{osp}(2m+2|2n) && \text{if } m,\, n \in \mathbb{N}, \\ C(n+1) &= \mathfrak{osp}(2|2n) && \text{if } n \in \mathbb{N}. \end{aligned}$$

We also note that $\mathfrak{osp}(2m+1|0)$, $\mathfrak{osp}(0|2n)$, and $\mathfrak{osp}(2m|0)$ are isomorphic to the simple Lie algebras of type B_m (if $m \geq 2$), C_n (if $n \geq 3$) and D_m (if $m \geq 4$) respectively, so $\mathfrak{osp}(2m+1|0) \cong \mathfrak{o}_{2m+1}$, $\mathfrak{osp}(0|2n) \cong \mathfrak{sp}_{2n}$ and $\mathfrak{osp}(2m|0) \cong \mathfrak{o}_{2m}$. As for the even part $\mathfrak{osp}(m|2n)(0)$ of $\mathfrak{osp}(m|2n)$, we have

$$\mathfrak{osp}(m|2n)(0) \cong \mathfrak{osp}(m|0) \oplus \mathfrak{osp}(0|2n). \tag{2.4}$$

§ 2.3. Definition of Iwahori-Hecke type algebras

Definition 2.1. Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the groupoid introduced in (1.1). Assume that A is finite. Let $q \in \mathbb{C}$. Let $H_q(W)$ be the $\mathbb{C}$-algebra (with 1) generated by

$$\{E_a, T_{i,a} | a \in A, i \in N\} \tag{2.5}$$

and defined by the relations

$$E_a^2 = E_a, \tag{2.6}$$

$$E_{i \triangleright a} T_{i,a} E_a = T_{i,a}, \tag{2.7}$$

$$\sum_{a \in A} E_a = 1 \tag{2.8}$$

$$E_aE_b = 0 \qquad \text{if } a \neq b, \tag{2.9}$$

$$(T_{i,a} - qE_a)(T_{i,a} + E_a) = 0 \qquad \text{if } i \triangleright a = a, \tag{2.10}$$

$$T_{i,i\triangleright a}T_{i,a} = E_a \qquad \text{if } i \triangleright a \neq a, \tag{2.11}$$

$$T_iT_j \cdots T_jT_{i,a} = T_jT_i \cdots T_iT_{j,a} \ (m_{i,j;a} \text{ factors}) \quad \text{if } m_{i,j;a} \text{ is finite and odd}, \tag{2.12}$$

$$T_jT_i \cdots T_jT_{i,a} = T_iT_j \cdots T_iT_{j,a} \ (m_{i,j;a} \text{ factors}) \quad \text{if } m_{i,j;a} \text{ is finite and even}, \tag{2.13}$$

where, in (2.12)-(2.13), we use the same convention as that of (1.5) with $s_{i,a}$ in place of $T_{i,a}$.

Lemma 2.2. *Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the Coxeter groupoid associated with an element $(R, N, A, \triangleright)$ of $\mathcal{R}$ (see Definition 1.4). Assume that A is finite. Then there exists a map $f : W \to H_q(W)$ such that*

$$f(0) = 0, \ f(e_a) = E_a, \tag{2.14}$$

$$f(s_{i,a}w) = T_{i,a}f(w) \qquad \textit{if } w \in W \setminus \{0\} \textit{ and } \ell(s_{i,a}w) = 1 + \ell(w). \tag{2.15}$$

Further, as a $\mathbb{C}$-linear space, $H_q(W)$ is spanned by $f(W \setminus \{0\})$. In particular, if W is finite, then

$$\dim H_q(W) \leq |W| - 1. \tag{2.16}$$

Proof. Let $\widetilde{W}$ be the semigroup introduced in Definition 1.6 for W. It is easy to show that there exists a unique semigroup homomorphism $\widetilde{f} : \widetilde{W} \to H_q(W)$ such that $\widetilde{f}(0) = 0$, $\widetilde{f}(\tilde{e}_a) = E_a$ and $\widetilde{f}(\tilde{s}_{i,a}) = T_{i,a}$. By Theorem 1.7, there exists a unique map $f : W \to H_q(W)$ such that $f(0) = 0$ and $f(w) = \widetilde{f}(\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m,a})$ if $w \in W \setminus \{0\}$, $\ell(w) = m$ and $w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m,a}$. Then f satisfies (2.14)-(2.15), as desired.

We show

$$\forall w \in W, \forall i \in N, \forall a \in A, T_{i,a} f(w) \in \mathbb{C} f(s_{i,a} w) + \mathbb{C} f(w). \tag{2.17}$$

If $s_{i,a} w = 0$, then clearly $T_{i,a} f(w) = 0$ holds. If $w \neq 0$, $s_{i,a} w \neq 0$ and $\ell(s_{i,a} w) = 1 + \ell(w)$, then (2.17) follows from (2.15). Assume that $w \neq 0$, $s_{i,a} w \neq 0$ and $\ell(s_{i,a} w) \neq 1 + \ell(w)$. Then by (1.8) and Proposition 1.9, we have $\ell(s_{i,a} w) = \ell(w) - 1$, so $f(w) = T_{i,i \triangleright a} f(s_{i,a} w)$. Since $T_{i,a} f(w) = T_{i,a} T_{i,i \triangleright a} f(s_{i,a} w)$, we have $T_{i,a} f(w) = f(s_{i,a} w)$ if $i \triangleright a \neq a$, and $T_{i,a} f(w) = (q-1) f(w) + q f(s_{i,a} w)$ otherwise. Hence we have (2.17), as desired.

It is clear from (2.17) that the rest of the statement follows. □

Notation. Let $r \in \mathbb{N}$. Let $V_0^{(r)}$ be the r-dimensional $\mathbb{R}$-linear space with a basis $\{\varepsilon_i | 1 \leq i \leq r\}$. Let $V_0^{(r),\prime}$ be the subspace of $V_0^{(r)}$ formed by the elements $\sum_{i=1}^r x_i \varepsilon_i$ with $x_i \in \mathbb{R}$ and $\sum_{i=1}^r x_i = 0$, so $\dim V_0^{(r),\prime} = r - 1$. For a non-zero element $x = \sum_{i=1}^{|N|} x_i \varepsilon_i$ of $V_0^{(r)}$ with $x_i \in \mathbb{R}$, define $\widetilde{\sigma}_x \in \mathrm{GL}(V_0^{(r)})$ by $\widetilde{\sigma}_x(\varepsilon_j) = \varepsilon_j - 2x_j (\sum_{i=1}^{|N|} x_i^2)^{-1} x$, that is, $\widetilde{\sigma}_x$ is the reflection of $V_0^{(r)}$ with respect to the hyperplane of $V_0^{(r)}$ orthogonal to x. Note that if $x \in V_0^{(r),\prime}$, then $\widetilde{\sigma}_x(V_0^{(r),\prime}) = V_0^{(r),\prime}$.

§ 2.4. Basic of Iwahori-Hecke algebras

For the basic facts about the Iwahori-Hecke algebras, we refer to [GU]. Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the groupoid introdued in (1.1). In this subsection we always assume that

$$|A| = 1 \textit{ and } N \textit{ and } W \textit{ are finite.} \tag{2.18}$$

Let $a \in A$, so $A = \{a\}$. Then $W \setminus \{0\}$ is nothing but the Coxeter group associated with the Coxeter system $(W \setminus \{0\}, \{s_{i,a}|, i \in N\})$. In this case, we also denote $H_q(W)$ and $T_{i,a}$ by $H_q(W \setminus \{0\})$ and T_i respectively. That is, $H_q(W \setminus \{0\})$ is the $\mathbb{C}$-algebra (with 1) generated by T_i ($i \in N$) and defined by the relations

$$(T_i - q)(T_i + 1) = 0, \tag{2.19}$$

$$\begin{aligned} T_i T_j \cdots T_j T_i &= T_j T_i \cdots T_i T_j \ (m_{i,j;a} \text{ factors}) && \text{if } m_{i,j;a} \text{ is odd,} \\ T_j T_i \cdots T_j T_i &= T_i T_j \cdots T_i T_j \ (m_{i,j;a} \text{ factors}) && \text{if } m_{i,j;a} \text{ is even.} \end{aligned} \tag{2.20}$$

It is well-known that $\dim H_q(W \setminus \{0\}) = |W \setminus \{0\}|$. In this paper we fix a complete set of non-equivalent irreducible representations of $H_q(W \setminus \{0\})$ by

$$\{\rho_{H_q(W\setminus\{0\}),\lambda} : H_q(W \setminus \{0\}) \to \mathrm{End}_{\mathbb{C}}(V_{H_q(W\setminus\{0\}),\lambda}) | \lambda \in \Lambda_{H_q(W\setminus\{0\})}\}, \tag{2.21}$$

where $\Lambda_{H_q(W\setminus\{0\})}$ is an index set. Define the polynomial $P_{W\setminus\{0\}}(q)$ in q by

$$P_{W\setminus\{0\}}(q) := \sum_{w\in W\setminus\{0\}} q^{\ell(w)}. \tag{2.22}$$

This is called the *Poincaré polynomial* of $W \setminus \{0\}$.

It is well-known [GU] (see also [CR, (25.22) and (27.4)]) that for $q \in \mathbb{C} \setminus \{0\}$, the following three conditions are equivalent.

(i) $P_{W\setminus\{0\}}(q) \neq 0$ holds.
(ii) $H_q(W \setminus \{0\})$ is a semisimle algebra.
(iii) The map

$$\bigoplus_{\lambda\in\Lambda_{H_q(W\setminus\{0\})}} \rho_{H_q(W\setminus\{0\}),\lambda} : H_q(W \setminus \{0\}) \to \bigoplus_{\lambda\in\Lambda_{H_q(W\setminus\{0\})}} \mathrm{End}_{\mathbb{C}}(V_{H_q(W\setminus\{0\}),\lambda}) \tag{2.23}$$

defined by $X \mapsto \oplus_{\lambda\in\Lambda_{H_q(W\setminus\{0\})}} \rho_{H_q(W\setminus\{0\}),\lambda}(X)$ is a $\mathbb{C}$-algebra isomorphism.

In particular,

$$q \cdot P_{W\setminus\{0\}}(q) \neq 0 \Longrightarrow \dim H_q(W \setminus \{0\}) = \sum_{\lambda\in\Lambda_{H_q(W\setminus\{0\})}} (\dim V_{H_q(W\setminus\{0\}),\lambda})^2. \tag{2.24}$$

Assume that $N = \{1, 2, \ldots, n\}$ and $m_{i,i+1;a} = 3$ and $m_{i,j;a} = 2$ ($|j - i| \geq 2$). Then W is the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $V_0 = V_0^{(n+1),\prime}$, $R_a^+ = \{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n + 1\}$, $\alpha_{i,a} = \varepsilon_i - \varepsilon_{i+1}$ and $\sigma_{i,a} = \widetilde{\sigma}_{\alpha_{i,a}}$. As a group, $W \setminus \{0\}$ is isomorphic to the symmetric group S_{n+1}, so we also denote $W \setminus \{0\}$ by S_{n+1} by abuse of notation. Note that $\dim H_q(S_{n+1}) = (n + 1)!$.

Assume that $N = \{1, 2, \ldots, n\}$ and $m_{i,i+1;a} = 3$ ($1 \leq i \leq n - 3$), $m_{n-1,n;a} = 4$ and $m_{i,j;a} = 2$ ($|j-i| \geq 2$). Then W is the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $V_0 = V_0^{(n)}$, $R_a^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq n\} \cup \{\varepsilon_i | 1 \leq i \leq n\}$, $\alpha_{i,a} = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq n - 1$), $\alpha_{n,a} = \varepsilon_n$ and $\sigma_{i,a} = \widetilde{\sigma}_{\alpha_{i,a}}$. We also denote $W \setminus \{0\}$ by $W(B_n)$ and $W(C_n)$. Note that $\dim H_q(W(B_n)) = 2^n n!$.

Assume that $N = \{1, 2, \ldots, n\}$ and $m_{i,i+1;a} = 3$ ($1 \leq i \leq n - 2$), $m_{n-1,n;a} = 2$, $m_{n-2,n;a} = 3$ and $m_{i,j;a} = 2$ ($|j - i| \geq 2$ and $1 \leq i \leq n - 3$). We also denote $W \setminus \{0\}$ by $W(D_n)$. Note that $\dim H_q(W(D_n)) = 2^{n-1} n!$. Then W is the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $V_0 = V_0^{(n)}$, $R_a^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq n\}$, $\alpha_{i,a} = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq n - 1$), $\alpha_{n,a} = \varepsilon_{n-1} + \varepsilon_n$ and $\sigma_{i,a} = \widetilde{\sigma}_{\alpha_{i,a}}$.

It is well-known (cf. [C, Theoerem 10.2.3 and Proposition 10.2.5]) that

$$P_{S_{n+1}}(q) = \prod_{r=1}^{n} \frac{q^{r+1}-1}{q-1}, \tag{2.25}$$

$$P_{W(B_n)}(q) = \prod_{r=1}^{n} \frac{q^{2r}-1}{q-1}, \tag{2.26}$$

$$P_{W(D_n)}(q) = \frac{q^n-1}{q-1} \prod_{r=1}^{n-1} \frac{q^{2r}-1}{q-1}. \tag{2.27}$$

§ 2.5. Iwahori-Hecke type algebra $H_q(A(m,n))$ associated with the Lie superalgebra $A(m,n)$

Let

$$\triangleright : S_{m+n+2} \times \mathcal{D}_{m+1|n+1} \to \mathcal{D}_{m+1|n+1}$$

denote the usual (left) action of the symmetric group S_{m+n+2} on $\mathcal{D}_{m+1|n+1}$ by permutations, that is, for $\sigma \in S_{m+n+2}$,

$$\sigma \triangleright (p_1, \dots, p_{m+n+2}) = (p_{\sigma^{-1}(1)}, \dots, p_{\sigma^{-1}(m+n+2)}).$$

Let $\sigma_i := (i, i+1) \in S_{m+n+2}$. Let W be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $N = \{1, 2, \dots, m+n+1\}$, $A = \mathcal{D}_{m+1|n+1}$, $i \triangleright d = \sigma_i \triangleright d$, $V_0 = V_0^{(m+n+2),\prime}$, $R_d^+ = \{\varepsilon_i - \varepsilon_j | 1 \le i < j \le m+n+1\}$, $\alpha_{i,d} = \varepsilon_i - \varepsilon_{i+1}$ and $\sigma_{i,d} = \widetilde{\sigma}_{\alpha_{i,d}}$. Denote $H_q(W)$ by $H_q(A(m,n))$. Then $H_q(A(m,n))$ is the $\mathbb{C}$-algebra (with 1) generated by

$$\{E_d \,|\, d \in \mathcal{D}_{m+1|n+1}\} \cup \{T_{i,d} \,|\, 1 \le i \le m+n+1,\, d \in \mathcal{D}_{m+1|n+1}\} \tag{2.28}$$

and defined by the relations (2.6)-(2.11) and the relations

$$T_{i,\sigma_j\sigma_i\triangleright d} T_{j,\sigma_i\triangleright d} T_{i,d} = T_{j,\sigma_i\sigma_j\triangleright d} T_{i,\sigma_j\triangleright d} T_{j,d} \qquad \text{if } |i-j| = 1, \tag{2.29}$$

$$T_{i,\sigma_j\triangleright d} T_{j,d} = T_{j,\sigma_i\triangleright d} T_{i,d} \qquad \text{if } |i-j| \ge 2. \tag{2.30}$$

Define d_e, $d_o \in \mathcal{D}_{m+1|n+1}$ by

$$d_e := (\overbrace{0, \dots, 0}^{m+1}, \overbrace{1, \dots, 1}^{n+1}), \;\; d_o := (\overbrace{1, \dots, 1}^{n+1}, \overbrace{0, \dots, 0}^{m+1}). \tag{2.31}$$

For $d = (p_1, \dots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$, define the two elements

$$\tau_{+.d}, \tau_{-.d} \in S_{m+n+2} \tag{2.32}$$

by

$$p_{\tau_{\pm.d}(i)} = \frac{1 \mp 1}{2} \quad \text{and} \quad \tau_{\pm.d}(i) \le \tau_{\pm.d}(j) \qquad \text{if } 1 \le i \le j \le m+1, \tag{2.33}$$

$$p_{\tau_{\pm.d}(i)} = \frac{1 \pm 1}{2} \quad \text{and} \quad \tau_{\pm.d}(i) \le \tau_{\pm.d}(j) \qquad \text{if } m+2 \le i \le j \le m+n+2. \tag{2.34}$$

Then $\tau_{+,d}$ (resp. $\tau_{-,d}$) is the minimal length element among the elements $\sigma \in S_{m+n+2}$ satisfying the condition that for any i, i-th component of d_e (resp. d_o) is the same as $\sigma(i)$-th component $p_{\sigma(i)}$ of d.

Example 2.3. Assume that $m = n = 1$. Then $\mathcal{D}_{2|2} = \{d_e = (0,0,1,1), d_1 = (0,1,0,1), d_2 = (1,0,0,1), d_3 = (0,1,1,0), d_4 = (1,0,1,0), d_o = (1,1,0,0)\}$. Then $\tau_{+.d_e} = \left[\begin{smallmatrix}1234\\1234\end{smallmatrix}\right]$, $\tau_{-.d_e} = \left[\begin{smallmatrix}1234\\3412\end{smallmatrix}\right]$, $\tau_{+.d_1} = \left[\begin{smallmatrix}1234\\1324\end{smallmatrix}\right]$, $\tau_{-.d_1} = \left[\begin{smallmatrix}1234\\2413\end{smallmatrix}\right]$, $\tau_{+.d_2} = \left[\begin{smallmatrix}1234\\2314\end{smallmatrix}\right]$, $\tau_{-.d_2} = \left[\begin{smallmatrix}1234\\1423\end{smallmatrix}\right]$, $\tau_{+.d_3} = \left[\begin{smallmatrix}1234\\1423\end{smallmatrix}\right]$, $\tau_{-.d_3} = \left[\begin{smallmatrix}1234\\2314\end{smallmatrix}\right]$, $\tau_{+.d_4} = \left[\begin{smallmatrix}1234\\2413\end{smallmatrix}\right]$, $\tau_{-.d_4} = \left[\begin{smallmatrix}1234\\1324\end{smallmatrix}\right]$, $\tau_{+.d_o} = \left[\begin{smallmatrix}1234\\3412\end{smallmatrix}\right]$, $\tau_{-.d_o} = \left[\begin{smallmatrix}1234\\1234\end{smallmatrix}\right]$. See also Figure 2.

Now we consider $|W|$. Recall ρ and d_e from Theorem 1.5 and (2.32) respectively. It is easy to see that $P_{d_e}\rho(e_{d_e}We_{d_e})\iota_{d_e} \subset \{(\sum_{i=1}^{m+n+2}\mathbf{E}_{\sigma(i)i})_{|V_0^{(m+n+2),\prime}}|\sigma \in S_{m+n+2}, \sigma(\{1,\ldots,m+1\}) = \{1,\ldots,m+1\}\}$. Hence $|e_{d_e}We_{d_e}| \le (m+1)!(n+1)!$ by Theorem 1.5, so $|W \setminus \{0\}| = |\mathcal{D}_{m+1|n+1}|^2|e_{d_e}We_{d_e}| \le \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}$. Hence by (2.16), we conclude

$$\dim H_q(A(m,n)) \le \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}. \tag{2.35}$$

Proposition 2.4. *Let V and W be finite dimensional $\mathbb{C}$-linear spaces, and let $\mathbf{l} : H_q(S_{m+1}) \to \mathrm{End}_{\mathbb{C}}(V)$ and $\mathbf{r} : H_q(S_{n+1}) \to \mathrm{End}_{\mathbb{C}}(W)$ be $\mathbb{C}$-algebra homomorphisms, i.e., representations. Let $\mathbf{l} \otimes \mathbf{r} : H_q(S_{m+1}) \otimes H_q(S_{n+1}) \to \mathrm{End}_{\mathbb{C}}(V \otimes W)$ denote the tensor representation of $\mathbf{l}$ and $\mathbf{r}$ in the ordinary sense. Let $C_{V\otimes W;d}$ be copies of the $\mathbb{C}$-linear space $V \otimes W$, indexed by $d \in \mathcal{D}_{m+1|n+1}$. Let $C_{V\otimes W} := \oplus_{d\in\mathcal{D}_{m+1|n+1}} C_{V\otimes W;d}$. Let $P_d : C_{V\otimes W} \to C_{V\otimes W;d}$ and $\iota_d : C_{V\otimes W;d} \to C_{V\otimes W}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unigue $\mathbb{C}$-algebra homomorphism $\mathbf{l} \boxtimes^{A(m,n)} \mathbf{r} : H_q(A(m,n)) \to \mathrm{End}_{\mathbb{C}}(C_{V\otimes W})$ satisfying the following conditions:*

(i) *For each $d \in \mathcal{D}_{m+1|n+1}$, one has $(\mathbf{l} \boxtimes^{A(m,n)} \mathbf{r})(E_d) = \iota_d \circ P_d$,*

(ii) *For each $i \in \{1,\ldots,m+n+1\}$ and each $d = (p_1,\ldots,p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$, one has*

$$(\mathbf{l} \boxtimes^{A(m,n)} \mathbf{r})(T_{i,d}) = \begin{cases} P_{\sigma_i \triangleright d} \circ \iota_d & \text{if } p_i \ne p_{i+1}, \\ \iota_d \circ (\mathbf{l}(T_{\tau_{+.d}^{-1}(i)}) \otimes \mathrm{id}_W) \circ P_d & \text{if } p_i = p_{i+1} = 0, \\ \iota_d \circ (\mathrm{id}_V \otimes \mathbf{r}(T_{\tau_{-.d}^{-1}(i)})) \circ P_d & \text{if } p_i = p_{i+1} = 1. \end{cases} \tag{2.36}$$

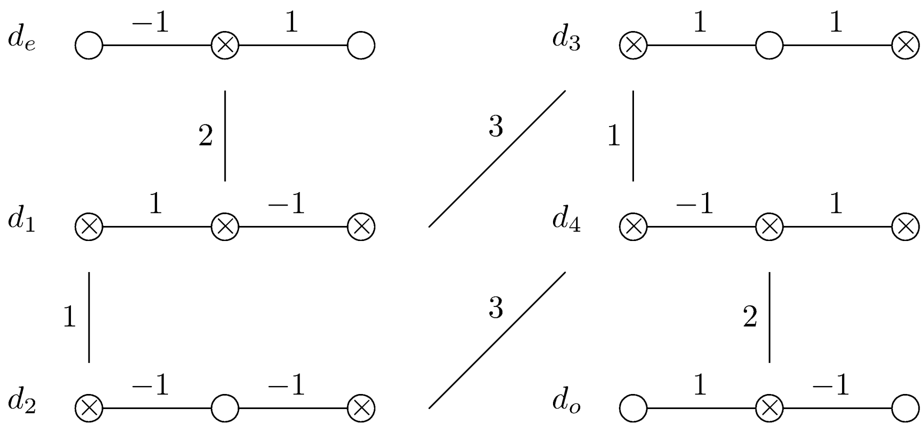


Figure 2. Dynkin diagrams of the Lie superalgebra $A(1,1)$

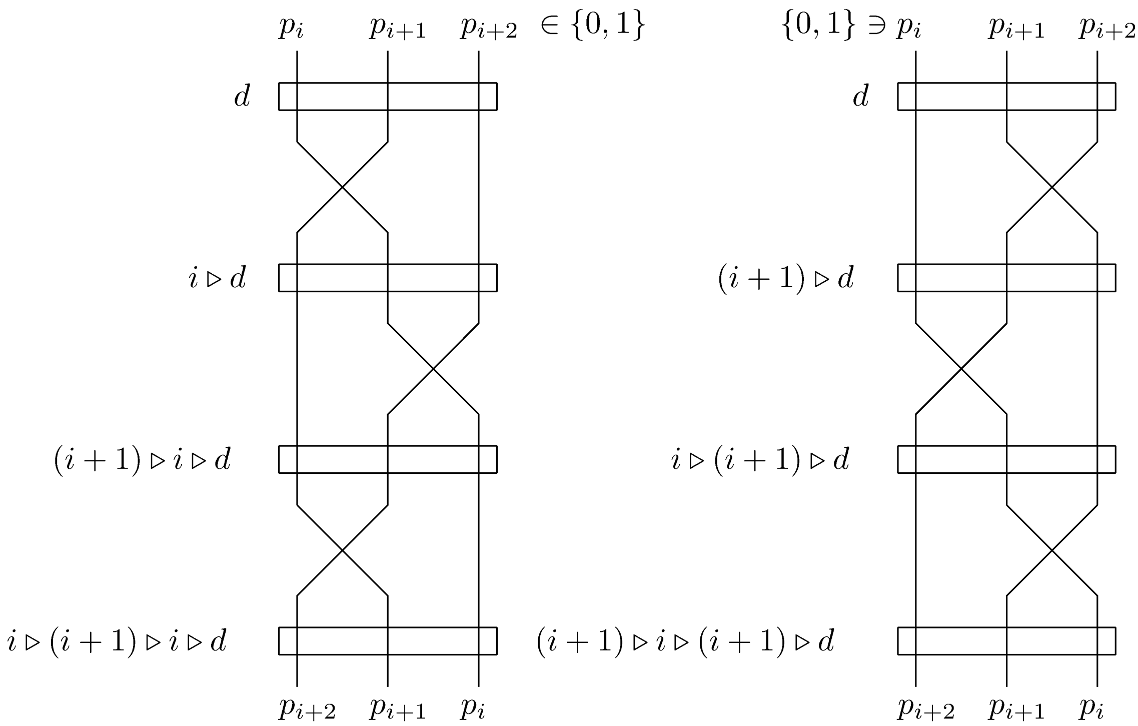


Figure 3. Braid relation

Proof. This can be checked directly. Refer to Figure 3. We explain by using an example. Denote $(\mathbf{l}\boxtimes^{A(m,n)}\mathbf{r})(T_{i',d'})$ by $S_{i',d'}$ for any d' and i'. Let $d=(p_1,\dots,p_{m+n+2})\in\mathcal{D}_{m+1|n+1}$ and $i\in\{1,\dots,m+n\}$ and assume $p_i=p_{i+1}=0$ and $p_{i+2}=1$. Let $d_1:=i\triangleright d(=\sigma_i\triangleright d)$, $d_2:=(i+1)\triangleright d_1$, $d_3:=i\triangleright d_2$, $d_4:=(i+1)\triangleright d$, $d_5:=i\triangleright d_4$ and $d_6:=(i+1)\triangleright d_4$. Then

$$d=d_1=(p_1,\dots,p_{i-1},0,0,1,p_{i+2},\dots,p_{m+n+2}), \tag{2.37}$$

$$d_2=d_4=(p_1,\dots,p_{i-1},0,1,0,p_{i+2},\dots,p_{m+n+2}), \tag{2.38}$$

$$d_3=d_5=d_6=(p_1,\dots,p_{i-1},1,0,0,p_{i+2},\dots,p_{m+n+2}). \tag{2.39}$$

Note that $\tau_{+.d_5}=\sigma_i\sigma_{i+1}\tau_{+.d}$. Hence $\tau^{-1}_{+.d_5}(i+1)=\tau^{-1}_{+.d}(i)$. Then we have $S_{i,d}=\iota_d\circ(\mathbf{l}(T_{\tau^{-1}_{+.d}(i)})\otimes\mathrm{id}_W)\circ P_d$, $S_{i+1,d_1}=\iota_{d_2}\circ P_d$, $S_{i,d_2}=\iota_{d_3}\circ P_{d_2}$, $S_{i+1,d}=\iota_{d_2}\circ P_d$, $S_{i,d_4}=\iota_{d_3}\circ P_{d_2}$ and $S_{i+1,d_5}=\iota_{d_3}\circ(\mathbf{l}(T_{\tau^{-1}_{+.d}(i)})\otimes\mathrm{id}_W)\circ P_{d_3}$. Hence we have $S_{i,d_2}S_{i+1,d_1}S_{i,d}=S_{i+1,d_5}S_{i,d_4}S_{i+1,d}=\iota_{d_3}\circ(\mathbf{l}(T_{\tau^{-1}_{+.d}(i)})\otimes\mathrm{id}_W)\circ P_d$, as desired. $\square$

For $\lambda\in\Lambda_{H_q(S_{m+1})}$ and $\mu\in\Lambda_{H_q(S_{n+1})}$, we denote $\rho_{H_q(S_{m+1}),\lambda}\boxtimes^{A(m,n)}\rho_{H_q(S_{n+1}),\mu}$ by $\rho^{A(m,n)}_{q;\lambda,\mu}$ and we denote $C_{V\otimes W}$, P_d, ι_d for $V=V_{H_q(S_{m+1}),\lambda}$ and $W=V_{H_q(S_{n+1}),\mu}$ by $C^{A(m,n)}_{q;\lambda,\mu}$, $P^{\lambda,\mu}_d$, $\iota^{\lambda,\mu}_d$ respectively.

Theorem 2.5. *Let $q\in\mathbb{C}$ and assume that*

$$qP_{S_{m+1}}(q)P_{S_{n+1}}(q)\neq 0. \tag{2.40}$$

Then the $\mathbb{C}$-algebra homomorphism

$$\bigoplus_{(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}}\rho^{A(m,n)}_{q;\lambda,\mu}:\ H_q(A(m,n))\to\bigoplus_{(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}}\mathrm{End}_{\mathbb{C}}(C^{A(m,n)}_{q;\lambda,\mu}) \tag{2.41}$$

is an isomorphism. Further we have

$$\dim H_q(A(m,n))=\frac{((m+n+2)!)^2}{(m+1)!(n+1)!}. \tag{2.42}$$

Moreover $H_q(A(m,n))$ is a semisimple $\mathbb{C}$-algebra and a complete set of non-equivalent irreducible representations of $H_q(A(m,n))$ is given by $\{\rho^{A(m,n)}_{q;\lambda,\mu}|(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}\}$.

Proof. Define the $\mathbb{C}$-algebra homomorphism

$$f_1:H_q(S_{m+1})\otimes H_q(S_{n+1})\to H_q(A(m,n))$$

by $f_1(T_i \otimes 1) = T_{i,e}$ and $f_1(1 \otimes T_j) = T_{m+1+j,e}$. Let

$$R_{\lambda,\mu} := (\iota_{d_e}^{\lambda,\mu} \circ P_{d_e}^{\lambda,\mu})\mathrm{End}_{\mathbb{C}}(C_{q;\lambda,\mu}^{A(m,n)})(\iota_{d_e}^{\lambda,\mu} \circ P_{d_e}^{\lambda,\mu}).$$

Let f_2 denote the homomorphism of (2.41). It follows from (2.40) that $H_q(S_{m+1}) \otimes H_q(S_{n+1})$ is a semisimple $\mathbb{C}$-algebra. This implies

$$\mathrm{Im}(f_2 \circ f_1) = \bigoplus_{(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}} R_{\lambda,\mu}.$$

On the other hand, we have

$$\mathrm{End}_{\mathbb{C}}(C_{q;\lambda,\mu}^{A(m,n)}) = \bigoplus_{d_1,d_2\in\mathcal{D}_{m+1|n+1}} (\iota_{d_1}^{\lambda,\mu} \circ P_{d_e}^{\lambda,\mu})R_{\lambda,\mu}(\iota_{d_e}^{\lambda,\mu} \circ P_{d_2}^{\lambda,\mu}).$$

Hence by (2.36) we can easily see that f_2 is surjective. In particular, we have

$$\begin{aligned}
&\dim H_q(A(m,n)) \\
&\geq \sum_{(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}} |\mathcal{D}_{m+1|n+1}|^2 \dim R_{\lambda,\mu} \\
&= |\mathcal{D}_{m+1|n+1}|^2 \sum_{(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}} \dim R_{\lambda,\mu} \\
&= (\frac{(m+n+2)!}{(m+1)!(n+1)!})^2 \\
&\qquad \sum_{(\lambda,\mu)\in\Lambda_{H_q(S_{m+1})}\times\Lambda_{H_q(S_{n+1})}} (\dim V_{H_q(S_{m+1}),\lambda})^2(\dim V_{H_q(S_{n+1}),\mu})^2 \\
&= (\frac{(m+n+2)!}{(m+1)!(n+1)!})^2 \dim H_q(S_{m+1}) \dim H_q(S_{n+1}) \\
&= \frac{((m+n+2)!)^2}{(m+1)!(n+1)!}.
\end{aligned}$$

Hence by (2.35), we have (2.42). Hence f_2 is an isomorphism. Then the rest of the statement follows from well-known facts concerning semisimple algebras (cf. [CR, (25.22) and (27.4)]). □

§ 2.6. Iwahori-Hecke type algebra associated with the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$

Let $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Ler $\ell := m+n$. For $1 \leq i \leq \ell$, define $\hat{\sigma}_i \in S_\ell$ by $\hat{\sigma}_i := \sigma_i$ $(1 \leq i \leq \ell-1)$. and $\hat{\sigma}_\ell := \mathrm{id}$.

Let W be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $N = \{1, 2, \ldots, \ell\}$, $A = \mathcal{D}_{m|n}$, $i \triangleright d = \hat{\sigma}_i \triangleright d$, $V_0 = V_0^{(m+n)}$, $R_d^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq$

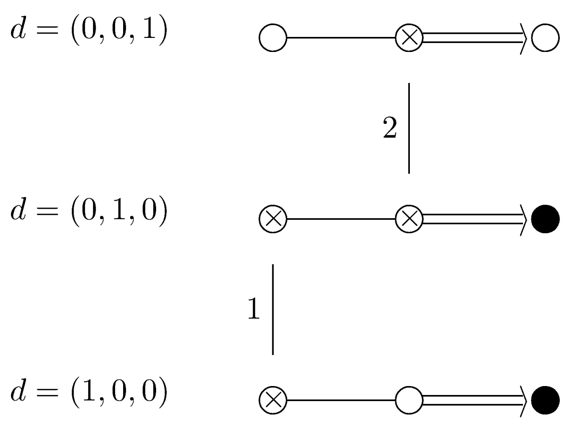


Figure 4. Dynkin diagrams of the Lie superalgebra $B(1,2)$

$\ell\} \cup \{\varepsilon_i | 1 \leq i \leq \ell\}$, $\alpha_{i,d} = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq \ell-1)$, $\alpha_{\ell,d} = \varepsilon_\ell$ and $\sigma_{i,d} = \widetilde{\sigma}_{\alpha_{i,d}}$. Denote $H_q(W)$ by $H_q(B(m,n))$. Then $H_q(B(m,n))$ is the $\mathbb{C}$-algebra (with 1) generated by

$$\{E_d \,|\, d \in \mathcal{D}_{m|n}\} \cup \{T_{i,d} \,|\, 1 \leq i \leq \ell,\, d \in \mathcal{D}_{m|n}\} \tag{2.43}$$

and defined by the relations (2.6)-(2.11) and the relations

(2.44)
$$T_{\ell-1,\hat{\sigma}_{\ell-1} \triangleright d} T_{\ell,\hat{\sigma}_{\ell-1} \triangleright d} T_{\ell-1,d} T_{\ell,d} = T_{\ell,d} T_{\ell-1,\hat{\sigma}_{\ell-1} \triangleright d} T_{\ell,\hat{\sigma}_{\ell-1} \triangleright d} T_{\ell-1,d}$$

$$T_{i,\hat{\sigma}_{i+1}\hat{\sigma}_i \triangleright d} T_{i+1,\hat{\sigma}_i \triangleright d} T_{i,d} = T_{i+1,\hat{\sigma}_i\hat{\sigma}_{i+1} \triangleright d} T_{i,\hat{\sigma}_{i+1} \triangleright d} T_{i+1,d} \quad \text{if } 1 \leq i \leq \ell-1, \tag{2.45}$$

$$T_{i,\hat{\sigma}_j \triangleright d} T_{j,d} = T_{j,\hat{\sigma}_i \triangleright d} T_{i,d} \quad \text{if } |i-j| \geq 2. \tag{2.46}$$

Recall ρ and $d_e \in \mathcal{D}_{m|n}$ from Theorem 1.5 and (2.31) respectively. Then

$$\begin{aligned} &P_{d_e} \rho(e_{d_e} W e_{d_e}) \iota_{d_e} \\ &\subset \{(\textstyle\sum_{i=1}^{m+n} z_i \mathbf{E}_{\sigma(i)i}) | \sigma \in S_{m+n}, z_i \in \{-1,1\}, \sigma(\{1,\ldots,m\}) = \{1,\ldots,m\}\}. \end{aligned}$$

Hence $|e_{d_e} W e_{d_e}| \leq 2^{m+n} m!n!$ by Theorem 1.5, so $|W \setminus \{0\}| = |\mathcal{D}_{m|n}|^2 |e_{d_e} W e_{d_e}| \leq \frac{2^{m+n}((m+n)!)^2}{m!n!}$. Hence by (2.16), we conclude

$$\dim H_q(B(m,n)) \leq \frac{2^{m+n}((m+n)!)^2}{m!n!}. \tag{2.47}$$

Proposition 2.6. *Let $V_{\mathbf{l}}$ and $V_{\mathbf{r}}$ be finite dimensional $\mathbb{C}$-linear spaces, and let $\mathbf{l} : H_q(W(B_m)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{l}})$ and $\mathbf{r} : H_q(W(B_n)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{r}})$ be $\mathbb{C}$-algebra homomorphisms, i.e., representations. Let $\mathbf{l} \otimes \mathbf{r} : H_q(W(B_m)) \otimes H_q(W(B_n)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{l}} \otimes V_{\mathbf{r}})$ denote the tensor representation of $\mathbf{l}$ and $\mathbf{r}$ in the ordinary sense. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};d}$ be copies* *of the $\mathbb{C}$-linear space $V_{\mathbf{l}} \otimes V_{\mathbf{r}}$, indexed by $d \in \mathcal{D}_{m|n}$. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}} := \oplus_{d \in \mathcal{D}_{m|n}} C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};d}$. Let $P_d : C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}} \to C_{V \otimes W;d}$ and $\iota_d : C_{V \otimes W;d} \to C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique $\mathbb{C}$-algebra homomorphism $\mathbf{l} \boxtimes \mathbf{r} = \mathbf{l} \boxtimes^{B(m,n)} \mathbf{r} : H_q(B(m,n)) \to \mathrm{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}})$ satisfying the following conditions:*

(i) *For each $d \in \mathcal{D}_{m|n}$, one has $(\mathbf{l} \boxtimes \mathbf{r})(E_d) = \iota_d \circ P_d$,*
(ii) *For each $i \in \{1, \ldots, \ell = m+n\}$ and each $d = (p_1, \ldots, p_\ell) \in \mathcal{D}_{m|n}$, one has*

(2.48)
$$(\mathbf{l} \boxtimes \mathbf{r})(T_{i,d}) = \begin{cases} P_{\hat{\sigma}_i \triangleright d} \circ \iota_d & \text{if } 1 \le i \le \ell - 1 \text{ and } p_i \neq p_{i+1}, \\ \iota_d \circ (\mathbf{l}(T_{\tau^{-1}_{+.d}(i)}) \otimes \mathrm{id}_{V_{\mathbf{r}}}) \circ P_d & \text{if } 1 \le i \le \ell - 1 \text{ and } p_i = p_{i+1} = 0, \\ \iota_d \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{\tau^{-1}_{-.d}(i)})) \circ P_d & \text{if } 1 \le i \le \ell - 1 \text{ and } p_i = p_{i+1} = 1, \\ \iota_d \circ (\mathbf{l}(T_m) \otimes \mathrm{id}_{V_{\mathbf{r}}}) \circ P_d & \text{if } i = \ell \text{ and } p_\ell = 0, \\ \iota_d \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_n)) \circ P_d & \text{if } i = \ell \text{ and } p_\ell = 1, \end{cases}$$

where $\tau_{\pm.d}$ are the ones of (2.32).

Proof. We can check out this directly in a way similar to that for Proof of Proposition 2.4. □

For $\lambda \in \Lambda_{H_q(W(B_m))}$ and $\mu \in \Lambda_{H_q(W(B_n))}$, we denote $\rho_{H_q(W(B_m)),\lambda} \boxtimes^{B(m,n)} \rho_{W(B_n)),\mu}$ by $\rho^{B(m,n)}_{q;\lambda,\mu}$ and we denote $C_{V \otimes W}$ for $V = V_{H_q(W(B_m)),\lambda}$ and $W = V_{H_q(W(B_n)),\mu}$ by $C^{B(m,n)}_{q;\lambda,\mu}$.

Theorem 2.7. *Let $q \in \mathbb{C}$ and assume that*

$$qP_{W(B_m)}(q)P_{W(B_n)}(q) \neq 0. \tag{2.49}$$

Then the $\mathbb{C}$-algebra homomorphism

$$\bigoplus_{(\lambda,\mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}} \rho^{B(m,n)}_{q;\lambda,\mu} : \tag{2.50}$$
$$H_q(B(m,n)) \to \bigoplus_{(\lambda,\mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}} \mathrm{End}_{\mathbb{C}}(C^{B(m,n)}_{q;\lambda,\mu})$$

is an isomorphism. Further we have

$$\dim H_q(B(m,n)) = \frac{2^{m+n}((m+n)!)^2}{m!n!}. \tag{2.51}$$

Moreover $H_q(B(m,n))$ is a semisimple $\mathbb{C}$-algebra and a complete set of non-equivalent irreducible representations of $H_q(B(m,n))$ is given by $\{\rho^{B(m,n)}_{q;\lambda,\mu} | (\lambda,\mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}\}$.

Proof. Let $\mathbf{l} : H_q(W(B_m)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{l}})$ and $\mathbf{r} : H_q(W(B_n)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{r}})$ be irreducible representations. Further, let $\mathbf{l} \boxtimes \mathbf{r} : H_q(B(m,n)) \to \mathrm{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}})$ be the representaion introduced in Proposition 2.6 for these $\mathbf{l}$ and $\mathbf{r}$. By (2.48), we can easily see that

$$\forall d', \forall d'' \in \mathcal{D}_{m|n}, \quad P_{d'} \circ \iota_{d''} \in \mathrm{Im}(\mathbf{l} \boxtimes \mathbf{r}). \tag{2.52}$$

Define the representation $f_1 : H_q(W(B_m)) \otimes H_q(W(B_n)) \to \mathrm{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};d_e})$ by $f_1(T_i \otimes 1) = (P_{d_e} \circ \iota_{d_o})((\mathbf{l} \boxtimes \mathbf{r})(T_{n+i,d_o}))(P_{d_o} \circ \iota_{d_e})$ and $f_1(1 \otimes T_j) = (\mathbf{l} \boxtimes \mathbf{r})(T_{m+j,d_e})$. The condition (2.49) implies that f_1 is an irreducible representaion of $H_q(W(B_m)) \otimes H_q(W(B_n))$. Moreover, using (2.52), we can easily see that $\mathbf{l} \boxtimes \mathbf{r}$ is an irreducible representation of $H_q(B(m,n))$.

By the above argument, together with (2.47), in the same way as that for Proof of Theorem 2.5, we can complete the proof of this theorem. $\square$

§ 2.7. Iwahori-Hecke type algebra associated with the Lie superalgebra $\mathfrak{osp}(2m|2n)$

Let m, $n \in \mathbb{N}$. Define the set $\mathcal{D}^{CD}_{m|n}$ by

$$\begin{aligned} \mathcal{D}^{CD}_{m|n} :=& \{d^D | d = (p_1, \ldots, p_{m+n}) \in \mathcal{D}_{m|n},\, p_{m+n} = 0\} \\ & \cup \{d^C_+, d^C_- | d = (p_1, \ldots, p_{m+n}) \in \mathcal{D}_{m|n},\, p_{m+n} = 1\}, \end{aligned} \tag{2.53}$$

so that

$$|\mathcal{D}^{CD}_{m|n}| = \frac{(m+n-1)!}{(m-1)!n!} + 2\frac{(m+n-1)!}{m!(n-1)!} = \frac{(m+n-1)!(m+2n)}{m!n!}. \tag{2.54}$$

Let $\ell := m+n$ and $N = \{1, \ldots, \ell\}$. Define the action $\triangleright$ of $F_2(N)$ on $\mathcal{D}^{CD}_{m|n}$ by

$$i \triangleright a = \begin{cases} (\sigma_i \triangleright d)^D & \text{if } a = d^D,\ 1 \le i \le \ell-2 \text{ and } p_i \ne p_{i+1}, \\ (\sigma_i \triangleright d)^C_+ & \text{if } a = d^D,\ i = \ell-1 \text{ and } p_i \ne p_{i+1}, \\ (\sigma_{i-1} \triangleright d)^C_- & \text{if } a = d^D,\ i = \ell \text{ and } p_{i-1} \ne p_i, \\ (\sigma_i \triangleright d)^C_\pm & \text{if } a = d^C_\pm,\ 1 \le i \le \ell-2 \text{ and } p_i \ne p_{i+1}, \\ (\sigma_i \triangleright d)^D & \text{if } a = d^C_+,\ i = \ell-1 \text{ and } p_i \ne p_{i+1}, \\ (\sigma_{i-1} \triangleright d)^D & \text{if } a = d^C_-,\ i = \ell \text{ and } p_{i-1} \ne p_i, \\ a & \text{otherwise.} \end{cases} \tag{2.55}$$

Now we define $R = (R, N, \mathcal{D}^{CD}_{m|n}, \triangleright) \in \mathcal{R}$ as follows. Let N be as above. Let $A = \mathcal{D}^{CD}_{m|n}$. Let $V_0 = V_0^{(\ell)}$. Let $a = d^D$, d^C_+ or $d^C_- \in \mathcal{D}^{CD}_{m|n}$ with $d = (p_1, \ldots, p_{m+n}) \in \mathcal{D}_{m|n}$. Let

R_a^+ be the subset of V_0 formed by the elements $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, $(1 \le i < j \le \ell)$ and $(\pm 1)^{\delta_{k\ell}} 2\varepsilon_k$ ($1 \le k \le \ell$ and $p_k = 1$, and $\pm$ is the one of $d^C_\pm$). Define

$$\alpha_{i,a} := \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } a = d^D \text{ or } d^C_+ \text{ and } 1 \le i \le \ell - 1, \\ \varepsilon_i - \varepsilon_{i+1} & \text{if } a = d^C_- \text{ and } 1 \le i \le \ell - 2, \\ \varepsilon_{\ell-1} + \varepsilon_\ell & \text{if } a = d^D \text{ and } i = \ell, \\ 2\varepsilon_\ell & \text{if } a = d^C_+ \text{ and } i = \ell, \\ -2\varepsilon_\ell & \text{if } a = d^C_- \text{ and } i = \ell - 1, \\ \varepsilon_{\ell-1} + \varepsilon_\ell & \text{if } a = d^C_- \text{ and } i = \ell. \end{cases} \tag{2.56}$$

Define $\sigma_{i.a} := \widetilde{\sigma}_{\alpha_{i.a}}$. Let W be the Coxeter groupoid associated with R. Recall ρ and $d_e \in \mathcal{D}_{m|n}$ from Theorem 1.5 and (2.31) respectively. It is easy to show that

$$\rho(e_{(d_e)^D} W e_{(d_e)^D}) =$$
$$\{\sum_{j=1}^{\ell} z_j \mathbf{E}_{\sigma(j)j} | \sigma \in S_\ell, z_j \in \{-1,1\}, \prod_{j=n+1}^{\ell} z_j = 1, \sigma(\{1,\ldots,n\}) = \{1,\ldots,n\}\},$$

so $|e_{(d_e)^D} W e_{(d_e)^D}| \le m!n!2^{\ell-1}$ by Theorem 1.5. Hence $|W \setminus \{0\}| \le |\mathcal{D}^{CD}_{m|n}|^2 m!n!2^{\ell-1}$. Denote $H_q(W)$ by $H_q(\mathfrak{osp}(2m|2n))$. By (2.16) and (2.54), we have

$$\dim H_q(\mathfrak{osp}(2m|2n)) \le \frac{2^{m+n-1}((m+n-1)!(m+2n))^2}{m!n!}. \tag{2.57}$$

Recall that $H_q(\mathfrak{osp}(2m|2n))$ is the $\mathbb{C}$-algebra (with 1) generated by

$$\{E_a \,|\, a \in \mathcal{D}^{CD}_{m|n}\} \cup \{T_{i,a} \,|\, 1 \le i \le m+n, \, a \in \mathcal{D}^{CD}_{m|n}\} \tag{2.58}$$

and defined by the relations (2.6)-(2.11) and the relations

$$(T_{i,a}T_{j,a})^2 = (T_{j,a}T_{i,a})^2 \text{ if } a = d^C_\pm,\ p_{\ell-1} = p_\ell \text{ and } i = \ell-1,\ j = \ell, \tag{2.59}$$

$$T_{i,a}T_{j,a} = T_{j,a}T_{i,a} \text{ if } a = d^D,\ p_{\ell-1} = p_\ell \text{ and } i = \ell-1,\ j = \ell, \tag{2.60}$$

$$T_{i,ji\triangleright d}T_{j,i\triangleright d}T_{i,d} = T_{j,ij\triangleright d}T_{i,j\triangleright d}T_{j,d} \text{ if } p_{\ell-1} \ne p_\ell \text{ and } i = \ell-1,\ j = \ell, \tag{2.61}$$

$$T_{i,ji\triangleright d}T_{j,i\triangleright d}T_{i,d} = T_{j,ij\triangleright d}T_{i,j\triangleright d}T_{j,d} \quad \text{if } 1 \le i \le \ell-3,\ j = i+1, \tag{2.62}$$

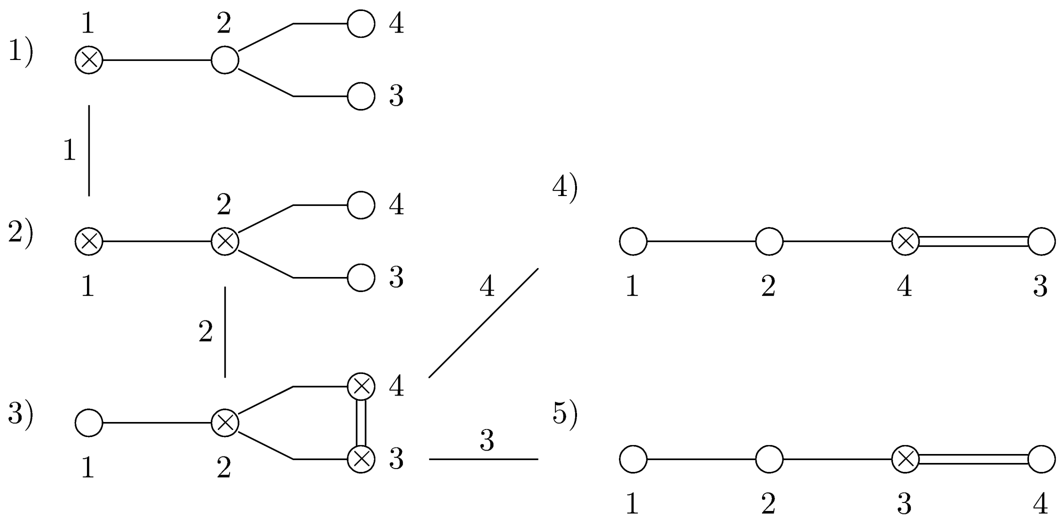
$$T_{i,ji\triangleright d}T_{j,i\triangleright d}T_{i,d} = T_{j,ij\triangleright d}T_{i,j\triangleright d}T_{j,d} \quad \text{if } a = d^C_+ \text{ and } i = \ell-2,\ j = \ell-1, \tag{2.63}$$

$$T_{i,ji\triangleright d}T_{j,i\triangleright d}T_{i,d} = T_{j,ij\triangleright d}T_{i,j\triangleright d}T_{j,d} \quad \text{if } a = d^C_- \text{ and } i = \ell-2,\ j = \ell, \tag{2.64}$$

$$T_{i,ji\triangleright d}T_{j,i\triangleright d}T_{i,d} = T_{j,ij\triangleright d}T_{i,j\triangleright d}T_{j,d} \text{ if } a = d^D \text{ and } i = \ell-2,\ j \in \{\ell-1,\ell\}, \tag{2.65}$$

$$T_{j,i\triangleright d}T_{i,d} = T_{i,j\triangleright d}T_{j,d} \text{ if } i < j, \text{ and } i,\ j \text{ are not the ones in (2.64)-(2.65).} \tag{2.66}$$

Recall that $W(C_k) = W(B_k)$ and $H_q(W(C_k)) = H_q(W(B_k))$.



where 1) $(1,0,0,0)^D$, 2) $(0,1,0,0)^D$, 3) $(0,0,1,0)^D$, 4) $(0,0,0,1)^C_-$, 5) $(0,0,0,1)^C_+$

Figure 5. Dynkin diagrams of the Lie superalgebra $D(3,1)$

Proposition 2.8. *Let $V_{\mathbf{l}}$ and $V_{\mathbf{r}}$ be finite dimensional $\mathbb{C}$-linear spaces, and let $\mathbf{l} : H_q(W(D_m)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{l}})$ and $\mathbf{r} : H_q(W(C_n)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{r}})$ be $\mathbb{C}$-algebra homomorphisms, i.e., representations. Let $\mathbf{l} \otimes \mathbf{r} : H_q(W(D_m)) \otimes H_q(W(C_n)) \to \mathrm{End}_{\mathbb{C}}(V_{\mathbf{l}} \otimes V_{\mathbf{r}})$ denote the tensor representation of $\mathbf{l}$ and $\mathbf{r}$ in the ordinary sense. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};a}$ be copies of the $\mathbb{C}$-linear space $V_{\mathbf{l}} \otimes V_{\mathbf{r}}$, indexed by $a \in \mathcal{D}^{CD}_{m|n}$. Let $C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}} := \oplus_{d \in \mathcal{D}^{CD}_{m|n}} C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}};d}$. Let $P_a : C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}} \to C_{V \otimes W;a}$ and $\iota_a : C_{V \otimes W;a} \to C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique $\mathbb{C}$-algebra homomorphism $\mathbf{l} \boxtimes \mathbf{r} = \mathbf{l} \boxtimes^{CD} \mathbf{r} : H_q(\mathfrak{osp}(2m|2n)) \to \mathrm{End}_{\mathbb{C}}(C_{V_{\mathbf{l}} \otimes V_{\mathbf{r}}})$ satisfying the following conditions:*

(i) *For each $a \in \mathcal{D}^{CD}_{m|n}$, one has $(\mathbf{l} \boxtimes \mathbf{r})(E_a) = \iota_a \circ P_a$.*

(ii) *For each $i \in \{1, \ldots, \ell = m+n\}$ and each $a \in \mathcal{D}^{CD}_{m|n}$ with $d = (p_1, \ldots, p_\ell) \in \mathcal{D}_{m|n}$ such that $a = d^C_+$, d^C_- or d^D, one has*

(2.67)
$$(\mathbf{l} \boxtimes \mathbf{r})(T_{i,a}) = \begin{cases} P_{i \triangleright a} \circ \iota_a & \text{if } 1 \le i \le \ell \text{ and } i \triangleright a \ne a, \\ \iota_a \circ (\mathbf{l}(T_{\tau^{-1}_{+.d}(i)}) \otimes \mathrm{id}_{V_{\mathbf{r}}}) \circ P_a & \text{if } 1 \le i \le \ell - 1 \text{ and } p_i = p_{i+1} = 0, \\ \iota_a \circ (\mathbf{l}(T_m) \otimes \mathrm{id}_{V_{\mathbf{r}}}) \circ P_a & \text{if } i = \ell \text{ and } p_\ell = 0, \\ \iota_a \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{\tau^{-1}_{-.d}(i)})) \circ P_a & \text{if } 1 \le i \le \ell - 2 \text{ and } p_i = p_{i+1} = 1, \\ \iota_a \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{n-1})) \circ P_a & \text{if } i = \ell - 1 \text{ and } p_{\ell-1} = p_\ell = 1,\ a = d^C_+. \\ \iota_a \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_n)) \circ P_a & \text{if } i = \ell - 1 \text{ and } p_\ell = 1,\ a = d^C_-. \\ \iota_a \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_n)) \circ P_a & \text{if } i = \ell \text{ and } p_\ell = 1,\ a = d^C_+. \\ \iota_a \circ (\mathrm{id}_{V_{\mathbf{l}}} \otimes \mathbf{r}(T_{n-1})) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell-1} = p_\ell = 1,\ a = d^C_-, \end{cases}$$

where $\tau_{\pm.d} \in S_{m+n}$ are the ones of (2.32).

Proof. We can check out this directly in a way similar to that for Proof of Proposition 2.4. □

For $\lambda \in \Lambda_{H_q(W(D_m))}$ and $\mu \in \Lambda_{H_q(W(C_n))}$, we denote $\rho_{H_q(W(D_m)),\lambda} \boxtimes^{CD} \rho_{W(C_n)),\mu}$ by $\rho^{CD}_{q;\lambda,\mu}$ and we denote $C_{V \otimes W}$ for $V = V_{H_q(W(D_m)),\lambda}$ and $W = V_{H_q(W(C_n)),\mu}$ by $C^{CD}_{q;\lambda,\mu}$.

Theorem 2.9. *Let $q \in \mathbb{C}$ and assume that*

(2.68)
$$qP_{W(D_m)}(q)P_{W(C_n)}(q) \ne 0.$$

Then the $\mathbb{C}$-algebra homomorphism

$$\bigoplus_{(\lambda,\mu)\in\Lambda_{H_q(W(D_m))}\times\Lambda_{H_q(W(C_n))}} \rho^{CD}_{q;\lambda,\mu} : \quad H_q(\mathfrak{osp}(2m|2n)) \to \bigoplus_{(\lambda,\mu)\in\Lambda_{H_q(W(D_m))}\times\Lambda_{H_q(W(C_n))}} \mathrm{End}_{\mathbb{C}}(C^{CD}_{q;\lambda,\mu}) \tag{2.69}$$

is an isomorphism. Further we have

$$\dim H_q(\mathfrak{osp}(2m|2n)) = \frac{2^{m+n-1}((m+n-1)!(m+2n))^2}{m!n!}. \tag{2.70}$$

Moreover $H_q(\mathfrak{osp}(2m|2n))$ is a semisimple $\mathbb{C}$-algebra and a complete set of non-equivalent irreducible representations of $H_q(\mathfrak{osp}(2m|2n))$ is given by $\{\rho^{CD}_{q;\lambda,\mu}|(\lambda,\mu)\in\Lambda_{H_q(W(D_m))}\times\Lambda_{H_q(W(C_n))}\}$.

Proof. Note that $W(D_m)\times W(C_n) = 2^{m+n-1}m!n!$. Then we can prove this theorem in the same way as that for Proof of Theorem 2.7. □

Remark 1. Now, by (2.3), (2.4) and Theorems 2.5, 2.7 and 2.9, it has turned out that if q is non-zero and not any primitive root of unity, then as a $\mathbb{C}$-algebra, $H_q(\mathbf{g}) = H_q(W)$ introduced in this section for the Lie superalgebra $\mathbf{g} = A(m,n)$ or $\mathfrak{osp}(m|2n)$ is very similar to the Iwahori-Hecke algebra $H_q(W_0)$ associated with the Weyl group W_0 of the Lie algebra $\mathbf{g}(0)$ given as the even part of $\mathbf{g}$, that is, Morita equivalence.

Remark 2. Assume q to be an element of $\mathbb{C}$ transcendental over $\mathbb{Q}$. Then the $\mathbb{Z}$-subalgebra (with identity) of $\mathbb{C}$ generated by q can also be regarded as the polynomial ring $\mathbb{Z}[q]$ in the variable q over $\mathbb{Z}$. Let W be one of the Coxeter groupoids treated in Subsections 2.5, 2.6 and 2.7. By Lemma 2.2 and (2.42), (2.51), (2.70), one see that $\{f(w)|w\in W\setminus\{0\}\}$ is a $\mathbb{C}$-basis of $H_q(W)$, that is, $H_q(W) = \oplus_{w\in W\setminus\{0\}}\mathbb{C}f(w)$. Define $H_{\mathbb{Z}[q],q}(W)$ to be the $\mathbb{Z}[q]$-submodule generated by $\{f(w)|w\in W\setminus\{0\}\}$. Using Theorem 1.7 and Corollary 1.8, one see that $H_{\mathbb{Z}[q],q}(W)$ is also a $\mathbb{Z}[q]$-subalgebra of $H_q(W)$. Let $\mathbb{A}$ be any commutative ring (with identity). Let ζ be any element of $\mathbb{A}$. Regard $\mathbb{A}$ as a $\mathbb{Z}[q]$-algebra via the $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}[q]\to\mathbb{A}$ that takes q to ζ. Let $H_{\mathbb{A},\zeta}(W)$ be the $\mathbb{A}$-algebra (with identity) defined by $H_{\mathbb{A},\zeta}(W) := H_{\mathbb{Z}[q],q}(W)\otimes_{\mathbb{Z}[q]}\mathbb{A}$. For $X\in H_{\mathbb{Z}[q],q}(W)$, we also denote the element $X\otimes 1$ of $H_{\mathbb{A},\zeta}(W)$ by X. Then $H_{\mathbb{A},\zeta}(W)$ is a free $\mathbb{A}$-module with an $\mathbb{A}$-basis $\{f(w)|w\in W\setminus\{0\}\}$, that is,

$$\mathrm{rank}_{\mathbb{A}} H_{\mathbb{A},\zeta}(W) = |W| - 1. \tag{2.71}$$

Using Theorem 1.7 and Corollary 1.8 again, one see that $H_{\mathbb{A},\zeta}(W)$ can also be defined by the same generators as (2.5) and the same relations as (2.6)-(2.13) with ζ in place of q.

The same properties as above seem to be true for many Coxeter groupoids, which might be able to be proved in a way similar to that of the proof of [L, Proposition 3.3].

References

[AS1] N. Andruskiewitsch and H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3, *J. Algebra* **209** (1998) 658–691.

[AS2] N. Andruskiewitsch and H.-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Accepted for publication in Ann. Math., [arXiv:math.QA/0502157].

[C] R.W. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics, Vol. 28. John Wiley & Sons, London-New York-Sydney, 1972.

[CR] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Interscience Publishers, a division of John Wiley & Sons, New York-London 1962

[GU] A. Gyoja and K. Uno, On the semisimplicity of Hecke algabras, *J. Math. Soc. Japan* **41** (1989), no. 1, 75–79

[H1] I. Heckenberger, Classification of arithmetic root systems, Accepted for publication in Adv. Math. [math.QA/0605795]

[H2] ———, The Weyl groupoid of a Nichols algebra of diagonal type, *Invent. Math.* **164** (2006) 175–188.

[HSTY] I. Heckenberger, F. Spill, A. Torrielli, H. Yamane, Drinfeld second realization of quantum affine superalgebras of $D^{(1)}(2,1;x)$ via the Weyl groupoid, *RIMS Kôkyûroku Bessatsu* **B8** (2008) 171-216

[HY] I. Heckenberger and H. Yamane, A generalization of Coxeter groups, root systems, and Matsumoto's theorem, *Math. Z.* **259** (2008) 255-276

[Hu] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990

[K] V.G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977) 8–96

[L] G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monograph Series, 18. American Mathematical Society, Providence, RI, 2003.

[S] V. Serganova, On generalizations of root systems, *Commun. Algebra* **24** (1996), 4281–4299.

[SV] A.N. Sergeev and A.P. Veselov, Grothendieck rings of basic classical Lie superalgebras, Preprint arXiv:0704.2250