RIMS Kôkyûroku Bessatsu
B12 (2009), 57–66

Cycle classes, Lefschetz trace formula and integrality for *p*-adic cohomology

By

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§ 1. Introduction

This article is an announcement of my recent work on p-adic cohomology. Detailed proofs will appear elsewhere. Let k be a field of characteristic $p > 0$ and X a purely d-dimensional scheme which is separated and smooth over k. An algebraic correspondence on X is a closed subscheme Γ of $X \times X$ with pure dimension d (here we simply write $X \times X$ for $X \times_k X$). We are interested in the trace of the action of Γ on the p-adic cohomology of X (the precise setting will be introduced later).

Although we are interested in p-adic cohomology, our first motivating problem is about ℓ-adic cohomology. We will begin by explaining this problem. Here we will assume that k is algebraically closed and remove the restriction that the characteristic of k is positive. Let ℓ be a prime number which is different from p. Denote by γ the closed immersion $\Gamma \hookrightarrow X \times X$ and put $\gamma_i = \mathrm{pr}_i \circ \gamma$. If γ_1 is proper, then we may define the homomorphism $\Gamma^*\colon H^i_c(X, \mathbb{Q}_\ell) \longrightarrow H^i_c(X, \mathbb{Q}_\ell)$ as the composite

$$H^i_c(X, \mathbb{Q}_\ell) \xrightarrow{\gamma_1^*} H^i_c(\Gamma, \mathbb{Q}_\ell) \xrightarrow{\gamma_{2*}} H^i_c(X, \mathbb{Q}_\ell).$$

Here γ_1^* can be defined since γ_1 is proper, and γ_{2*} can be defined since X is smooth over k. Note that a proper k-morphism $f\colon X \longrightarrow X$ can be regarded as the correspondence $\gamma_f = f \times \mathrm{id}_X\colon \Gamma_f = X \longrightarrow X \times X$ and Γ_f^* coincides with f^*.

We are interested in the alternating sum of the traces of Γ^*.

Problem 1.1. Under the assumptions above, does the alternating sum of the traces $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_c(X, \mathbb{Q}_\ell))$ lie in $\mathbb{Z}$?

Received March 31, 2008. Revised December 17, 2008.
2000 Mathematics Subject Classification(s): 14F30.
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First we should remark that this problem has actually been solved in [BE] by using relative motivic cohomology defined by Levine. However the author wanted a more standard cohomology-theoretic proof.

If the characteristic of k is 0, then $\mathrm{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell))$ lies in $\mathbb{Z}$ for each i. This is because we may assume $k = \mathbb{C}$ and may use compactly supported Betti cohomology $H_c^i(X, \mathbb{Q})$ and $H_c^i(\Gamma, \mathbb{Q})$ in place of ℓ-adic cohomology. We have comparison results $H_c^i(X, \mathbb{Q}) \otimes \mathbb{Q}_\ell \cong H_c^i(X, \mathbb{Q}_\ell)$ and $H_c^i(\Gamma, \mathbb{Q}) \otimes \mathbb{Q}_\ell \cong H_c^i(\Gamma, \mathbb{Q}_\ell)$, which lead to the equality $\mathrm{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}))$. Since Γ^* preserves the $\mathbb{Z}$-lattice $\mathrm{Im}(H_c^i(X, \mathbb{Z}) \longrightarrow H_c^i(X, \mathbb{Q}))$ of $H_c^i(X, \mathbb{Q})$, the trace $\mathrm{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}))$ lies in $\mathbb{Z}$.

If X is proper over k, by the Lefschetz trace formula, we have

$$\sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}\bigl(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell)\bigr) = (\Gamma, \Delta_X)_{X \times X},$$

where $\Delta_X \subset X \times X$ denotes the diagonal and $(-,-)_{X\times X}$ the intersection number in $X \times X$. Since the right hand side is an integer which is independent of ℓ ($\neq p$), so is the left hand side.

By Fujiwara's trace formula [Fu], which is in some sense a generalization of the Lefschetz trace formula used above, we have the following result for X which is not necessarily proper over k:

Proposition 1.2. *The alternating sum* $\sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell))$ *is a rational number which is independent of* ℓ ($\neq p$).

Proof. This is proved in [Mi, Theorem 2.1.2], but we include its proof for the reader's convenience. By the standard specialization argument, we may assume that $k = \overline{\mathbb{F}}_q$ and that X and Γ come from $\mathbb{F}_q$. Fix a scheme X_0 over $\mathbb{F}_q$ such that $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and denote by $\mathrm{Fr}_{X_0}\colon X_0 \longrightarrow X_0$ the qth power Frobenius morphism of X_0. By the base change from $\mathbb{F}_q$ to $\overline{\mathbb{F}}_q$, we obtain the morphism $X \longrightarrow X$, which is also denoted by Fr_{X_0}. By Fujiwara's trace formula ([Fu, Proposition 5.3.4, 5.4.1]), there exists an integer N_ℓ such that for every integer n with $n \geq N_\ell$ we have

$$(*) \qquad \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_{X_0}^*)^n; H_c^i(X, \mathbb{Q}_\ell)\bigr) = (\gamma_*^{(n)}\Gamma, \Delta_X)_{X \times X},$$

where $\gamma^{(n)}\colon \Gamma \longrightarrow X \times X$ denotes the unique morphism that satisfies $\mathrm{pr}_1 \circ \gamma^{(n)} = \mathrm{Fr}_{X_0}^n \circ \gamma_1$ and $\mathrm{pr}_2 \circ \gamma^{(n)} = \gamma_2$. In particular, $\sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}(\Gamma^* \circ (\mathrm{Fr}_{X_0}^*)^n; H_c^i(X, \mathbb{Q}_\ell))$ is an integer.

Let $\alpha_{i,1,\ell}, \ldots, \alpha_{i,m_i,\ell}$ be the eigenvalues of Γ^* on $H_c^i(X, \mathbb{Q}_\ell)$ and $\lambda_{i,1,\ell}, \ldots, \lambda_{i,m_i,\ell}$ those of $\mathrm{Fr}_{X_0}^*$ on $H_c^i(X, \mathbb{Q}_\ell)$. Since Γ^* and $\mathrm{Fr}_{X_0}^*$ commutes with each other, the left hand side of $(*)$ is equal to $\sum_i (-1)^i \sum_{j=1}^{m_i} \alpha_{i,j,\ell}\lambda_{i,j,\ell}^n$ with $\lambda_{i,1,\ell}, \dots, \lambda_{i,m_i,\ell}$ permuted suitably. By simplifying this sum, we have an expression of the form $\sum_i \alpha'_{i,\ell}\lambda'^n_{i,\ell}$ where each $\lambda'_{i,\ell}$ is one of $\lambda_{i,j,\ell}$ and $\lambda'_{i,\ell} \neq \lambda'_{j,\ell}$ for $i \neq j$. Since $\lambda'_{i,\ell}$ is a non-zero algebraic integer, by the subsequent lemma, every $\alpha'_{i,\ell}$ is an algebraic number and $\sum_i \alpha'_{i,\ell}\lambda'^n_{i,\ell} \in \mathbb{Q}$ for every n.

Moreover, since the right hand side of $(*)$ is independent of ℓ, for two primes ℓ and ℓ' which are different from p, we have $\sum_i \alpha'_{i,\ell}\lambda'^n_{i,\ell} = \sum_i \alpha'_{i,\ell'}\lambda'^n_{i,\ell'}$ for $n \geq N_\ell + N_{\ell'}$. Therefore by the van der Mond argument, we have this equality for every n.

Hence we conclude that $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^* \circ (\mathrm{Fr}_{X_0}^*)^n; H_c^i(X, \mathbb{Q}_\ell))$ is a rational number which is independent of ℓ for every n. In particular, $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell))$ is a rational number which is independent of ℓ. $\square$

Lemma 1.3. *Let L be a field of characteristic 0 and $\alpha_1, \dots, \alpha_m$, $\lambda_1, \dots, \lambda_m$ elements of L such that $\lambda_i \neq \lambda_j$ for $i \neq j$, λ_i is algebraic over $\mathbb{Q}$ and $\lambda_i \neq 0$ for every i. Assume that there exists an integer N such that $\sum_{i=1}^m \alpha_i\lambda_i^n \in \mathbb{Q}$ for every $n \geq N$. Then α_i is algebraic over $\mathbb{Q}$ and $\sum_{i=1}^m \alpha_i\lambda_i^n \in \mathbb{Q}$ for every n.*

Proof. This is an easy exercise of linear algebra. See [Mi, Lemma 2.1.3] for example. $\square$

Remark. In the proof above, we cannot conclude that $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell))$ is an integer, since $\lambda_{i,j,\ell}$ is not necessarily a unit of $\overline{\mathbb{Z}}$, the ring of algebraic integers.

By Proposition 1.2 and its proof, we have the following integrality result:

Corollary 1.4. *The alternating sum $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell))$ lies in $\mathbb{Z}[1/p]$.*

There are (at least) two ways to derive this corollary:

① Use the fact that $\lambda_{i,j,\ell}$ in the proof of Proposition 1.2 is the unit of $\overline{\mathbb{Z}}[1/p]$ ([SGA7-II, Exposé XXI, Corollaire 5.5.3]). For a detailed proof, see [Mi, Lemma 2.1.3].

② Use the integral structure of $H_c^i(X, \mathbb{Q}_\ell)$. Actually Γ induces the action on the integral ℓ-adic cohomology of X; $\Gamma^* \colon H_c^i(X, \mathbb{Z}_\ell) \longrightarrow H_c^i(X, \mathbb{Z}_\ell)$. Therefore Γ^* on $H_c^i(X, \mathbb{Q}_\ell)$ preserves the $\mathbb{Z}_\ell$-lattice $\operatorname{Im}(H_c^i(X, \mathbb{Z}_\ell) \longrightarrow H_c^i(X, \mathbb{Q}_\ell))$ and thus we have $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell)) \in \mathbb{Z}_\ell \cap \mathbb{Q} = \mathbb{Z}_{(\ell)}$. By the ℓ-independence proved in Proposition 1.2, we have $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H_c^i(X, \mathbb{Q}_\ell)) \in \bigcap_{\ell \neq p} \mathbb{Z}_{(\ell)} = \mathbb{Z}[1/p]$ as desired.

These proofs are apparently different. The crucial fact in ① is that the Frobenius eigenvalues $\lambda_{i,j,\ell}$ are units in $\overline{\mathbb{Z}}[1/p]$. By the Weil conjecture, this can be deduced from some properties (integrality of the coefficients, the functional equation) of the congruence zeta function of X, which is inherent in X itself. Therefore, ① is the proof which is independent of cohomology theory; that is, we may apply this method to any Weil cohomology (*cf.* [KM, Corollary 1]). On the other hand, the proof ② uses the existence of a $\mathbb{Z}_\ell$-structure of the cohomology group, which is peculiar to ℓ-adic cohomology.

Now we are at the starting point of the research reported in this article; if we have an analogue of Fujiwara's trace formula and a good integral structure for "p-adic cohomology",

- by ①, the alternating sum of the traces $\sum_i (-1)^i \operatorname{Tr}$ should lie in $\mathbb{Z}[1/p]$,
- by ②, $\sum_i (-1)^i \operatorname{Tr}$ should lie in a "p-adic ring" like $\mathbb{Z}_p$,

and thus $\sum_i (-1)^i \operatorname{Tr}$ should be an integer. Furthermore, by comparing the ℓ-adic and p-adic traces, we expect to have an analogous result for ℓ-adic cohomology (= Problem 1.1). These are exactly what the author did in this work.

The outline of the remaining part of this article is as follows. In the next section, after giving a short review of p-adic cohomology, we will deal with the part corresponding to ②. The essential ingredient is the construction of refined cycle classes in partially supported integral log crystalline cohomology. This is needed to define the action of Γ on integral log crystalline cohomology. In Section 3, we consider an analogue of Fujiwara's trace formula for rigid cohomology by which we can make the same argument as in the proof of Proposition 1.2. It enables us to carry out the p-adic version of the proof ①, and to compare the ℓ-adic and p-adic traces.

§ 2. *p*-adic cohomology and an integrality result

In the sequel, let k be a perfect (not necessarily algebraically closed) field of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vectors of k, K_0 the fraction field of W, and K a finite totally ramified extension of K_0. We use the same notation X, Γ, γ, γ_i as in the previous section.

In this work, we will use two p-adic cohomology: rigid cohomology and log crystalline cohomology. Let us recall them briefly.

For a scheme Y which is separated of finite type over k, we may define the rigid cohomology $H^i_{\mathrm{rig}}(Y/K)$ and the compactly supported rigid cohomology $H^i_{\mathrm{rig},c}(Y/K)$ of Y ([Be1]). These are K-vector spaces. Rigid cohomology is a good cohomology theory even for schemes which are neither proper nor smooth over k (e.g., the K-vector spaces $H^i_{\mathrm{rig}}(Y/K)$ and $H^i_{\mathrm{rig},c}(Y/K)$ are known to be finite-dimensional). However, (as far as the author knows) there is no integral structure on rigid cohomology in general.

We can define the action of Γ on $H^i_{\mathrm{rig},c}(X/K)$ as follows. First we assume that Γ is integral. Then, by de Jong's alteration theorem, there exists a scheme Γ' which is smooth over k and a proper surjective generically finite k-morphism $\pi\colon \Gamma' \longrightarrow \Gamma$. Put $\gamma'_i = \gamma_i \circ \pi$ for $i = 1, 2$. By the Poincaré duality ([Be2]), $\gamma_2'^*\colon H^i_{\mathrm{rig}}(X/K) \longrightarrow H^i_{\mathrm{rig}}(\Gamma'/K)$ induces a K-linear map $H^i_{\mathrm{rig},c}(\Gamma'/K) \longrightarrow H^i_{\mathrm{rig},c}(X/K)$. We denote this map by γ'_{2*} and define $\Gamma^* = (\deg\pi)^{-1}(\gamma'_{2*} \circ \gamma_1'^*)$. It is easy to see that this is independent of the choice of π. For general Γ, let $\Gamma_1, \ldots, \Gamma_n$ be the irreducible components of Γ and put $m_i = \operatorname{length} \mathcal{O}_{\Gamma,\eta_i}$ where η_i denotes the generic point of Γ_i. We define Γ^* as $\sum_{i=1}^n m_i \Gamma_i^*$.

Next we recall log crystalline cohomology. Let Y be a scheme which is proper smooth over k, and D a strict normal crossing divisor on Y. Then we can define the log crystalline cohomology $H^i_{\mathrm{crys}}((Y,D)/W)$ and the "compactly supported" (or "with minus log pole") log crystalline cohomology $H^i_{\mathrm{crys}}((Y,-D)/W)$. These are W-modules. If we put $U = Y \setminus D$, then we have the following comparison result due to Shiho [Sh]:

$$H^i_{\mathrm{crys}}\big((Y,D)/W\big) \otimes_W K \cong H^i_{\mathrm{rig}}(U/K).$$

Moreover, combining this with the Poincaré duality for log crystalline cohomology ([Ts]) and rigid cohomology ([Be2]), we also have the comparison result for compactly supported cohomology (see [Na, Remark 5.4]):

$$H^i_{\mathrm{crys}}\big((Y,-D)/W\big) \otimes_W K \cong H^i_{\mathrm{rig},c}(U/K).$$

In other words, if a smooth scheme U has a compactification $U \hookrightarrow Y$ such that the boundary is a strict normal crossing divisor, then the log crystalline cohomology $H^i_{\mathrm{crys}}((Y,D)/W)$ (resp. $H^i_{\mathrm{crys}}((Y,-D)/W)$) gives an integral structure of the rigid cohomology $H^i_{\mathrm{rig}}(U/K)$ (resp. $H^i_{\mathrm{rig},c}(U/K)$).

Now we state our integrality result.

Theorem 2.1. *The trace* $\operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ *lies in* $\mathcal{O}_K$*, the ring of integers of* K*.*

First, by de Jong's alteration theorem, we may assume that X has a compactification $X \hookrightarrow \overline{X}$ such that $D = \overline{X} \setminus X$ is a strict normal crossing divisor on $\overline{X}$. This reduction might not seem immediate, since after taking an alteration $\pi\colon X' \longrightarrow X$, the trace of the correspondence is multiplied by $\deg\pi$. If we have the desired result for X', we can only have $\operatorname{Tr}\Gamma^* \in (\deg\pi)^{-1}\mathcal{O}_K$. However, since $\deg\pi$ depends only on X and is independent of Γ, we may replace Γ by its composite and have $\operatorname{Tr}(\Gamma^*)^n \in (\deg\pi)^{-1}\mathcal{O}_K$ for every $n \geq 1$. By this, we may conclude $\operatorname{Tr}\Gamma^* \in \mathcal{O}_K$. For a detailed proof, see [Sa, p. 629] for example.

In this case, it suffices to show the following proposition:

Proposition 2.2. *We can define a* W*-homomorphism*

$$\Gamma^*\colon H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big) \longrightarrow H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big)$$

which makes the following diagram commutative:

$$\begin{tikzcd}
H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big)\otimes_W K \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\Gamma^*\otimes\mathrm{id}_K"] & H^i_{\mathrm{rig},c}(X/K) \arrow[d, "\Gamma^*"] \\
H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big)\otimes_W K \arrow[r, leftrightarrow, "\cong"] & H^i_{\mathrm{rig},c}(X/K).
\end{tikzcd}$$

In order to define Γ^*, we introduce partially supported log crystalline cohomology. Let Y be a scheme which is proper smooth over k and D_1+D_2 a strict normal crossing divisor on Y such that D_1 and D_2 have no common irreducible component. Then we can define the partially supported log crystalline cohomology $H^i_{\mathrm{crys}}((Y,D_1-D_2)/W)$. This is a crystalline analogue of $H^i(Y\setminus D_1, j_!\mathbb{Z}_\ell)$, where j denotes the open immersion $Y\setminus(D_1\cup D_2)\hookrightarrow Y\setminus D_1$. The definition of $H^i_{\mathrm{crys}}((Y,D_1-D_2)/W)$ using the de Rham-Witt complex is essentially given in [Hy] (roughly speaking, replace "$\Omega^\bullet_{Y/k}$" in the theory of usual de Rham-Witt complexes by "$\Omega^\bullet_{Y/k}(D_1,D_2)$" appearing in [DI, (4.2.1.2)]). We may also give the definition within the framework of log crystalline site.

Put $Z=\overline{X}\times\overline{X}$, $D_1=D\times\overline{X}$ and $D_2=\overline{X}\times D$. Then we have the natural W-homomorphisms

$$\begin{aligned}
\mathrm{pr}_1^*\colon H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big) &\longrightarrow H^i_{\mathrm{crys}}\big((Z,-D_1+D_2)/W\big),\\
\mathrm{pr}_{2*}\colon H^{i+2d}_{\mathrm{crys}}\big((Z,-D_1-D_2)/W\big) &\longrightarrow H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big).
\end{aligned}$$

Let $\overline{\Gamma}$ be the closure of Γ in Z. Since γ_1 is assumed to be proper, we have $\overline{\Gamma}\cap D_2\subset\overline{\Gamma}\cap D_1$. Therefore, by Theorem 2.3 stated below, we have the cycle class $\mathrm{cl}(\overline{\Gamma})$ in the partially supported log crystalline cohomology $H^{2d}_{\mathrm{crys}}((Z,D_1-D_2)/W)$. Hence we may define $\Gamma^*\colon H^i_{\mathrm{crys}}((\overline{X},-D)/W)\longrightarrow H^i_{\mathrm{crys}}((\overline{X},-D)/W)$ as the composite

$$\begin{aligned}
H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big)\xrightarrow{\mathrm{pr}_1^*}H^i_{\mathrm{crys}}\big((Z,-D_1+D_2)/W\big)\xrightarrow{-\cup\mathrm{cl}(\overline{\Gamma})}H^{i+2d}_{\mathrm{crys}}\big((Z,-D_1-D_2)/W\big)\\
\xrightarrow{\mathrm{pr}_{2*}}H^i_{\mathrm{crys}}\big((\overline{X},-D)/W\big).
\end{aligned}$$

It can be shown that this Γ^* satisfies the commutativity in Proposition 2.2.

Here is our theorem on cycle classes in the partially supported log crystalline cohomology.

Theorem 2.3. *Let X be a purely d-dimensional scheme which is proper smooth over k, $D=D_1+D_2$ a strict normal crossing divisor on X such that D_1 and D_2 have no common irreducible component. Let Y be a closed subscheme of X with pure codimension c such that $Y\setminus D$ is dense in Y. If $Y\cap D_2\subset Y\cap D_1$, then we can define the cycle class $\mathrm{cl}(Y)\in H^{2c}_{\mathrm{crys}}((X,D_1-D_2)/W)$ which satisfies various functorialities (e.g., the image of $\mathrm{cl}(Y)$ under the natural map $H^{2c}_{\mathrm{crys}}((X,D_1-D_2)/W)\longrightarrow H^{2c}_{\mathrm{crys}}((X\setminus D)/W)$ coincides with the cycle class of $Y\setminus D$ defined in* [Gr]).

We will explain the outline of the proof. As in [Gr], we construct the cycle class $\mathrm{cl}_{\mathrm{HW}}(Y)$ in the local Hodge-Witt cohomology $H^c_Y(X, W\Omega^c_X(\log(D_1 - D_2)))$. If Y does not intersect D, then we have $H^c_Y(X, W\Omega^c_X(\log(D_1 - D_2))) \cong H^c_Y(X, W\Omega^c_X)$. Thus we may define $\mathrm{cl}_{\mathrm{HW}}(Y)$ as Gros' cycle class of Y. In the general case, we want to remove the intersection $Y \cap D$. Consider the natural map

$$H^c_Y\big(X, W\Omega^c_X\big(\log(D_1 - D_2)\big)\big) \longrightarrow H^c_{Y\setminus D}(X \setminus D, W\Omega^c_X).$$

This is not an isomorphism in general, but induces an isomorphism on their Frobenius fixed parts:

Theorem 2.4. *Under the setting of Theorem 2.3, the natural map*

$$H^c_Y\big(X, W\Omega^c_X\big(\log(D_1 - D_2)\big)\big)^{F=1} \longrightarrow H^c_{Y\setminus D}(X \setminus D, W\Omega^c_X)^{F=1}$$

is an isomorphism (F *denotes the Frobenius map on Hodge-Witt cohomology*).

The proof of this theorem is the hardest part of this work. Note that $Y \cap D$ is contained in D_1 by the assumption that $Y \cap D_2 \subset Y \cap D_1$, and that the codimension of $Y \cap D$ in X is greater than c since $Y \setminus D$ is dense in Y. Therefore it suffices to show the following vanishing results of local cohomology sheaves: for every closed subscheme Z of D_1, we have

$$\begin{aligned} \underline{H}^i_Z\big(X, W\Omega^c_X\big(\log(D_1 - D_2)\big)\big) &= 0 \quad \text{for } i < \operatorname{codim}_X Z, \\ \underline{H}^i_Z\big(X, W\Omega^c_X\big(\log(D_1 - D_2)\big)\big)^{F=1} &= 0 \quad \text{for } i = \operatorname{codim}_X Z. \end{aligned}$$

The former is easy. For the latter, the crucial case is where Z is smooth and contained in all irreducible components of $D = D_1 + D_2$. The proof of this case involves direct local calculation as in the proof of [Gr, II, Théorème 3.5.8].

In fact, the former is true for every closed subscheme Z of X. However, if we drop the assumption that Z is contained in D_1 and only assume that Z is contained in D, the latter becomes false. In other words, the assumption $Y \cap D_2 \subset Y \cap D_1$ in Theorem 2.3 is essential. To see this, let us consider the case where $D_1 = \varnothing$, D_2 is a smooth divisor and $Z \subset D_2$ is smooth of pure codimension $c + 1$ in X. Then we have the following exact sequence:

$$\cdots \longrightarrow \underline{H}^c_Z(X, W\Omega^c_X) \longrightarrow \underline{H}^c_Z(D_2, W\Omega^c_{D_2}) \longrightarrow \underline{H}^{c+1}_Z\big(X, W\Omega^c_X\big(\log(-D_2)\big)\big) \longrightarrow \cdots.$$

Since $c < \operatorname{codim}_Z X$, the sheaf $\underline{H}^c_Z(X, W\Omega^c_X)$ is zero. Hence we have an injection $\underline{H}^c_Z(D_2, W\Omega^c_{D_2})^{F=1} \hookrightarrow \underline{H}^{c+1}_Z(X, W\Omega^c_X(\log(-D_2)))^{F=1}$. On the other hand, by Gros' purity result ([Gr, II, Théorème 3.5.8]), we have $\underline{H}^c_Z(D_2, W\Omega^c_{D_2})^{F=1} \cong W\Omega^0_{Z,\log} \neq 0$. Therefore we conclude that $\underline{H}^{c+1}_Z(X, W\Omega^c_X(\log(-D_2)))^{F=1} \neq 0$.

By Theorem 2.4, we may define $\mathrm{cl}_{\mathrm{HW}}(Y)$ as the unique element that is mapped to Gros' cycle class of $Y \setminus D$ under the isomorphism there. Various functorialities of $\mathrm{cl}_{\mathrm{HW}}$ are immediately derived from the corresponding results for Gros' cycle classes.

Once we have $\mathrm{cl}_{\mathrm{HW}}(Y)$, we can define $\mathrm{cl}(Y)$ as the image of $\mathrm{cl}_{\mathrm{HW}}(Y)$ under the following natural maps

$$\begin{aligned} H^c_Y\bigl(X, W\Omega^c_X\bigl(\log(D_1 - D_2)\bigr)\bigr) &\longrightarrow H^{2c}_Y\bigl(X, W\Omega^\bullet_X\bigl(\log(D_1 - D_2)\bigr)\bigr) \\ &\longrightarrow H^{2c}\bigl(X, W\Omega^\bullet_X\bigl(\log(D_1 - D_2)\bigr)\bigr) \cong H^{2c}_{\mathrm{crys}}\bigl((X, D_1 - D_2)/W\bigr). \end{aligned}$$

The existence of the first homomorphism above is a consequence of the vanishing of the local cohomology sheaf $\underline{H}^i_Y(W\Omega^r_X\bigl(\log(D_1 - D_2)\bigr))$ for $i < c$. The functorialities of $\mathrm{cl}(Y)$ follow from those of $\mathrm{cl}_{\mathrm{HW}}(Y)$.

Remark. The ℓ-adic version of Theorem 2.3 is immediate. See [Mi, 4.1.13].

§ 3. Lefschetz trace formula and its consequences

We again use the notation introduced in the beginning of Section 2. Assume that $k = \mathbb{F}_q$ where q is a power of p. The statement of our Lefschetz trace formula is as follows:

Theorem 3.1 (an analogue of Fujiwara's trace formula). *There exists an integer N such that for every n with $n \geq N$ we have*

$$\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H^i_{\mathrm{rig},c}(X/K)\bigr) = (\gamma^{(n)}_* \Gamma, \Delta_X)_{X \times X}.$$

(*The notation is the same as in the proof of* Proposition 1.2.)

This theorem can be proved by a method similar to that used in [KS]. We can establish a log crystalline version of [KS, Theorem 2.3.4] and deduce Theorem 3.1 from it as in [KS, Proposition 2.3.6].

This theorem gives an integrality result as ① in Section 1.

Corollary 3.2. *The alternating sum $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ is an integer.*

Proof. By [KM], the Frobenius eigenvalues of $H^i_{\mathrm{rig},c}(X/K)$ are units of $\overline{\mathbb{Z}}[1/p]$. Therefore, we can prove that $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ lies in $\mathbb{Z}[1/p]$ by the same way as ① in Section 1. On the other hand, by Theorem 2.1, it lies in $\mathcal{O}_K$. Thus we have $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)) \in \mathbb{Z}[1/p] \cap \mathcal{O}_K = \mathbb{Z}$. □

Theorem 3.1 also enables us to compare the ℓ-adic and p-adic traces.

Corollary 3.3. *We have the following equality:*

$$\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)\bigr) = \sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^*; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)\bigr).$$

Proof. By Fujiwara's trace formula and Theorem 3.1, there exists an integer N such that for every n with $n \geq N$ we have

$$(*) \quad \sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H^i_{\mathrm{rig},c}(X/K)\bigr) = \sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)\bigr).$$

Moreover, as in the proof of Proposition 1.2, each side is of the form $\sum_i \alpha_i \lambda_i^n$ where $\alpha_i, \lambda_i \in \overline{\mathbb{Q}}$ (we use the commutativity of Γ^* and Fr_X^* on both cohomology). Therefore, by the van der Mond argument, we have the equality $(*)$ for every $n \geq 0$. In particular, setting $n = 0$, we have the desired equality. □

These corollaries give a positive answer to Problem 1.1:

Corollary 3.4. *Here let k be an arbitrary field. Then, the alternating sum of traces $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell))$ is an integer.*

Proof. By the standard specialization argument, we may assume $k = \mathbb{F}_q$. In this case, the corollary is clear from Corollary 3.2 and Corollary 3.3. □

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