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On the anabelian geometry of hyperbolic curves over finite fields

By

Mohamed SAÏDI* and Akio TAMAGAWA**

Abstract

In this partly survey, partly original paper, we review some recent results and progress on the anabelian geometry of hyperbolic curves over finite fields. We also give another proof of a certain prime-to-characteristic version of Uchida's theorem on isomorphisms between Galois groups of global fields in positive characteristics which is different from the one given in [ST1].

Contents

§0. Introduction.

In this partly survey, partly original paper, we review some recent results and progress on the anabelian geometry of hyperbolic curves over finite fields. Let U be a smooth and geometrically connected hyperbolic curve over a finite field k of characteristic $p > 0$.

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*School of Engineering, Computer Science, and Mathematics, University of Exeter, Harrison Building, North Park Road, EXETER EX4 4QF, United Kingdom.
e-mail: M.Saidi@exeter.ac.uk
**Research Institute for Mathematical Sciences, Kyoto University, KYOTO 606-8502, Japan.
e-mail: tamagawa@kurims.kyoto-u.ac.jp

We have the following exact sequence of profinite groups:

$$1 \to \pi_1(U \times_k \bar{k}, *) \to \pi_1(U, *) \to G_k \to 1.$$

Here, G_k is the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$, $*$ means a suitable geometric point, and π_1 stands for the étale fundamental group. In [G] Grothendieck laid down the foundation of his anabelian geometry program and philosophy. According to this philosophy the isomorphy type of U above, as a scheme, should be determined by the isomorphy type of $\pi_1(U, *)$ as a profinite group. This was proved to be true by Tamagawa and Mochizuki (cf. Theorem 1.1). Thus, a deep connection between the geometry of hyperbolic curves over finite fields and profinite group theory has been established. As a consequence one can embed a suitable category of hyperbolic curves over finite fields into the category of profinite groups via the fundamental group functor. It is essential in the anabelian philosophy of Grothendieck, as was formulated in [G], to be able to determine the image of this functor. Unfortunately this seems to be out of reach for the moment, the reason being that the structure of $\pi_1(U, *)$, as a profinite group, is unknown for any single example of U. This motivates the following question:

Question 0.1. Is it possible to reconstruct the isomorphy type of U solely from the isomorphy type of any quotients of $\pi_1(U, *)$ which are better understood?

The first quotients that come into mind are the following. Let $\mathfrak{Primes}$ be the set of all prime numbers. Let $\Sigma = \Sigma_U \subset \mathfrak{Primes}$ be a set of prime numbers not containing the characteristic p. For a profinite group Γ, Γ^{Σ} stands for the maximal pro-Σ quotient of Γ. The structure of $\pi_1(U \times_k \bar{k}, *)^{\Sigma}$ is well understood: it is isomorphic to the pro-Σ completion of a certain well-known finitely generated discrete group (i.e., either a free group or a surface group). Let

$$\Pi_U^{(\Sigma)} \stackrel{\mathrm{def}}{=} \pi_1(U, *)/\operatorname{Ker}(\pi_1(U \times_k \bar{k}, *) \twoheadrightarrow \pi_1(U \times_k \bar{k}, *)^{\Sigma})$$

be the corresponding quotient of $\pi_1(U)$.

Question 0.2. Is it possible to reconstruct the isomorphy type of U solely from the isomorphy type of $\Pi_U^{(\Sigma)}$, for a given set of primes Σ not containing p?

In a recent work we proved that this is indeed possible in the case where $\Sigma = \mathfrak{Primes} \setminus \{p\}$ (cf. Theorem 1.3). As a consequence one deduces a prime-to-characteristic version of Uchida's theorem on isomorphisms between absolute Galois groups of global fields in positive characteristic (cf. Theorem 1.4). In more recent work (not yet written at the time of writing this paper) we proved that one can reconstruct the isomorphy type of U solely from the isomorphy type of $\Pi_U^{(\Sigma)}$ (in the case where U is proper) for a certain type of infinite set of primes Σ (cf. Theorem 1.6, and Theorem 1.5 in the birational case). One can ultimately ask the following question.

Question 0.3. Is it possible to reconstruct the isomorphy type of U solely form the isomorphy type of $\Pi_U^{(\Sigma)}$ in the case where $\Sigma = \{l\}$ consists of a single prime number l which is different from p?

At the moment of writing this paper we do not know if Question 0.3 has an affirmative answer.

In §1 we review the above results concerning the Isom-form of the Grothendieck (birational) anabelian conjecture for hyperbolic curves over finite fields. In §2 we state the Hom-form of the Grothendieck anabelian (respectively, birational anabelian) conjecture for hyperbolic curves over finite fields. According to these conjectures one can reconstruct a non-constant morphism between hyperbolic curves over finite fields from the corresponding open homomorphism between corresponding fundamental groups (respectively, Galois groups) (cf. Conjecture 2.1 and Conjecture 2.2). These conjectures seem to be quite difficult to prove, for the time being, mainly because of the lack of a suitable "local theory" for continuous homomorphisms between fundamental groups and Galois groups. In §2 we discuss these difficulties in some detail. Our approach to the Hom-form of the Grothendieck (birational) anabelian conjecture for hyperbolic curves over finite fields is to attempt to prove it after imposing some extra conditions, which mainly ensure the existence of a suitable local theory. As a result we can prove the Hom-form of the Grothendieck birational anabelian conjecture for hyperbolic curves over finite fields under some suitable "local conditions" (cf. Theorem 2.6). In Conjecture 2.10 we state a revised (realistic) form of the Hom-form of the Grothendieck anabelian conjecture for hyperbolic curves over finite fields which one hopes to be able to prove. §3 is the most original part of this paper. It contains another proof of the prime-to-characteristic version of Uchida's theorem on isomorphisms between Galois groups of global fields in positive characteristics, which is different from the one given in [ST1], Corollary 3.11. This proof is very much inspired by Uchida's proof of Theorem 1.2, and uses class field theory.

§1. The Isom-form of the Grothendieck anabelian conjecture.

Let X be a proper, smooth, and geometrically connected curve over a finite field $k = k_X$ of characteristic $p = p_X > 0$. Write $K = K_X$ for the function field of X. Let S be a (possibly empty) finite set of closed points of X, and set $U = U_S \overset{\text{def}}{=} X - S$. We assume that U is hyperbolic. Let ξ be a base point of X with value in the generic point of X. Then ξ determines an algebraic closure $\bar{k}$ of k, and a separable closure K^{sep} of K. Denote by $\overline{U} \overset{\text{def}}{=} U \times_k \bar{k}$ the geometric fiber of U, by $G_k \overset{\text{def}}{=} \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group of k, and by $\pi_1(U)$ the étale fundamental group of U with base point ξ. Then $\pi_1(U)$ sits naturally in the following exact sequence:

$$1 \to \pi_1(\overline{U}) \to \pi_1(U) \to G_k \to 1,$$

where $\pi_1(\overline{U})$ is the étale fundamental group of $\overline{U}$ with base point ξ. Also, denote by $\overline{X} \overset{\text{def}}{=} X \times_k \bar{k}$ the geometric fiber of X, and by $K_{\overline{X}}$ the function field of $\overline{X}$. (Thus, $K_{\overline{X}} = K\bar{k}$.) Let $G_{K_X} \overset{\text{def}}{=} \mathrm{Gal}(K^{\text{sep}}/K)$ be the absolute Galois group of K_X. Then G_{K_X} sits naturally in the following exact sequence:

$$1 \to G_{K_{\overline{X}}} \to G_{K_X} \to G_k \to 1,$$

where $G_{K_{\overline{X}}} \overset{\text{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K_{\overline{X}})$ is the absolute Galois group of $K_{\overline{X}}$.

According to the anabelian (respectively, birational anabelian) philosophy of Grothendieck (cf. [G]) the isomorphy type of U as a scheme (respectively, K_X as a field) should be determined by the isomorphy type of $\pi_1(U)$ as a profinite group (respectively, G_{K_X} as a profinite group). The following result is fundamental in the anabelian geometry of hyperbolic curves over finite fields.

Theorem 1.1 (Tamagawa, Mochizuki). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Let*

$$\alpha : \pi_1(U) \xrightarrow{\sim} \pi_1(V)$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the profinite étale universal coverings corresponding to $\pi_1(U)$, $\pi_1(V)$, respectively.

Theorem 1.1 implies in particular the following birational version of the Grothendieck anabelian conjecture for hyperbolic curves over finite fields, which was already proved by Uchida.

Theorem 1.2 (Uchida). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively. Let*

$$\alpha : G_{K_X} \xrightarrow{\sim} G_{K_Y}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} \tilde{K}_X & \xrightarrow{\sim} & \tilde{K}_Y \\ \uparrow & & \uparrow \\ K_X & \xrightarrow{\sim} & K_Y \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the field extensions corresponding to the Galois groups G_{K_X}, G_{K_Y}, respectively (i.e., $\tilde{K}_X = K_X^{\mathrm{sep}}$, $\tilde{K}_Y = K_Y^{\mathrm{sep}}$).

Theorem 1.2 was first proved by Uchida (cf. [U]). Theorem 1.1 was proved by Tamagawa (cf. [T], Theorem (4.3)) in the affine case (together with a certain tame version), and more recently by Mochizuki (cf. [M], Theorem 3.2) in the proper case. It implies in particular that one can embed a suitable category of hyperbolic curves over finite fields into the category of profinite groups via the fundamental group functor. It is essential in the anabelian philosophy of Grothendieck, as was formulated in [G], to be able to determine the image of this functor. Recall that the full structure of the profinite group $\pi_1(\overline{U})$ is unknown (for any single example of U which is hyperbolic). Hence, a fortiori, the structure of $\pi_1(U)$ is unknown (the closed subgroup $\pi_1(\overline{U})$ of $\pi_1(U)$ can be reconstructed group-theoretically from the isomorphy type of $\pi_1(U)$ (cf. [T], Proposition (3.3)). (Even if we replace the fundamental groups $\pi_1(\overline{U})$, $\pi_1(U)$ by the tame fundamental groups $\pi_1^{\mathrm{t}}(\overline{U})$, $\pi_1^{\mathrm{t}}(U)$, respectively, the situation is just the same.) The full structure of the absolute Galois group G_{K_X} is also unknown, though one knows the structure of the closed subgroup $G_{K_{\overline{X}}}$ of G_{K_X} by a result of Pop and Harbater. (Namely $G_{K_{\overline{X}}}$ is a free profinite group of countable rank, cf. [P], [H].) Thus, the problem of determining the image of the above functor seems to be quite difficult, at least for the moment. Now, it is quite natural to address the following question:

Question 1. Is it possible to prove any result analogous to the above Theorem 1.1 (respectively, 1.2) where $\pi_1(U)$ (respectively, G_{K_X}) is replaced by some quotient of $\pi_1(U)$ (respectively, G_{K_X}) whose structure is better understood?

The first quotients that come into mind are the following. Let $\mathfrak{Primes}$ be the set of all prime numbers. Let $\Sigma = \Sigma_U \subset \mathfrak{Primes}$ be a set of prime numbers not containing p. For a profinite group Γ, Γ^Σ stands for the maximal pro-Σ quotient of Γ. The structure of $\pi_1(\overline{U})^\Sigma$ is well understood: it is isomorphic to the pro-Σ completion of a certain well-known finitely generated discrete group (i.e., either a free group or a surface group). Let $\Pi_U^{(\Sigma)} \stackrel{\text{def}}{=} \pi_1(U)/\operatorname{Ker}(\pi_1(\overline{U}) \twoheadrightarrow \pi_1(\overline{U})^\Sigma)$ be the corresponding quotient of $\pi_1(U)$. We shall refer to $\Pi_U^{(\Sigma)}$ as the geometrically pro-Σ étale fundamental group of U. In a similar way we can define the maximal pro-Σ quotient $G_{K_{\overline{X}}}^\Sigma$ of $G_{K_{\overline{X}}}$ and the corresponding quotient $G_{K_X}^{(\Sigma)}$ of G_{K_X}, which we will refer to as the geometrically pro-Σ quotient of the absolute Galois group G_{K_X}.

Question 2. Is it possible to prove any result analogous to the above Theorem 1.1 (respectively, 1.2) where $\pi_1(U)$ (respectively, G_{K_X}) is replaced by $\Pi_U^{(\Sigma)}$ (respectively, $G_{K_X}^{(\Sigma)}$), for a given set of prime numbers $\Sigma \subset \mathfrak{Primes}$ not containing p?

The first set Σ to consider is the set $\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \smallsetminus \{p\}$. In this case, we set $\Pi_U^{(\prime)} \stackrel{\text{def}}{=} \Pi_U^{(\Sigma)}$ and $G_K^{(\prime)} \stackrel{\text{def}}{=} G_K^{(\Sigma)}$, and we shall refer to them as the geometrically prime-to-characteristic quotients. We have the following results.

Theorem 1.3 (A Prime-to-p Version of Grothendieck's Anabelian Conjecture for Hyperbolic Curves over Finite Fields). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Write $\Pi_U^{(\prime)}$, $\Pi_V^{(\prime)}$ for the geometrically prime-to-characteristic quotients of $\pi_1(U)$, $\pi_1(V)$, respectively. Let*

$$\alpha : \Pi_U^{(\prime)} \xrightarrow{\sim} \Pi_V^{(\prime)}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U}^{(\prime)} & \xrightarrow{\sim} & \tilde{V}^{(\prime)} \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to $\Pi_U^{(\prime)}$, $\Pi_V^{(\prime)}$, respectively.

Theorem 1.3 was proved by the authors (cf. [ST1], Corollary 3.10). As a consequence of Theorem 1.3 one can deduce the following prime-to-characteristic version of Uchida's theorem (cf. [ST1], Corollary 3.11).

Theorem 1.4 (A Prime-to-p Version of Uchida's Theorem on Isomorphisms between Galois Groups of Function Fields). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively, and let $G_{K_X}^{(\prime)}$, $G_{K_Y}^{(\prime)}$ be their geometrically prime-to-characteristic quotients. Let*

$$\alpha : G_{K_X}^{(\prime)} \xrightarrow{\sim} G_{K_Y}^{(\prime)}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} \tilde{K}_X^{(\prime)} & \xrightarrow{\sim} & \tilde{K}_Y^{(\prime)} \\ \big\uparrow & & \big\uparrow \\ K_X & \xrightarrow{\sim} & K_Y \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the extensions corresponding to $G_{K_X}^{(\prime)}$, $G_{K_Y}^{(\prime)}$, respectively.

See §3 for another proof of Theorem 1.4.

In a more recent joint work, the authors proved the following refined version of Uchida's theorem which is stronger than Theorem 1.4.

Theorem 1.5 (A Refined Version of Uchida's Theorem on Isomorphisms between Galois Groups of Function Fields). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y of characteristics p_X, p_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively. Let $\Sigma_X \subset \mathfrak{Primes} \setminus \{p_X\}$, $\Sigma_Y \subset \mathfrak{Primes} \setminus \{p_Y\}$ be sets of primes, and set $\Sigma'_X \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus (\Sigma_X \cup \{p_X\})$, $\Sigma'_Y \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus (\Sigma_Y \cup \{p_Y\})$. Assume that neither the Σ'_X-cyclotomic character $\chi_{\Sigma'_X} : G_{k_X} \to \prod_{l \in \Sigma'_X} \mathbb{Z}_l^\times$ nor the Σ'_Y-cyclotomic character $\chi_{\Sigma'_Y} : G_{k_Y} \to \prod_{l \in \Sigma'_Y} \mathbb{Z}_l^\times$*

are injective. Let $G_{K_X}^{(\Sigma_X)}$, $G_{K_Y}^{(\Sigma_Y)}$ be the geometrically pro-Σ_X quotient of G_{K_X} and the geometrically pro-Σ_Y quotient of G_{K_Y}, respectively. Let

$$\alpha : G_{K_X}^{(\Sigma_X)} \xrightarrow{\sim} G_{K_Y}^{(\Sigma_Y)}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} \tilde{K}_X^{(\Sigma_X)} & \xrightarrow{\sim} & \tilde{K}_Y^{(\Sigma_Y)} \\ \uparrow & & \uparrow \\ K_X & \xrightarrow{\sim} & K_Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the field extensions corresponding to $G_{K_X}^{(\Sigma_X)}$, $G_{K_Y}^{(\Sigma_Y)}$, respectively.

We also proved the following refined version of Theorem 1.3.

Theorem 1.6 (A Refined Version of Grothendieck's Anabelian Conjecture for Hyperbolic Curves over Finite Fields). *Let X, Y be proper hyperbolic curves over finite fields k_X, k_Y of characteristics p_X, p_Y, respectively. Let $\Sigma_X \subset \mathfrak{Primes} \setminus \{p_X\}$, $\Sigma_Y \subset \mathfrak{Primes} \setminus \{p_Y\}$ be sets of primes, and set $\Sigma'_X \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus (\Sigma_X \cup \{p_X\})$, $\Sigma'_Y \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus (\Sigma_Y \cup \{p_Y\})$. Assume that neither the Σ'_X-adic representation $\rho_{\Sigma'_X} : G_{k_X} \to \prod_{l \in \Sigma'_X} GL_{2g_X}(\mathbb{Z}_l)$ nor the Σ'_Y-adic representation $\rho_{\Sigma'_Y} : G_{k_Y} \to \prod_{l \in \Sigma'_Y} GL_{2g_Y}(\mathbb{Z}_l)$, which arise from the actions on the Tate modules of the Jacobian varieties of $\overline{X}$, $\overline{Y}$, respectively, are injective (where g_X, g_Y denote the genus of X, Y, respectively). Write $\Pi_X^{(\Sigma_X)}$, $\Pi_Y^{(\Sigma_Y)}$ for the geometrically pro-Σ_X étale fundamental group of X and the geometrically pro-Σ_Y étale fundamental group of Y, respectively. Let*

$$\alpha : \Pi_X^{(\Sigma_X)} \xrightarrow{\sim} \Pi_Y^{(\Sigma_Y)}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{X}^{(\Sigma_X)} & \xrightarrow{\sim} & \tilde{Y}^{(\Sigma_Y)} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to $\Pi_X^{(\Sigma_X)}$, $\Pi_Y^{(\Sigma_Y)}$, respectively.

Note that the extra assumptions for Σ_X, Σ_Y in Theorems 1.5 and 1.6 are satisfied if Σ'_X, Σ'_Y are finite.

At the moment of writing this paper we do not know if a pro-l version of the above theorems holds, namely if the above Theorems 1.5 and 1.6 hold (under certain Frobenius-preserving assumptions) in the case where $\Sigma = \{l\}$ consists of a single prime number l which is different from p. It is very important for the anabelian geometry of hyperbolic curves over finite fields to know whether such a version holds or not.

§2. The Hom-form of the Grothendieck anabelian conjecture.

In this section we use the same notations as in §1, unless we specify otherwise. We expect that a Hom-form of the Grothendieck (birational) anabelian conjecture holds for hyperbolic curves over finite fields. These conjectures can be formulated as follows.

Conjecture 2.1 (The Hom-Form of the Grothendieck Anabelian Conjecture). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Let*

$$\alpha : \pi_1(U) \to \pi_1(V)$$

be a continuous open homomorphism between profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \tilde{V} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

in which the horizontal arrows are generically (pro-)étale morphisms and the vertical arrows are the profinite étale coverings corresponding to $\pi_1(U)$, $\pi_1(V)$, respectively.

Conjecture 2.2 (The Hom-Form of the Grothendieck Birational Anabelian Conjecture). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively. Let*

$$\alpha : G_{K_X} \to G_{K_Y}$$

be a continuous open homomorphism between profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} \tilde{K}_Y & \longrightarrow & \tilde{K}_X \\ \uparrow & & \uparrow \\ K_Y & \longrightarrow & K_X \end{array}$$

in which the horizontal arrows are separable embeddings and the vertical arrows are the extensions corresponding to G_{K_Y}, G_{K_X}, respectively.

Note that if Conjecture 2.2 is true then every continuous open homomorphism between absolute Galois groups $\alpha : G_{K_X} \to G_{K_Y}$ would be injective since it would arise from an embedding $K_Y \to K_X$ between corresponding fields. Clearly, in order to prove the above conjectures it suffices to prove them for continuous surjective homomorphisms.

The above conjectures seem to be quite difficult to prove for the moment. One of the main difficulties in proving these conjectures is the lack of a suitable "local theory" for continuous open homomorphisms between absolute Galois groups or fundamental groups. Consider for instance Conjecture 2.2, and let

$$\alpha : G_{K_X} \to G_{K_Y}$$

be a continuous open homomorphism between absolute Galois groups. One would like to show that the image $\alpha(D_x)$ of a decomposition subgroup D_x of G_{K_X} associated to a closed point $x \in X$ is contained and open in a decomposition subgroup D_y of G_{K_Y} associated to some closed point $y \in Y$. As a consequence one would obtain a set-theoretic map $X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$ between the sets of closed points of X and Y, respectively; one would hope to prove eventually that this map arises from a geometric map $X \to Y$. The "local theory" is in the heart of the proof of Uchida's theorem (Theorem 1.2) and its refined versions (Theorems 1.4 and 1.5). Such a "local theory" seems to be difficult to establish in the above context for the time being. In a recent joint work, the authors investigated the possibility of proving the above conjectures under suitable "local conditions", mainly assuming that such a "local theory" exists, and we succeeded in doing so with Conjecture 2.2.

More specifically, in [ST2] we investigated a class of homomorphisms between absolute Galois groups of function fields of curves over finite fields which we call proper. In what follows we use the following notations. Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y of characteristics p_X, p_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Write $G_{K_X} \overset{\mathrm{def}}{=} \mathrm{Gal}(K_X^{\mathrm{sep}}/K_X)$, $G_{K_Y} \overset{\mathrm{def}}{=} \mathrm{Gal}(K_Y^{\mathrm{sep}}/K_Y)$ for the absolute Galois groups of K_X, K_Y, respectively. Let $\tilde{X} \to X$, $\tilde{Y} \to Y$ be the profinite, generically étale covers corresponding to the absolute Galois groups G_{K_X}, G_{K_Y}, respectively. From now on, for a scheme S, we denote by S^{cl} the set of closed points of S.

Definition 2.3 (Well-Behaved Homomorphisms between Absolute Galois Groups). A continuous homomorphism

$$\alpha : G_{K_X} \to G_{K_Y}$$

between profinite groups is *well-behaved* if there exists a map

$$\tilde{\psi} : \tilde{X}^{\mathrm{cl}} \to \tilde{Y}^{\mathrm{cl}},\ \tilde{x} \mapsto \tilde{y},$$

such that

$$\alpha(D_{\tilde{x}}) \underset{\text{open}}{\subseteq} D_{\tilde{y}}$$

for any $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$, where $D_{\tilde{x}}$, $D_{\tilde{y}}$ denote the decomposition subgroups of G_{K_X}, G_{K_Y} at $\tilde{x}, \tilde{y}$, respectively.

In particular, given a well-behaved homomorphism $\alpha : G_{K_X} \to G_{K_Y}$ the map $\tilde{\psi} : \tilde{X}^{\mathrm{cl}} \to \tilde{Y}^{\mathrm{cl}},\ \tilde{x} \mapsto \tilde{y}$, is α-equivariant and induces naturally a map

$$\psi : X^{\mathrm{cl}} \to Y^{\mathrm{cl}},\ x \mapsto y,$$

where x, y denote the images of $\tilde{x}, \tilde{y}$ in $X^{\mathrm{cl}}, Y^{\mathrm{cl}}$, respectively.

Definition 2.4 (Proper Homomorphisms between Absolute Galois Groups). A well-behaved continuous homomorphism $\alpha : G_{K_X} \to G_{K_Y}$ between profinite groups is called *proper* if the induced map

$$\psi : X^{\mathrm{cl}} \to Y^{\mathrm{cl}},\ x \mapsto y,$$

has finite fibers, i.e., for each $y \in Y^{\mathrm{cl}}$ the pre-image $\psi^{-1}(y)$ is either an empty or a finite set.

Let $\alpha : G_{K_X} \to G_{K_Y}$ be a well-behaved homomorphism, and let x be a closed point of X. Then α induces naturally an open homomorphism $\alpha : D_x \to D_{\psi(x)}$ between the decomposition groups, an open homomorphism $\alpha : I_x \to I_{\psi(x)}$ between the inertia groups, and an open homomorphism $\alpha^{\mathrm{t}} : I_x^{\mathrm{t}} \to I_{\psi(x)}^{\mathrm{t}}$ between the tame inertia groups, at the points x and $\psi(x)$. (In particular, we have $p \stackrel{\mathrm{def}}{=} p_X = p_Y$.) Then α^{t} induces an isomorphism $I_x^{\mathrm{t}} \xrightarrow{\sim} (I_{\psi(x)}^{\mathrm{t}})^{e_x}$, where $e_x \stackrel{\mathrm{def}}{=} [I_{\psi(x)} : \alpha(I_x)]$. Taking the composite of this isomorphism $I_x^{\mathrm{t}} \xrightarrow{\sim} (I_{\psi(x)}^{\mathrm{t}})^{e_x}$ and the e_x-multiplication isomorphism $(I_{\psi(x)}^{\mathrm{t}})^{e_x} \xleftarrow{\sim} I_{\psi(x)}^{\mathrm{t}}$, we obtain an isomorphism

$$\tau_{x,\psi(x)} : I_x^{\mathrm{t}} \xrightarrow{\sim} I_{\psi(x)}^{\mathrm{t}}.$$

It is well-known that the tame inertia groups $I_x^{\mathrm{t}}, I_{\psi(x)}^{\mathrm{t}}$ are naturally isomorphic to the global modules of roots of unity $M_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (K_X^{\mathrm{sep}})^\times)$, $M_Y \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (K_Y^{\mathrm{sep}})^\times)$, respectively.

Definition 2.5 (Inertia-Rigid Homomorphisms between Absolute Galois Groups). A well-behaved continuous homomorphism $\alpha : G_{K_X} \to G_{K_Y}$ as above is called *inertia-rigid* if the above isomorphisms $\tau_{x,\psi(x)}$, where x runs over all closed points of X, are all identical to some (single) isomorphism

$$\tau : M_X \xrightarrow{\sim} M_Y$$

between the global modules of roots of unity.

In [ST2] we proved the following theorem.

Theorem 2.6. *Let*

$$\alpha : G_{K_X} \to G_{K_Y}$$

be a continuous open homomorphism between profinite groups, and assume that α is proper and inertia-rigid. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} \tilde{K}_Y & \longrightarrow & \tilde{K}_X \\ \uparrow & & \uparrow \\ K_Y & \longrightarrow & K_X \end{array}$$

in which the horizontal arrows are separable embeddings and the vertical arrows are the extensions corresponding to the Galois groups G_{K_Y}, G_{K_X}, respectively.

Next, we would like to discuss Conjecture 2.1, assuming for simplicity that the curves in question are proper. So, let X, Y be proper hyperbolic curves over finite fields k_X, k_Y of characteristics p_X, p_Y, respectively. Let

$$\alpha : \pi_1(X) \to \pi_1(Y)$$

be a continuous open homomorphism between profinite groups. We would like to show that α arises from some morphism of schemes $X \to Y$. The first major obstacle in proving this is, as in the Hom-form of the birational conjecture, the lack of a suitable "local theory" for continuous open homomorphisms between fundamental groups. Let x be a closed point of X and let $D_x \subset \pi_1(X)$ be a decomposition subgroup of $\pi_1(X)$ associated to x. (Observe that D_x is isomorphic to $\hat{\mathbb{Z}}$.) One would like to show that the image $\alpha(D_x)$ of D_x in $\pi_1(Y)$ is contained and open in a decomposition subgroup D_y of $\pi_1(Y)$ associated to some closed point $y \in Y$. As a consequence one would obtain a set-theoretic map $X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$ between the sets of closed points of X and Y, respectively; one would hope to prove eventually that this map arises from a geometric map $X \to Y$. The "local theory" is in the heart of the proof of Theorem 1.1 by Tamagawa and Mochizuki (and also of the authors' proof of Theorems 1.3 and 1.6) and was established for the Isom-form by Tamagawa in [T]. Unfortunately, Tamagawa's arguments fail for the Hom-form. Such a "local theory" seems to be difficult to establish in the above context for the time being.

Another obstacle in proving Conjecture 2.1 is the following problem related to the global modules of roots of unity. Let $M_X \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (K_X^{\mathrm{sep}})^\times)$, $M_Y \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (K_Y^{\mathrm{sep}})^\times)$ be the global modules of roots of unity associated to X, Y, respectively. (M_X, M_Y are isomorphic to the prime-to-p_X, prime-to-p_Y parts of $\hat{\mathbb{Z}}$, respectively.) Then we have natural identifications $M_X = \mathrm{Hom}_{\hat{\mathbb{Z}}}(H^2(\pi_1(\overline{X}), \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$, $M_Y = \mathrm{Hom}_{\hat{\mathbb{Z}}}(H^2(\pi_1(\overline{Y}), \hat{\mathbb{Z}}), \hat{\mathbb{Z}})$. Thus, α induces naturally a map

$$\tau : M_X \to M_Y,$$

which is Galois-equivariant with respect to α. One would like to show that the map τ is an open homomorphism (hence, in particular, that $p_X = p_Y$), which is necessarily the case if α arises from a non-constant morphism of schemes $X \to Y$. At the moment of writing this paper, we are not even able to prove that the map τ is non-zero.

In order to prove the assertion of Conjecture 2.1 it seems reasonable, for the time being, to impose some conditions on the homomorphism α. The following conditions seem to be quite natural in light of the above discussion.

Definition 2.7 (Proper Homomorphisms between Fundamental Groups). Let $\tilde{X} \to X$ (respectively, $\tilde{Y} \to Y$) be the pro-étale cover corresponding to $\pi_1(X)$ (respectively, $\pi_1(Y)$). A continuous homomorphism $\alpha : \pi_1(X) \to \pi_1(Y)$ between profinite groups is called *proper*, if there exists a map

$$\tilde{\psi} : \tilde{X}^{\mathrm{cl}} \to \tilde{Y}^{\mathrm{cl}},\ \tilde{x} \mapsto \tilde{y},$$

such that

$$\alpha(D_{\tilde{x}}) \underset{\text{open}}{\subseteq} D_{\tilde{y}}$$

for any $\tilde{x} \in \tilde{X}^{\mathrm{cl}}$, where $D_{\tilde{x}}$ $D_{\tilde{y}}$ denote the decomposition subgroups of $\pi_1(X), \pi_1(Y)$ at $\tilde{x}, \tilde{y}$, respectively, and that the map

$$\psi : X^{\mathrm{cl}} \to Y^{\mathrm{cl}},\ x \mapsto y,$$

induced naturally by $\tilde{\psi}$, has finite fibers.

Definition 2.8 (Non-Degenerate Homomorphisms between Fundamental Groups). A continuous homomorphism $\alpha : \pi_1(X) \to \pi_1(Y)$ between profinite groups is called *non-degenerate* if the natural map $\tau : M_X \to M_Y$, induced by α, is an open homomorphism.

Let $\alpha : \pi_1(X) \to \pi_1(Y)$ be a continuous open homomorphism between profinite groups, and assume that α is proper. Thus, we have a finite-to-one map $\psi : X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$, $x \mapsto y$. One would like to show that ψ (hence α) arises from a non-constant scheme morphism $f : X \to Y$. This expected scheme morphism $f : X \to Y$ may not be étale in general. One would like to reconstruct, group-theoretically from the homomorphism α, and the map ψ, what should be the ramification indices at the points of Y in the morphism $f : X \to Y$. This could be done as follows. Assume further that α is non-degenerate, thus giving rise to an open homomorphism $\tau : M_X \to M_Y$, and set $p \overset{\mathrm{def}}{=} p_X = p_Y$. Then τ induces an isomorphism $M_X \xrightarrow{\sim} \delta_\alpha M_Y$, where $\delta_\alpha \overset{\mathrm{def}}{=} [M_Y : \tau(M_X)]$. Taking the composite of this isomorphism $M_X \xrightarrow{\sim} \delta_\alpha M_Y$ and the δ_α-multiplication isomorphism $\delta_\alpha M_Y \xleftarrow{\sim} M_Y$, we obtain an isomorphism $\tilde{\tau} : M_X \xrightarrow{\sim} M_Y$, Galois-equivariant with respect to α. The maps $\tilde{\tau}^{-1} : M_Y \xrightarrow{\sim} M_X$ and $\alpha : \pi_1(X) \to \pi_1(Y)$ induce naturally a homomorphism

$$\beta : H^2(\pi_1(Y), M_Y) \to H^2(\pi_1(X), M_X)$$

between cohomology groups. Note that $H^2(\pi_1(X), M_X)$, $H^2(\pi_1(Y), M_Y)$ can be naturally identified with the étale cohomology groups $H^2(X, M_X)$, $H^2(Y, M_Y)$, respectively. Now, the ramification indices should appear as follows. Let y be a closed point of Y. Write $\psi^{-1}(y) = \{x_1, x_2, ..., x_n\}$, and let $c(y) \in H^2(\pi_1(Y), M_Y)$ (respectively, $c(x_i) \in H^2(\pi_1(X), M_X)$) be the Chern class of the line bundle $\mathcal{O}_Y(y)$ (respectively, $\mathcal{O}_X(x_i)$). If α arises from a non-constant morphism $f : X \to Y$ then we should have:

$$p^{a_f} \beta(c(y)) = \sum_{i=1}^{n} e_i c(x_i),$$

where p^{a_f} is the maximal p-power dividing $\deg(f)$ (so that $\deg(f) = p^{a_f}\delta_\alpha$) and e_i is the ramification index at x_i. (In particular, e_i is a positive integer.) This motivates the following definition.

Definition 2.9 (Chern Class Compatible Homomorphisms between Fundamental Groups). Let $\alpha : \pi_1(X) \to \pi_1(Y)$ be a continuous open homomorphism between profinite groups which is proper and non-degenerate. Thus, we have a natural map $\tilde{\psi} : \tilde{X}^{\mathrm{cl}} \to \tilde{Y}^{\mathrm{cl}}$ and a natural isomorphism $\tilde{\tau} : M_X \xrightarrow{\sim} M_Y$. The map α is called

Chern class compatible if there exists an $a \geq 0$ and a function $X^{\mathrm{cl}} \to \mathbb{Z}_{>0}, x \mapsto e_x$, such that for any finite sub-coverings $X' \to X$, $Y' \to Y$ of $\tilde{X} \to X$, $\tilde{Y} \to Y$ satisfying $\alpha(\pi_1(X')) \subset \pi_1(Y')$, for any $y' \in (Y')^{\mathrm{cl}}$, $(\psi')^{-1}(y') = \{x'_1, x'_2, ..., x'_n\}$ (where $\psi' : (X')^{\mathrm{cl}} \to (Y')^{\mathrm{cl}}$ is the map induced by $\tilde{\psi}$), we have:

$$p^a \beta'(c(y')) = \sum_{i=1}^{n} e_{x_i} c(x'_i)$$

where $c(y') \in H^2(\pi_1(Y'), M_Y)$ (respectively, $c(x'_i) \in H^2(\pi_1(X'), M_X)$) denotes the Chern class of the line bundle $\mathcal{O}_{Y'}(y')$ (respectively, $\mathcal{O}_{X'}(x'_i)$), x_i is the image of x'_i in X, and $\beta' : H^2(\pi_1(Y'), M_Y) \to H^2(\pi_1(X'), M_X)$ is the natural map between cohomology groups induced by $\tilde{\tau}^{-1} : M_Y \xrightarrow{\sim} M_X$ and $\alpha : \pi_1(X') \to \pi_1(Y')$.

It is plausible that one could prove the following.

Conjecture 2.10. *Let X, Y be proper hyperbolic curves over finite fields k_X, k_Y, respectively. Let*

$$\alpha : \pi_1(X) \to \pi_1(Y)$$

be a continuous open homomorphism between profinite groups, and assume that α is proper, non-degenerate, and Chern class compatible. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in which the horizontal arrows are generically (pro-)étale morphisms and the vertical arrows are the profinite étale coverings corresponding to $\pi_1(X)$, $\pi_1(Y)$, respectively.

§3. Another proof of the prime-to-characteristic version of Uchida's theorem.

In this section we give a proof of the prime-to-characteristic version of Uchida's theorem on isomorphisms between Galois groups of global fields in positive characteristics (cf. Theorem 1.4) which is different from the one given in [ST1], Corollary 3.11.

First we start by investigating isomorphisms between tame local Galois groups. For $i = 1, 2$, let $p_i > 0$ be a prime number. Let L_i be a complete discrete valuation field of equal characteristic p_i, with finite residue field ℓ_i. We denote the ring of integers of L_i by $\mathcal{O}_{L_i}$. We choose a separable closure L_i^{sep} of L_i, and write $G_{L_i} \overset{\mathrm{def}}{=} \mathrm{Gal}(L_i^{\mathrm{sep}}/L_i)$ for the corresponding absolute Galois group of L_i. Thus, by local class field theory (cf., e.g., [Se]), we have a natural isomorphism $(L_i^{\times})^{\wedge} \xrightarrow{\sim} G_{L_i}^{\mathrm{ab}}$, where $(L_i^{\times})^{\wedge}$ denotes the completion of the multiplicative group $L_i^{\times}$, and $G_{L_i}^{\mathrm{ab}}$ denotes the maximal abelian quotient of G_{L_i}. In particular, $G_{L_i}^{\mathrm{ab}}$ fits into an exact sequence

$$0 \to \mathcal{O}_{L_i}^{\times} \to G_{L_i}^{\mathrm{ab}} \to \hat{\mathbb{Z}} \to 0$$

(arising from a similar exact sequence for $(L_i^\times)^\wedge$), where $\mathcal{O}_{L_i}^\times$ is the group of multiplicative units in $\mathcal{O}_{L_i}$. Moreover, we obtain natural inclusions

$$\ell_i^\times \hookrightarrow \mathcal{O}_{L_i}^\times \subset L_i^\times \hookrightarrow G_{L_i}^{\mathrm{ab}},$$

and

$$\mathbb{Z} \xleftarrow{\sim} L_i^\times/\mathcal{O}_{L_i}^\times \hookrightarrow G_{L_i}^{\mathrm{ab}}/\operatorname{Im}(\mathcal{O}_{L_i}^\times),$$

where $\xleftarrow{\sim}$ is induced by the valuation and $1 \in \mathbb{Z}$ maps to the Frobenius element in $G_{L_i}^{\mathrm{ab}}/\operatorname{Im}(\mathcal{O}_{L_i}^\times)$. For $i = 1, 2$, write $D_i \overset{\mathrm{def}}{=} \mathrm{Gal}(L_i^{\mathrm{t}}/L_i)$ for the Galois group of the maximal tamely ramified sub-extension L_i^{t}/L_i of L_i^{sep}/L_i. The above isomorphism $(L_i^\times)^\wedge \xrightarrow{\sim} G_{L_i}^{\mathrm{ab}}$ induces a natural isomorphism $(L_i^\times)^\wedge/U_i^1 \xrightarrow{\sim} D_i^{\mathrm{ab}}$, where U_i^1 is the group of principal units in $\mathcal{O}_{L_i}^\times$, and D_i^{ab} is the maximal abelian quotient of D_i.

Proposition 3.1 (Invariants of Arbitrary Isomorphisms between Tame Decomposition Groups). *Let*

$$\tau : D_1 \xrightarrow{\sim} D_2$$

be an isomorphism between profinite groups. Then the followings hold:
(i) *The equality $p_1 = p_2$ holds. (Set $p \overset{\mathrm{def}}{=} p_1 = p_2$.)*
(ii) *The isomorphism τ induces a natural isomorphism $\ell_1^\times \xrightarrow{\sim} \ell_2^\times$ between the multiplicative groups of residue fields. In particular, ℓ_1 and ℓ_2 have the same cardinality.*
(iii) *We have $\tau(I_1) = I_2$, where I_i is the inertia subgroup of D_i.*
(iv) *The natural isomorphism $\tau^{\mathrm{ab}} : D_1^{\mathrm{ab}} \to D_2^{\mathrm{ab}}$ induced by τ preserves the images* $\operatorname{Im}(\ell_i^\times)$, $\operatorname{Im}(L_i^\times/U_i^1)$, *of the natural homomorphisms discussed above. Further, the isomorphism $D_1^{\mathrm{ab}}/\operatorname{Im}(\ell_1^\times) \to D_2^{\mathrm{ab}}/\operatorname{Im}(\ell_2^\times)$ induced by τ preserves the respective Frobenius elements.*
(v) *τ induces naturally an isomorphism $M_{\ell_1} \xrightarrow{\sim} M_{\ell_2}$, which is Galois-equivariant with respect to τ, where $M_{\ell_i} \overset{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (\overline{\ell_i})^\times)$ is the module of roots of unity. (Here $\overline{\ell_i}$ denotes the algebraic closure of ℓ_i in L_i^{sep}.) In particular, τ commutes with the cyclotomic characters $\chi_i : D_i \to (\hat{\mathbb{Z}}')^\times$ of D_i, where $\hat{\mathbb{Z}}'$ stands for the prime-to-p part of $\hat{\mathbb{Z}}$.*

Proof. This is (well-known and) easy to prove. For more details, see, e.g., [ST2]. □

Next we will consider isomorphisms between geometrically prime-to-characteristic quotients of absolute Galois groups of function fields of curves over finite fields. We will use the following notations. For $i = 1, 2$, let X_i be a proper, smooth, and geometrically connected curve over a finite field k_i of characteristic $p_i > 0$. Let $K_i = K_{X_i}$ be the function field of X_i. Write $G_i \overset{\mathrm{def}}{=} \mathrm{Gal}(K_i^{\mathrm{sep}}/K_i)$ for the absolute Galois group of K_i. For $i = 1, 2$, let $G_i^{(\prime)}$ be the geometrically prime-to-p_i quotient of G_i. Let $\tilde{X}_i \to X_i$ be the profinite, generically étale cover corresponding to $G_i^{(\prime)}$. Our aim is to prove the following.

Theorem 3.2 (= Theorem 1.4 with different notations)**.** *Let*

$$\alpha : G_1^{(\prime)} \xrightarrow{\sim} G_2^{(\prime)}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} \tilde{K}_1 & \xrightarrow{\sim} & \tilde{K}_2 \\ \uparrow & & \uparrow \\ K_1 & \xrightarrow{\sim} & K_2 \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the extensions corresponding to $G_1^{(\prime)}$, $G_2^{(\prime)}$, respectively. (*Thus, $\tilde{K}_1$, $\tilde{K}_2$ are the function fields of $\tilde{X}_1$, $\tilde{X}_2$, respectively.*)

Theorem 3.2 was proved in [ST1] as a corollary of Theorem 1.3. In what follows we will give another proof of Theorem 3.2, which is very much inspired by Uchida's proof of Theorem 1.2 and uses class field theory. For the rest of this section we will consider an isomorphism

$$\alpha : G_1^{(\prime)} \xrightarrow{\sim} G_2^{(\prime)}$$

between profinite groups.

Lemma 3.3 (Local Theory). *Let $\tilde{x}_1 \in \tilde{X}_1^{\mathrm{cl}}$. Then $\alpha(D_{\tilde{x}_1}) = D_{\tilde{x}_2}$ for a unique point $\tilde{x}_2 \in \tilde{X}_2^{\mathrm{cl}}$, where $D_{\tilde{x}_i}$ stands for the decomposition subgroup of $G_i^{(\prime)}$ at the point $\tilde{x}_i$, for $i = 1, 2$. Further, $\alpha(I_{\tilde{x}_1}) = I_{\tilde{x}_2}$, where $I_{\tilde{x}_i}$ stands for the inertia subgroup of $G_i^{(\prime)}$ at the point $\tilde{x}_i$, for $i = 1, 2$.*

Accordingly, α induces naturally a bijection $\tilde{\psi} : \tilde{X}_1^{\mathrm{cl}} \to \tilde{X}_2^{\mathrm{cl}}$, $\tilde{x}_1 \mapsto \tilde{x}_2$, such that $\alpha(D_{\tilde{x}_1}) = D_{\tilde{x}_2}$. Further, $\tilde{\psi}$ induces naturally a bijection $\psi : X_1^{\mathrm{cl}} \to X_2^{\mathrm{cl}}$.

Proof. Similar to the proof of [U], Lemmas 3 and 4. See also [ST1], Remark 3.12. □

Lemma 3.4 (Invariance of the Characteristic). *The equality $p_1 = p_2$ holds.* (*Set $p \stackrel{\mathrm{def}}{=} p_1 = p_2$.*)

Proof. This follows for example by considering the p_i-torsion in $G_i^{(\prime),\mathrm{ab}}$. Indeed, for $i = 1, 2$, p_i can be characterized as the unique prime number p such that, for any open subgroup H_i of $G_i^{(\prime)}$, the p-torsion in H_i^{ab} is trivial, as follows easily from class field theory. □

Lemma 3.5. *The isomorphism α commutes with the canonical projections $\mathrm{pr}_i : G_i^{(\prime)} \to G_{k_i}$, $i = 1, 2$, i.e., we have a commutative diagram:*

$$\begin{array}{ccc} G_1^{(\prime)} & \xrightarrow{\alpha} & G_2^{(\prime)} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle \mathrm{pr}_2}\downarrow \\ G_{k_1} & \longrightarrow & G_{k_2} \end{array}$$

where the horizontal arrows are isomorphisms. Further, the following diagram is commutative:

$$\begin{tikzcd} G_1^{(\prime)} \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_1"'] & G_2^{(\prime)} \arrow[d, "\mathrm{pr}_2"'] \\ G_{k_1} \arrow[r] \arrow[d, "\chi_{k_1}"'] & G_{k_1} \arrow[d, "\chi_{k_2}"'] \\ (\hat{\mathbb{Z}}')^\times \arrow[r, equal] & (\hat{\mathbb{Z}}')^\times \end{tikzcd} \tag{3.1}$$

where χ_{k_i} is the cyclotomic character of G_{k_i}, $i = 1, 2$. In particular, $\sharp(k_1) = \sharp(k_2)$, and the natural isomorphism $G_{k_1} \to G_{k_2}$ induced by α maps the $\sharp(k_1)$-th power Frobenius element φ_{k_1} of G_{k_1} to the $\sharp(k_2)$-th power Frobenius element φ_{k_2} of G_{k_2}.

Proof. First, observe that G_{k_i} is the unique (up to isomorphism) quotient of $G_i^{(\prime)}$ which is isomorphic to $\hat{\mathbb{Z}}$, as follows from the structure of G_i^{ab} given by global class field theory, $i = 1, 2$. The first assertion follows from this.

Next, we prove the second assertion. For each $\tilde{x}_1 \in \tilde{X}_1^{\mathrm{cl}}$, with $\tilde{x}_2 = \tilde{\psi}(\tilde{x}_1)$, we have the following diagram:

$$\begin{tikzcd} D_{\tilde{x}_1} \arrow[r] \arrow[d] & D_{\tilde{x}_2} \arrow[d] \\ G_1^{(\prime)} \arrow[r, "\alpha"] \arrow[d, "\chi_1"'] & G_2^{(\prime)} \arrow[d, "\chi_2"'] \\ (\hat{\mathbb{Z}}')^\times \arrow[r, equal] & (\hat{\mathbb{Z}}')^\times \end{tikzcd}$$

where the maps $D_{\tilde{x}_i} \to G_i^{(\prime)}$ are the inclusions, the upper square is commutative, and $\chi_i \stackrel{\mathrm{def}}{=} \chi_{k_i} \circ \mathrm{pr}_i$ is the cyclotomic character of $G_i^{(\prime)}$, $i = 1, 2$. Observe that the restriction of χ_i to $D_{\tilde{x}_i}$ coincides with the cyclotomic character $\chi_{\tilde{x}_i}$ of $D_{\tilde{x}_i}$, $i = 1, 2$. The isomorphism $D_{\tilde{x}_1} \xrightarrow{\sim} D_{\tilde{x}_2}$ commutes with the cyclotomic characters $\chi_{\tilde{x}_i}$, $i = 1, 2$, by Proposition 3.1 (v). Hence $\alpha : G_1^{(\prime)} \xrightarrow{\sim} G_2^{(\prime)}$ commutes with the cyclotomic characters χ_i, $i = 1, 2$, since $G_1^{(\prime)}$ is topologically generated by the decomposition subgroups $D_{\tilde{x}_1}$ ($\tilde{x}_1 \in \tilde{X}_1^{\mathrm{cl}}$), as follows from Chebotarev's density theorem. Further, we have a diagram of maps (3.1), where the exterior and the upper squares are commutative. Thus, the lower square in diagram (3.1) is also commutative. Further, as is well known, the cyclotomic character $\chi_{k_i} : G_{k_i} \to (\hat{\mathbb{Z}}')^\times$ is injective, and the image $\chi_{k_i}(< \varphi_{k_i} >)$ of the subgroup generated by the Frobenius element φ_{k_i} is contained in $p^{\mathbb{Z}}$. Moreover, the image $\chi_{k_i}(\varphi_{k_i})$ of φ_{k_i} equals $\sharp(k_i)$. The second assertion follows from this, since G_{k_i} is topologically generated by φ_{k_i}, $i = 1, 2$. $\square$

Lemma 3.6 (Invariance of the Module of Roots of Unity). *For $i = 1, 2$, let $M_{X_i} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (K_i^{\mathrm{sep}})^\times)$ be the global module of roots of unity associated to X_i. Then α induces a natural isomorphism $M_{X_1} \xrightarrow{\sim} M_{X_2}$ which is Galois-equivariant with respect to α.*

Proof. For $i = 1, 2$, let $G_i^{(\prime),\mathrm{ab}}$ be the maximal abelian quotient of $G_i^{(\prime)}$. We have the following commutative diagram:

(3.2)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & k_1^\times & \longrightarrow & \prod_{x_1\in X_1^{\mathrm{cl}}} k(x_1)^\times & \longrightarrow & G_1^{(\prime),\mathrm{ab}} & \longrightarrow & \pi_1(X_1)^{(\prime),\mathrm{ab}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k_2^\times & \longrightarrow & \prod_{x_2\in X_2^{\mathrm{cl}}} k(x_2)^\times & \longrightarrow & G_2^{(\prime),\mathrm{ab}} & \longrightarrow & \pi_1(X_2)^{(\prime),\mathrm{ab}} & \longrightarrow & 1
\end{array}$$

where the map $\prod_{x_i\in X_i^{\mathrm{cl}}} k(x_i)^\times \to G_i^{(\prime),\mathrm{ab}}$ is naturally induced by Artin's reciprocity map of global class field theory. Here $\pi_1(X_i)^{(\prime),\mathrm{ab}}$ denotes the maximal abelian quotient of the geometrically prime-to-p quotient $\pi_1(X_i)^{(\prime)}$ of $\pi_1(X_i)$. In particular, the image of $k(x_i)^\times$ in $G_i^{(\prime),\mathrm{ab}}$ coincides with the image of the inertia subgroup $I_{\tilde{x}_i}$, where $\tilde{x}_i \in \tilde{X}_i^{\mathrm{cl}}$ is any point above x_i, via the natural map $G_i^{(\prime)} \twoheadrightarrow G_i^{(\prime),\mathrm{ab}}$. The maps $G_i^{(\prime),\mathrm{ab}} \twoheadrightarrow \pi_1(X_i)^{(\prime),\mathrm{ab}}$ are the natural ones, and the maps $k_i^\times \to \prod_{x_i\in X_i^{\mathrm{cl}}} k(x_i)^\times$ are the natural diagonal embeddings, $i = 1, 2$. The map $\prod_{x_1\in X_1^{\mathrm{cl}}} k(x_1)^\times \to \prod_{x_2\in X_2^{\mathrm{cl}}} k(x_2)^\times$ maps each component $k(x_1)^\times$ onto $k(x_2)^\times$, where $x_2 \overset{\mathrm{def}}{=} \psi(x_1)$, via the natural isomorphism in Proposition 3.1 (ii), which is induced by α. In particular, this map is a bijection since ψ is a bijection. Thus, the far left vertical map gives a natural isomorphism $k_1^\times \xrightarrow{\sim} k_2^\times$. Passing to open subgroups of $G_i^{(\prime)}$, $i = 1, 2$, corresponding to extensions of the constant fields, and passing to the projective limit via the natural maps, we obtain the desired isomorphism $M_{X_1} \xrightarrow{\sim} M_{X_2}$, which is Galois-equivariant with respect to α by construction. □

Let $\tilde{x}_1 \in \tilde{X}_1^{\mathrm{cl}}$ and set $\tilde{x}_2 \overset{\mathrm{def}}{=} \tilde{\psi}(\tilde{x}_1) \in \tilde{X}_2^{\mathrm{cl}}$. Then α induces naturally an isomorphism $D_{\tilde{x}_1} \xrightarrow{\sim} D_{\tilde{x}_2}$ (cf. Lemma 3.3). In particular, α induces a natural isomorphism $M_{k(x_1)} \xrightarrow{\sim} M_{k(x_2)}$, Galois-equivariant with respect to the isomorphism $D_{\tilde{x}_1} \xrightarrow{\sim} D_{\tilde{x}_2}$, where x_i is the image of $\tilde{x}_i$ in X_i, $i = 1, 2$ (cf. Proposition 3.1 (v)). The following is a rigidity statement concerning the various isomorphisms of the local modules of roots of unity.

Lemma 3.7 (Inertia-Rigidity). *The following diagram is commutative:*

$$\begin{array}{ccc}
M_{X_1} & \longrightarrow & M_{X_2} \\
\downarrow & & \downarrow \\
M_{k(x_1)} & \longrightarrow & M_{k(x_2)}
\end{array}$$

where the upper horizontal arrow is the isomorphism in Lemma 3.6*, the lower horizontal arrow is the isomorphism in* Proposition 3.1 (v)*, and the vertical maps are the natural isomorphisms. Further, this diagram is Galois-equivariant with respect to the commutative diagram:*

$$\begin{array}{ccc} D_{\tilde{x}_1} & \longrightarrow & D_{\tilde{x}_2} \\ \downarrow & & \downarrow \\ G_1^{(\prime)} & \xrightarrow{\alpha} & G_2^{(\prime)} \end{array}$$

where the vertical maps are the inclusions.

Proof. Indeed, the far left square in diagram (3.2) induces a commutative diagram:

$$\begin{array}{ccc} k_1^{\times} & \longrightarrow & k_2^{\times} \\ \downarrow & & \downarrow \\ k(x_1)^{\times} & \longrightarrow & k(x_2)^{\times} \end{array}$$

where the vertical arrows are the natural inclusions. Passing to open subgroups of $G_i^{(\prime),\mathrm{ab}}$, $i = 1, 2$, corresponding to extensions (containing $k(x_i)$) of the constant fields, and passing to the projective limit via the natural maps, we obtain the desired Galois-equivariant diagram. $\square$

Lemma 3.8 (Recovering the Multiplicative Group). *Let p^{n_i} be the exponent of the p-primary part $\pi_1(X_i)^{\mathrm{ab,tor},p}$ of the torsion subgroup $\pi_1(X_i)^{\mathrm{ab,tor}}$ of $\pi_1(X_i)^{\mathrm{ab}}$, $i = 1, 2$. (Note that $\pi_1(X_i)^{\mathrm{ab,tor}}$ is naturally identified with $J_{X_i}(k_i)$, where J_{X_i} is the Jacobian variety of X_i.) Then α induces naturally an injective homomorphism*

$$\gamma : K_1^{\times} \to (K_2^{p^{-n_2}})^{\times}$$

between multiplicative groups, where $K_2^{p^{-n_2}}$ denotes the field of p^{n_2}-th roots of elements of K_2.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \operatorname{Ker}\psi_1' & \longrightarrow & I_{K_1}' & \xrightarrow{\psi_1'} & G_1^{(\prime),\mathrm{ab}} \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \operatorname{Ker}\psi_2' & \longrightarrow & I_{K_2}' & \xrightarrow{\psi_2'} & G_2^{(\prime),\mathrm{ab}} \end{array} \tag{3.3}$$

in which both rows are exact. Here, $I'_{K_i} \stackrel{\mathrm{def}}{=} \prod'_{x_i \in X_i^{\mathrm{cl}}} (K_i)_{x_i}^{\times}/U_{x_i}^1$ is a quotient of the idèle group $I_{K_i} \stackrel{\mathrm{def}}{=} \prod'_{x_i \in X_i^{\mathrm{cl}}} (K_i)_{x_i}^{\times}$ of K_i, $(K_i)_{x_i}$ denotes the x_i-adic completion of K_i, $\mathcal{O}_{x_i}$ is the ring of integers of $(K_i)_{x_i}$, and $U_{x_i}^1 \subset \mathcal{O}_{x_i}^{\times}$ is the group of principal units. The map $\psi_i' : I'_{K_i} \to G_i^{(\prime),\mathrm{ab}}$ is naturally induced by Artin's reciprocity map in global class field theory. The far right vertical map is naturally induced by α, and the middle vertical map $I'_{K_1} \to I'_{K_2}$ maps each component $(K_1)_{x_1}^{\times}/U_{x_1}^1$ to $(K_2)_{x_2}^{\times}/U_{x_2}^1$, $x_2 \stackrel{\mathrm{def}}{=} \phi(x_1)$, via the natural isomorphism in Proposition 3.1 (iv), which is naturally induced by α. In particular, the map $I'_{K_1} \to I'_{K_2}$ is an isomorphism. Thus, the far left vertical map is a natural isomorphism $\operatorname{Ker}\psi_1' \to \operatorname{Ker}\psi_2'$ between kernels of Artin's maps. We claim:

Claim 1. For $i = 1, 2$, $\operatorname{Ker}\psi_i'$ inserts naturally into the following exact sequence:

$$1 \to K_i^\times \to \operatorname{Ker}\psi_i' \to \pi_1(X_i)^{\mathrm{ab,tor},p} \to 1.$$

Assuming this claim, we then have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K_1^\times \arrow[r] & \operatorname{Ker}\psi_1' \arrow[r] \arrow[d] & \pi_1(X_1)^{\mathrm{ab,tor},p} \arrow[r] & 1 \\
1 \arrow[r] & K_2^\times \arrow[r] & \operatorname{Ker}\psi_2' \arrow[r] & \pi_1(X_2)^{\mathrm{ab,tor},p} \arrow[r] & 1
\end{tikzcd}$$

in which both rows are exact and the vertical arrow is the above isomorphism. This isomorphism has, a priori, no reason to map $K_1^\times$ into $K_2^\times$. However, since $\pi_1(X_2)^{\mathrm{ab,tor},p}$ is a finite group of exponent p^{n_2}, we can conclude that the above isomorphism $\operatorname{Ker}\psi_1' \to \operatorname{Ker}\psi_2'$ maps $(K_1^\times)^{p^{n_2}} = (K_1^{p^{n_2}})^\times$ injectively into $K_2^\times$. Thus, we obtain a natural injective map $\gamma : K_1^\times \to (K_2^{p^{-n_2}})^\times$. It remains to prove the above claim.

For $i = 1, 2$, we have the following commutative diagram:

$$\begin{tikzcd}
 & & 1 & 1 & \\
 & & \hat{\mathbb{Z}}/\mathbb{Z} \arrow[u] \arrow[r, equal] & \hat{\mathbb{Z}}/\mathbb{Z} \arrow[u] & \\
1 \arrow[r] & \operatorname{Ker}\rho_i \arrow[r] & G_i^{\mathrm{ab}} \arrow[u] \arrow[r, "\rho_i"] & G_i^{(\prime),\mathrm{ab}} \arrow[u] \arrow[r] & 1 \\
1 \arrow[r] & \prod_{x_i \in X_i^{\mathrm{cl}}} U_{x_i}^1 \arrow[u] \arrow[r] & I_{K_i} \arrow[u, "\psi_i"] \arrow[r] & I'_{K_i} \arrow[u, "\psi_i'"] \arrow[r] & 1 \\
 & 1 \arrow[u] \arrow[r] & K_i^\times \arrow[u] \arrow[r] & \operatorname{Ker}\psi_i' \arrow[u] & \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

in which the rows and the columns are all exact. Here, the map $\psi_i : I_{K_i} \to G_i^{\mathrm{ab}}$ is Artin's reciprocity map in global class field theory, and the map $\rho_i : G_i^{\mathrm{ab}} \twoheadrightarrow G_i^{(\prime),\mathrm{ab}}$ is the natural map. Further, $I_{K_i} \to I'_{K_i}$ is the natural map which maps each component $(K_i)_{x_i}^\times$ canonically onto $(K_i)_{x_i}^\times / U_{x_i}^1$. In particular, we deduce that the cokernel of the injective map $\prod_{x_i \in X_i^{\mathrm{cl}}} U_{x_i}^1 \to \operatorname{Ker}\rho_i$ is naturally isomorphic to the cokernel of $K_i^\times \to \operatorname{Ker}\psi_i'$. Further, we claim:

Claim 2. The cokernel of the above injective homomorphism $\prod_{x_i \in X_i^{\mathrm{cl}}} U_{x_i}^1 \to \operatorname{Ker}\rho_i$ is naturally isomorphic to $\pi_1(X_i)^{\mathrm{ab,tor},p}$.

Indeed, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & & k_i^{\times} & = & k_i^{\times} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \prod_{x_i\in X_i^{\mathrm{cl}}} U_{x_i}^1 & \longrightarrow & \prod_{x_i\in X_i^{\mathrm{cl}}} \mathcal{O}_{x_i}^{\times} & \longrightarrow & \prod_{x_i\in X_i^{\mathrm{cl}}} k(x_i)^{\times} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}(\rho_i) & \longrightarrow & G_i^{\mathrm{ab}} & \xrightarrow{\rho_i} & G_i^{(\prime),\mathrm{ab}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}(\nu_i) & \longrightarrow & \pi_1(X_i)^{\mathrm{ab}} & \xrightarrow{\nu_i} & \pi_1(X_i)^{(\prime),\mathrm{ab}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

in which the rows and the columns are all exact. Here, the maps $G_i^{\mathrm{ab}} \to \pi_1(X_i)^{\mathrm{ab}}$ and $G_i^{(\prime),\mathrm{ab}} \to \pi_1(X_i)^{(\prime),\mathrm{ab}}$ are the natural maps, the map $\prod_{x_i\in X_i^{\mathrm{cl}}} \mathcal{O}_{x_i}^{\times} \to G_i^{\mathrm{ab}}$ is the restriction of Artin's reciprocity map, and the map $k_i^{\times} \to \prod_{x_i\in X_i^{\mathrm{cl}}} \mathcal{O}_{x_i}^{\times}$ is the natural diagonal embedding which maps into $\prod_{x_i\in X_i^{\mathrm{cl}}} k(x_i)^{\times}$. Further, the kernel $\mathrm{Ker}(\nu_i)$ of the natural map $\nu_i : \pi_1(X_i)^{\mathrm{ab}} \to \pi_1(X_i)^{(\prime),\mathrm{ab}}$ coincides with the kernel of the natural map $\pi_1(X_i)^{\mathrm{ab,tor}} \to \pi_1(X_i)^{(\prime),\mathrm{ab,tor}}$, hence is canonically isomorphic to $\pi_1(X_i)^{\mathrm{ab,tor},p}$. Thus, our second claim, hence our first claim, are proved. $\square$

Let U_2 be an open subgroup of $G_2^{(\prime)}$, and let $U_1 \overset{\mathrm{def}}{=} \alpha^{-1}(U_2)$. Let F_i/K_i be the subextension of $\tilde{K}_i/K_i$ corresponding to U_i, for $i = 1, 2$. Then α induces, by restriction to U_1, an isomorphism $\alpha : U_1 \xrightarrow{\sim} U_2$, which induces an injective homomorphism

$$\gamma' : F_1^{\times} \to (F_2^{p^{-n_2'}})^{\times}$$

by Lemma 3.8. Here, $p^{n_i'}$ denotes the exponent of $\pi_1(Y_i)^{\mathrm{ab,tor},p}$, where Y_i is the normalization of X_i in F_i.

Lemma 3.9. *We have $n_i' \geq n_i$. Further, the map $\gamma' : F_1^{\times} \to (F_2^{p^{-n_2'}})^{\times}$ is an extension of the map $\gamma : K_1^{\times} \to (K_2^{p^{-n_2}})^{\times}$.*

Proof. For $i = 1, 2$, let k_i' be the constant field of Y_i. Set $X_i' \overset{\mathrm{def}}{=} X_i \times_{k_i} k_i'$ and let $K_i' = K_i k_i'$ be the function field of X_i'. Then $Y_i \to X_i$ naturally factors as $Y_i \to X_i' \to X_i$. Accordingly, we obtain natural maps $J_{X_i}(k_i)[p^{\infty}] \to J_{X_i}(k_i')[p^{\infty}] = J_{X_i'}(k_i')[p^{\infty}] \to J_{Y_i}(k_i')[p^{\infty}]$, where J_{X_i}, $J_{X_i'}$, J_{Y_i} are the Jacobian varieties of X_i, X_i', Y_i, respectively, and, for an abelian group M, $M[p^\infty]$ stands for the subgroup of p-primary torsion of M. Here, the first map is clearly injective. Since the degree of the extension F_i/K'_i is prime to p, the morphism $J_{X'_i} \to J_{Y_i}$ has finite kernel of order prime to p. Thus, the second map is also injective, and the inequality $n'_i \geq n_i$ follows.

For $i = 1, 2$, global class field theory gives a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & F_i^\times \arrow[r] & I_{F_i} \arrow[r, "\tilde{\psi}_i"] & H_i^{\mathrm{ab}} \\
1 \arrow[r] & K_i^\times \arrow[r] \arrow[u] & I_{K_i} \arrow[r, "\psi_i"] \arrow[u] & G_i^{\mathrm{ab}} \arrow[u, "\mathrm{tr}"]
\end{tikzcd}$$

in which both rows are exact, and ψ_i, $\tilde{\psi}_i$ are Artin's reciprocity maps. The map $I_{K_i} \to I_{F_i}$ is the natural embedding which maps each component $(K_i)_{x_i}^\times$ to $\prod_{y_{i,j}} (F_i)_{y_{i,j}}^\times$ via the natural diagonal embedding, where $y_{i,j}$ runs over all points of Y_i above x_i. In particular, the far left vertical map is the natural embedding $K_i^\times \to F_i^\times$. The map $\mathrm{tr} : G_i^{\mathrm{ab}} \to H_i^{\mathrm{ab}}$, where H_i denotes the open subgroup of G_i corresponding to U_i, is the transfer map.

Further, the transfer map $\mathrm{tr} : G_i^{(\prime),\mathrm{ab}} \to U_i^{\mathrm{ab}}$ inserts into the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Ker} \tilde{\psi}'_i \arrow[r] & I'_{F_i} \arrow[r, "\tilde{\psi}'_i"] & U_i^{\mathrm{ab}} \\
1 \arrow[r] & \operatorname{Ker} \psi'_i \arrow[r] \arrow[u] & I'_{K_i} \arrow[r, "\psi'_i"] \arrow[u] & G_i^{(\prime),\mathrm{ab}} \arrow[u, "\mathrm{tr}"]
\end{tikzcd}$$

in which the horizontal rows are exact, the middle vertical arrow is the natural embedding induced by the above map $I_{K_i} \to I_{F_i}$, and the maps ψ'_i, $\tilde{\psi}'_i$ are induced by Artin's reciprocity maps. Thus, the far left vertical arrow is a natural embedding $\operatorname{Ker} \psi'_i \to \operatorname{Ker} \tilde{\psi}'_i$ between kernels of Artin's maps. Moreover, we have the following commutative diagrams:

$$\begin{tikzcd}
H_i^{\mathrm{ab}} \arrow[r] & U_i^{\mathrm{ab}} \\
G_i^{\mathrm{ab}} \arrow[r] \arrow[u, "\mathrm{tr}"] & G_i^{(\prime),\mathrm{ab}} \arrow[u, "\mathrm{tr}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
I_{F_i} \arrow[r] & I'_{F_i} \\
I_{K_i} \arrow[r] \arrow[u] & I'_{K_i} \arrow[u]
\end{tikzcd}$$

which commute with Artin's reciprocity maps. Thus, the above natural embedding $\operatorname{Ker} \psi'_i \to \operatorname{Ker} \tilde{\psi}'_i$ inserts into the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & F_i^\times \arrow[r] & \operatorname{Ker} \tilde{\psi}'_i \arrow[r] & \pi_1(Y_i)^{\mathrm{ab,tor},p} \arrow[r] & 1 \\
1 \arrow[r] & K_i^\times \arrow[r] \arrow[u] & \operatorname{Ker} \psi'_i \arrow[r] \arrow[u] & \pi_1(X_i)^{\mathrm{ab,tor},p} \arrow[r] & 1
\end{tikzcd}$$

Further, we have commutative diagrams:

$$\begin{tikzcd} U_1^{\mathrm{ab}} \arrow[r] & U_2^{\mathrm{ab}} \\ G_1^{(\prime)\,\mathrm{ab}} \arrow[r] \arrow[u, "\mathrm{tr}"] & G_2^{(\prime),\mathrm{ab}} \arrow[u, "\mathrm{tr}"] \end{tikzcd}$$

and

$$\begin{tikzcd} I'_{F_1} \arrow[r] & I'_{F_2} \\ I'_{K_1} \arrow[r] \arrow[u] & I'_{K_2} \arrow[u] \end{tikzcd}$$

in which the horizontal arrows are isomorphisms induced by α (cf. diagram (3.3)). From this we deduce that the commutative diagram:

$$\begin{tikzcd} \operatorname{Ker} \tilde{\psi}'_1 \arrow[r] & \operatorname{Ker} \tilde{\psi}'_2 \\ \operatorname{Ker} \psi'_1 \arrow[r] \arrow[u] & \operatorname{Ker} \psi'_2 \arrow[u] \end{tikzcd}$$

in which the vertical (respectively, horizontal) arrows are the above natural embeddings (respectively, the isomorphisms induced by α), induces a commutative diagram:

$$\begin{tikzcd} (F_1^{p^{n'_2}})^{\times} \arrow[r] & F_2^{\times} \\ (K_1^{p^{n'_2}})^{\times} \arrow[r] \arrow[u] & K_2^{\times} \arrow[u] \end{tikzcd}$$

where the vertical arrows are the natural embeddings. $\square$

Let $\gamma : K_1^{\times} \to (K_2^{p^{-n_2}})^{\times}$ be the map obtained in Lemma 3.8. We define $\gamma(0) = 0$. Then γ gives an injective multiplicative map $K_1 \to K_2^{p^{-n_2}}$.

Lemma 3.10 (Recovering the Additive Structure). *The injective map $\gamma : K_1 \to K_2^{p^{-n_2}}$ is a radicial field homomorphism, and the image $\gamma(K_1)$ of K_1 equals K_2. Thus γ induces a field isomorphism $\gamma : K_1 \xrightarrow{\sim} K_2$.*

Proof. (Compare the proof of [ST1], Theorem 3.7.) Take a prime $l \neq p$. Let k_i^l be the (unique) $\mathbb{Z}_l$-extension of k_i and set $K_i^l \overset{\mathrm{def}}{=} K_i k_i^l$, $i = 1, 2$. Let $p^{n_i^l}$ denote the exponent of the p-primary abelian group $J_{X_i}(k_i^l)[p^{\infty}]$, which is finite by [R], Theorem 11.6. Then, by Lemma 3.9, we obtain a natural injective map $\gamma^l : K_1^l \to (K_2^l)^{p^{-n_2^l}}$ extending γ. Now, by applying [ST1], Proposition 4.4 (with $E_X = E_Y = \emptyset$) to γ^l, we see that γ^l is a radicial field homomorphism, from which the first assertion follows. Thus, γ maps K_1 isomorphically onto $K_2^{p^m}$ for some $m \in \mathbb{Z}$. Since $\gamma(K_1^{\times})$ and $K_2^{\times}$ are commensurate to each other in $(K_2^{p^{-n_2}})^{\times}$ (cf. [ST1], Theorem 3.6 (ii)), we have $m = 0$. $\square$

By considering various open subgroups of $G_i^{(\prime)}$ as above, $i = 1, 2$, and using Lemmas 3.9 and 3.10, we obtain a natural field isomorphism $\tilde{\gamma} : \tilde{K}_1 \xrightarrow{\sim} \tilde{K}_2$.

Lemma 3.11. *The above isomorphism $\tilde{\gamma} : \tilde{K}_1 \xrightarrow{\sim} \tilde{K}_2$ is Galois-equivariant with respect to α, and is unique with this property.*

Proof. Same as in the last part of [U], Section 3. □

This finishes the proof of Theorem 3.2. □

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