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Smallness of fundamental groups for arithmetic schemes: a resume (joint work with Toshiro Hiranouchi)

By

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A profinite group G is said to be *small* if there exist only finitely many open normal subgroups of G with index n for any positive integer n. By the Hermite-Minkowski theorem, the Galois group $G_{F,S}$ of the maximal Galois extension of an algebraic number field F unramified outside a finite set S of primes of F is a small profinite group. Note that $G_{F,S}$ is the étale fundamental group $\pi_1(\operatorname{Spec}\mathcal{O}_F \setminus S)$ of the open subscheme $\operatorname{Spec}\mathcal{O}_F \setminus S$ of the affine scheme $\operatorname{Spec}\mathcal{O}_F$, where $\mathcal{O}_F$ is the ring of integers of F, if S contains all the infinite primes of F. We have obtained a generalization of the Hermite-Minkowski theorem for higher dimensional arithmetic schemes as follows:[1]

Theorem 1. *Let X be a connected scheme of finite type and flat over the ring of integers $\mathbb{Z}$. Then the étale fundamental group $\pi_1(X)$ is small.*

The 1-dimensional case of Theorem 1 is nothing but the Hermite-Minkowski theorem.

For a smooth curve C over a finite field k, there is also an analogue of the Hermite-Minkowski theorem. However, we need an additional condition on the ramification along the boundary $\overline{C} \setminus C$ of C, where $\overline{C}$ is the regular compactification of C. In fact the étale fundamental group $\pi_1(C)$ is not necessarily a small profinite group. For a higher dimensional variety X (separated scheme of finite type over k) and a *modulus* $\mathfrak{m}$ on X (which is a collection of moduli associated with all curves on X), we consider a quotient $\pi_1(X, \mathfrak{m})$ of $\pi_1(X)$ which classifies étale coverings of X with ramification bounded by the modulus $\mathfrak{m}$. Then we have obtained the smallness of the fundamental group $\pi_1(X, \mathfrak{m})$ under a certain condition:

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[1] In the talk at RIMS, we stated Theorem 1 under the assumption that X is a connected normal scheme flat, of finite type over $\mathbb{Z}$. Our thanks are due to Professor Alexander Schmidt for his suggestion on the possibility of removing the assumption 'normal'.
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Theorem 2. *Let X be a connected variety over a finite field k. Assume that there exist an étale morphism $X' \to X$ and a proper morphism $X' \to Z$, where Z is a curve over k. Then $\pi_1(X, \mathfrak{m})$ is small for any modulus $\mathfrak{m}$ on X.*

As an application, for a normal arithmetic scheme X (normal separated scheme which is flat and of finite type over $\mathbb{Z}$) or a normal variety over a finite field k (normal separated scheme of finite type over k) with certain conditions, we have obtained some finiteness results of mod p representations of $\pi_1(X, \mathfrak{m})$ as follows.

Theorem 3. *Let $\mathbf{k}$ be an algebraically closed field with arbitrary characteristic and let d be a positive integer.*

(i) *Let X be a connected normal scheme which is flat, separated and of finite type over $\mathbb{Z}$. For any modulus $\mathfrak{m}$ on X there exist only finitely many isomorphism classes of semisimple continuous representations $\rho : \pi_1(X, \mathfrak{m}) \to \mathrm{GL}_d(\mathbf{k})$ with solvable images.*

(ii) *Let X be a connected normal variety over a finite field k. We assume that there exist an étale open $X' \to X$ and a proper generically smooth morphism $X' \to Z$ to a regular curve Z over k. For any modulus $\mathfrak{m}$ on X there exist only finitely many isomorphism classes of semisimple continuous geometric representations $\rho : \pi_1(X, \mathfrak{m}) \to \mathrm{GL}_d(\mathbf{k})$ with solvable images.*

Theorem 4. *Let $\mathbf{k}$ be an algebraically closed field with characteristic 0 and let d be a positive integer.*

(i) *Let X be a connected normal scheme which is flat, separated and of finite type over $\mathbb{Z}$. For any modulus $\mathfrak{m}$ on X there exist only finitely many isomorphism classes of semisimple continuous representations $\rho : \pi_1(X, \mathfrak{m}) \to \mathrm{GL}_d(\mathbf{k})$.*

(ii) *Let X be a connected normal variety over a finite field k. We assume that there exist an étale open $X' \to X$ and a proper generically smooth morphism $X' \to Z$ to a regular curve Z over k. For any modulus $\mathfrak{m}$ on X there exist only finitely many isomorphism classes of semisimple continuous geometric representations $\rho : \pi_1(X, \mathfrak{m}) \to \mathrm{GL}_d(\mathbf{k})$.*

These results contain the earlier results on the finiteness of mod p Galois representations which have been proved in [MT]. For more details, see our preprint [HT].

References

[MT] H. Moon and Y. Taguchi, *Mod p Galois representations of solvable image*, Proc. Amer. Math. Soc. **129** (2001), 2529–2534 (electronic).

[HT] S. Harada and T. Hiranouchi, *Smallness of fundamental groups for arithmetic schemes*, preprint available online at `http://arxiv.org/abs/0803.2096`.