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Analytic solutions to the sixth q-Painlevé equation around the origin

By

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Abstract

We classify analytic solutions of a q-analogue of the sixth Painlevé equation by Jimbo and Sakai around the origin. We determine the linear connection for one of analytic solutions.

§ 1. Introduction

Jimbo and Sakai [4] have obtained a q-analogue of the sixth Painlevé equation, so called q-P_{VI}, as a connection preserving deformation of a linear q-difference equation. The study of the q-Painlevé equations has been developed after Jimbo and Sakai's work, but most of them do not use connection preserving deformation, although the technique of monodromy preserving deformation is useful for the Painlevé *differential* equations. The reason why researchers do not use connection preserving deformations might be some difficulty to calculate connection data for q-difference equations.

In this paper we study linear connection formula for special solutions of q-P_{VI}, which are analytic around the origin. In our main theorem 4.3, we determine the connection matrix only for one special analytic around the origin. Other cases will be published elsewhere.

For the sixth Painlevé differential equation P_{VI}, Kaneko [5] has studied analytic solutions around the origin. For generic parameters of P_{VI} there exist four solutions which are analytic around the origin and Kaneko has determined the monodromy data of the corresponding linear equation for such type of solutions. P_{VI} can be expressed

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by monodromy preserving deformation of the linear equation

$$\frac{dY}{dx} = \left(\frac{A_0(t)}{x} + \frac{A_t(t)}{x-t} + \frac{A_1(t)}{x-1} \right) Y. \tag{1.1}$$

We substitute an analytic solution around the origin in (1.1) and take a limit $t \to 0$. Then we have

$$\frac{dY}{dx} = \left(\frac{A_0(0) + A_t(0)}{x} + \frac{A_1(0)}{x-1} \right) Y. \tag{1.2}$$

The monodromy data of (1.2) can be determined explicitly because (1.2) can be solved by hypergeometric functions. For generic solutions, the limit $t \to 0$ is not so easy [3].

We will study a q-analogue of Kaneko's work. The q-analogue of the sixth Painlevé equation q-P_{VI} is given by

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{(\bar{z}-b_3)(\bar{z}-b_4)}, \quad \frac{z\bar{z}}{b_3b_4} = \frac{(y-a_1t)(y-a_2t)}{(y-a_3)(y-a_4)},$$

$$\frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4},$$

$a_1, a_2, a_3, a_4, b_1,\ b_2,\ b_3,\ b_4$ are non-zero complex parameters. We set $\bar{f} = f(qt)$ for any function $f = f(t)$. q-P_{VI} is expressed as connection preserving deformation of a linear q-difference equation

$$Y(qx,t) = (A_0(t) + xA_1(t) + x^2A_2)Y(x,t). \tag{1.3}$$

q-P_{VI} has also four analytic solutions around $t = 0$. If we take a limit $t \to 0$ in (1.3), we can solve the limit equation by Heine's basic hypergeometric functions ${}_2\varphi_1(a,b,c;x)$ [2]:

$${}_2\varphi_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} x^n, \tag{1.4}$$

where $(a;q)_n = (a)_n = \prod_{j=0}^{n-1}(1-aq^j)$. We set $(a;q)_\infty = (a)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

The connection formula of basic hypergeometric functions is given by Watson [6]. We can determine the connection formula of (1.3) by Watson's formula (see (4.2), which is the same as the connection formula of (1.3).

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§ 2. Connection preserving deformation

In this section we review the work by Jimbo and Sakai [4] on the q-analogue of the sixth Painlevé equation q-P_{VI}.

Jimbo and Sakai derived q-P_{VI} as a connection preserving deformation

$$Y(qx,t) = A(x,t)Y(x,t), \tag{2.1}$$

$$Y(x,qt) = B(x,t)Y(x,t). \tag{2.2}$$

Here

$$A(x,t) = A_0(t) + xA_1(t) + x^2 A_2,$$

and

$$B(x,t) = \frac{x}{(x-a_1qt)(x-a_2qt)}(xI + B_0(t)).$$

We set $A_2 = \mathrm{diag}(\kappa_1, \kappa_2)$. The eigenvalues of $A_0(t)$ are $\theta_1 t, \theta_2 t$, and $\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)(x-a_4)$. We assume that

$$\frac{\theta_1}{\theta_2}, \quad \frac{\kappa_1}{\kappa_2} \notin \{q^{\pm 1}, q^{\pm 2}, ...\}.$$

Set

$$A_0 = C_0 \mathrm{diag}(\theta_1 t, \theta_2 t) C_0^{-1}.$$

We denote $\mathrm{lq}\, x = \log_q x$.

Birkhoff [1] has shown the following theorem:

Proposition 2.1. *There exist unique solutions $Y_0(x), Y_\infty(x)$ with the following properties:*

$$Y_0(x) = \hat{Y}_0(x)x^{D_0}, \qquad D_0 = \mathrm{diag}(\mathrm{lq}\,\theta_1 t, \mathrm{lq}\,\theta_2 t)$$
$$Y_\infty(x) = q^{u(u-1)}\hat{Y}_\infty(x)x^{D_\infty}, \qquad D_\infty = \mathrm{diag}(\mathrm{lq}\,\kappa_1, \mathrm{lq}\,\kappa_2),\ u = \log_q x.$$

Here $\hat{Y}_0(x)$ and $\hat{Y}_\infty(x)$ are holomorphic and invertible at $x = 0$ and $x = \infty$, respectively. And $\hat{Y}_0(0) = C_0$, and $\hat{Y}_0(\infty) = I$.

The connection matrix $P(x)$ is defined by

$$Y_\infty(x) = Y_0(x)P(x).$$

Since $P(qx) = P(x)$, $P(x)$ is expressed by theta functions. We call a function $f(x)$ *pseudo-constant*, if $f(xq) = f(x)$. Jimbo and Sakai has proved the following theorem:

Theorem 2.2. *The connection matrix $P(x,t)$ is pseudo-constant as a function of t, i.e. $P(x,qt) = P(x,t)$ if Y satisfies (2.1-2.2).*

$A(x,t)$ has the following form:

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & \kappa_2 w(x-y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2((x-y)(x-\beta)+z_2) \end{pmatrix}.$$

Here

$$\begin{aligned}
\alpha =&\frac{1}{\kappa_1-\kappa_2}[y^{-1}((\theta_1+\theta_2)t-\kappa_1 z_1-\kappa_2 z_2)-\kappa_2((a_1+a_2)t+a_3+a_4-2y)],\\
\beta =&\frac{1}{\kappa_1-\kappa_2}[-y^{-1}((\theta_1+\theta_2)t-\kappa_1 z_1-\kappa_2 z_2)+\kappa_1((a_1+a_2)t+a_3+a_4-2y)],\\
\gamma =&z_1+z_2+(y+\alpha)(y+\beta)+(\alpha+\beta)y-a_1a_2t^2-(a_1+a_2)(a_3+a_4)t-a_3a_4,\\
\delta =&y^{-1}(a_1a_2a_3a_4t^2-(\alpha y+z_1)(\beta y+z_2)),
\end{aligned}$$

$$b_1=\frac{a_1a_2}{\theta_1},\quad b_2=\frac{a_1a_2}{\theta_2},\quad b_3=\frac{1}{\kappa_1 q},\quad b_4=\frac{1}{\kappa_2}.$$

The matrix elements of $B_0(t)$ are given by

$$\begin{aligned}
B_{11}&=\frac{-\kappa_2 q\bar{z}}{1-\kappa_2\bar{z}}\left(-\beta+\frac{t(a_1+a_2)-y}{\kappa_2\bar{z}}\right),\quad B_{12}=\frac{\kappa_2 qw\bar{z}}{1-\kappa_2\bar{z}},\\
B_{21}&=\frac{\kappa_1 q\bar{z}}{w(1-\kappa_1 q\bar{z})}\left(a_1qt-\bar{\alpha}+\frac{a_2qt-\bar{y}}{\kappa_1 q\bar{z}}\right)\left(a_1t-\beta+\frac{a_2t-y}{\kappa_2\bar{z}}\right),\\
B_{22}&=\frac{-\kappa_1 q\bar{z}}{1-\kappa_1 q\bar{z}}\left(-\bar{\alpha}+\frac{qt(a_1+a_2)-\bar{y}}{\kappa_1 q\bar{z}}\right).
\end{aligned}$$

The compatibility condition (2.1-2.2) is equivalent to q-P_{VI} and one more equation on w:

$$\frac{\bar{w}}{w}=\frac{b_4}{b_3}\frac{\bar{z}-b_3}{\bar{z}-b_4}.$$

By Theorem 2.2, q-P_{VI} is expressed as a connection preserving deformation.

§ 3. Meromorphic solutions around the origin

In this section we will list up all of meromorphic solutions of q-P_{VI} around the origin.

If a solution y, z of q-P_{VI} is meromorphic around the origin, it is evident that the solution is developed either as

$$y(t)=\sum_{n=0}^{\infty}y_nt^n,\quad z(t)=\sum_{n=0}^{\infty}z_nt^n\quad (y_0, z_0\neq 0),$$

or as

$$y(t)=\sum_{n=1}^{\infty}y_nt^n,\quad z(t)=\sum_{n=1}^{\infty}z_nt^n\quad (y_1, z_1\neq 0).$$

If $y(0)\neq 0$ and $z(0)\neq 0$, we have two cases

$$\text{(3.1)}\qquad \text{Case I)}\quad y(0)=\frac{a_3b_3-a_4b_4}{b_3-b_4},\quad z(0)=\frac{a_3b_3-a_4b_4}{a_3-a_4},$$

$$\text{(3.2)}\qquad \text{Case II)}\quad y(0)=\frac{a_4b_3-a_3b_4}{b_3-b_4},\quad z(0)=\frac{a_3b_4-a_4b_3}{a_3-a_4}.$$

and higher terms can be determined recursively.

If $y(0)=0$ and $z(0)=0$, we have two cases

$$\text{(3.3)} \qquad \text{Case III)} \quad y'(0)=\frac{a_1a_2(b_1-b_2)}{a_2b_1-a_1b_2}, \quad z'(0)=-\frac{b_1b_2(a_1-a_2)}{(a_2b_1-a_1b_2)q},$$

$$\text{(3.4)} \qquad \text{Case IV)} \quad y'(0)=\frac{a_1a_2(b_1-b_2)}{a_1b_1-a_2b_2}, \quad z'(0)=\frac{b_1b_2(a_1-a_2)}{(a_1b_1-a_2b_2)q}.$$

and higher terms can be determined recursively. These results can be checked out by direct calculation. We remark that the four series above converge for $|t|<r$ for a positive number r. The convergence is proved by a q-analogue Briot-Bouquet theorem.

§ 4. Connection matrix for the solution III

Here we express the connection matrix only for the solution III. Other cases can be shown similarly. We use some simple transformations of the connection.

Lemma 4.1. *(1) Assume that $Y(xq)=A(x)Y(x)$ for a matrix $A(x)$. If a scalar function $y(x)$ satisfies $y(xq)=a(x)y(x)$ for a scalar function $a(x)$, $Z(x)=y(x)Y(x)$ satisfies $Z(xq)=a(x)A(x)Z(x)$.*

(2) The function $f_c(x)=x^{\mathrm{lq}\,c}$ satisfies $f_c(xq)=cf_c(x)$.

(3) The function $g_c(x)=(cx;q)_\infty$ satisfies $g_c(xq)=\dfrac{1}{1-cx}g_c(x)$.

(4) Set $\Theta(x)=(x;q)_\infty(q/x;q)_\infty(q;q)_\infty$. The function $h_c(x)=\Theta(cx)$ satisfies $h_c(xq)=-\dfrac{1}{cx}h_c(x)$.

(5) The function $k(x)=q^{u(u-1)/2}$ satisfies $k(xq)=xk(x)$. Here $u=\mathrm{lq}\,x$.

A q-difference equation can be reduced to a simple form by transformations in Lemma 4.1. We remark that $f_{c/d}(x)\Theta(cx)/\Theta(dx)$ is pseudo-constant.

We substitute the solution III in (2.1) and take a limit $t\to 0$. Then we have

$$A(x,0)=\tilde{A}_1x+\tilde{A}_2x^2,$$

where

$$\tilde{A}_1=\begin{pmatrix} \frac{a_3b_1b_2+a_4b_1b_2-a_2b_1b_4-a_1b_2b_4}{b_1b_2(b_4-b_3q)} & \frac{w(0)}{b_4} \\ \frac{b_4(a_1b_4-a_4b_1)(a_4b_2-a_2b_4)(a_4b_1-a_1b_3q)(a_4b_2-a_2b_3q)}{a_4^2b_1^2b_2^2(b_4-b_3q)^2w(0)} & \frac{a_3a_4b_2^2+a_2^2b_3b_4q-a_2a_3b_2b_4-a_2a_4b_2b_4}{a_2b_2b_4(b_4-b_3q)} \end{pmatrix}$$

and $\tilde{A}_2=\mathrm{diag}\,(1/(b_3q),1/b_4)$. The eigenvalues of $\tilde{A}_1$ are $-a_1/b_1,-a_2/b_2$.

The equation

$$\text{(4.1)} \qquad Y(xq)=A(x,0)Y(x)$$

has local solutions $Y^{(0)}(x)$ and $Y^{(\infty)}(x)$ around $x=0$ and $x=\infty$, respectively. We set $D_0 = \mathrm{diag}\,(\mathrm{lq}\,(-a_1/b_1), \mathrm{lq}\,(-a_2/b_2))$ and $D_\infty = \mathrm{diag}\,(\mathrm{lq}\,(-a_3/(b_3q)), \mathrm{lq}\,(-a_3/b_4))$.

The matrix

$$Z(x) = \frac{(x/a_3)_\infty}{q^{u(u-1)/2}} Y(x)$$

satisfies an equation

$$Z(xq) = \left(\hat{A}_0 + \frac{\hat{A}_1}{x-a_3} \right) Z(x)$$

and $Z(x)$ has a solution $Z_0(x) = \hat{Z}_0(x)x^{D_0}$ around $x=0$ and $Z_\infty(x) = \hat{Z}_\infty(x)x^{D_\infty}$ around $x=\infty$. $\hat{Z}_0(x)$ is invertible around $x=0$ and $\hat{Z}_\infty(x)$ is invertible around $x=\infty$.

Proposition 4.2. *The solution of* (4.1) *around* $x=0$

$$Y^{(0)}(x) = \frac{q^{u(u-1)/2}}{(x/a_3)_\infty} \begin{pmatrix} y_{11}^{(0)} & y_{12}^{(0)} \\ y_{21}^{(0)} & y_{22}^{(0)} \end{pmatrix} x^{D_0}$$

is represented by

$$\begin{aligned}
y_{11}^{(0)} &= C_{11} \cdot {}_2\varphi_1 \left(\frac{a_4b_2}{a_2b_3}, \frac{a_4b_2}{a_2b_4}, \frac{a_1b_2}{a_2b_1}q; \frac{x}{a_4} \right), \\
y_{12}^{(0)} &= C_{12} \cdot {}_2\varphi_1 \left(\frac{a_4b_1}{a_1b_3}, \frac{a_4b_1}{a_1b_4}, \frac{a_2b_1}{a_1b_2}q; \frac{x}{a_4} \right), \\
y_{21}^{(0)} &= C_{21} \cdot {}_2\varphi_1 \left(\frac{a_1b_4}{a_3b_1}, \frac{a_4b_2}{a_2b_4}q, \frac{a_1b_2}{a_2b_1}q; \frac{x}{a_4} \right), \\
y_{22}^{(0)} &= C_{22} \cdot {}_2\varphi_1 \left(\frac{a_2b_4}{a_3b_2}, \frac{a_4b_1}{a_1b_4}q, \frac{a_2b_1}{a_1b_4}q; \frac{x}{a_4} \right),
\end{aligned}$$

where

$$\begin{aligned}
C_{11} &= b_2(a_1a_2b_4^2 - a_3a_4b_1b_2), \\
C_{12} &= b_1(a_1a_2b_4^2 - a_3a_4b_1b_2)w(0), \\
C_{21} &= a_1b_4(a_2b_4 - a_3b_2)(a_2b_4 - a_4b_2)/w(0), \\
C_{22} &= a_2b_4(a_1b_4 - a_3b_1)(a_1b_4 - a_4b_1).
\end{aligned}$$

(2) The solution (4.1) *around* $x=\infty$

$$Y^{(\infty)}(x) = \frac{q^{u(u-1)/2}}{(x/a_3)_\infty} \begin{pmatrix} y_{11}^{(\infty)} & y_{12}^{(\infty)} \\ y_{21}^{(\infty)} & y_{22}^{(\infty)} \end{pmatrix} x^{D_\infty}$$

is represented by

$$\begin{aligned} y_{11}^{(\infty)} &= {}_2\varphi_1\left(\frac{a_4b_2}{a_2b_4}, \frac{a_4b_1}{a_1b_4}, \frac{b_3}{b_4}q; \frac{a_3q}{x}\right),\\ y_{12}^{(\infty)} &= \frac{b_3qw(0)}{b_3-b_4}\cdot {}_2\varphi_1\left(\frac{a_4b_1}{a_1b_3}, \frac{a_4b_2}{a_2b_3}, \frac{b_4}{b_3}q; \frac{a_3q}{x}\right)\frac{1}{x},\\ y_{21}^{(\infty)} &= C\cdot {}_2\varphi_1\left(\frac{a_4b_2q}{a_2b_4}, \frac{a_4b_1q}{a_1b_4}, \frac{b_3q^3}{b_4}; \frac{a_3q}{x}\right)\frac{1}{x},\\ y_{22}^{(\infty)} &= {}_2\varphi_1\left(\frac{a_2b_4}{a_3b_2}, \frac{a_1b_4}{a_3b_1}, \frac{b_4}{b_3q}; \frac{a_3q}{x}\right), \end{aligned}$$

where

$$C = \frac{a_1a_2a_3a_4{b_4}^2q(a_1b_4-a_3b_1)(a_1b_4-a_4b_1)(a_2b_4-a_3b_2)(a_2b_4-a_4b_2)}{\left(a_3a_4b_1b_2-a_1a_2{b_4}^2\right)^2\left(a_3a_4b_1b_2q-a_1a_2{b_4}^2\right)w(0)}.$$

By Watson's connection formula [6]

$$\begin{aligned} {}_2\varphi_1(a,b;c;q;z) &= \frac{(b,c/a;q)_\infty(az,q/az;q)_\infty}{(c,b/a;q)_\infty(z,q/z;q)_\infty}{}_2\varphi_1(a,aq/c;aq/b;q;cq/abz)\\ &+\frac{(a,c/b;q)_\infty(bz,q/bz;q)_\infty}{(c,a/b;q)_\infty(z,q/z;q)_\infty}{}_2\varphi_1(b,bq/c;bq/a;q;cq/abz). \end{aligned} \tag{4.2}$$

We obtain the connection matrix $P(x)=Y^{(0)}(x)^{-1}Y^{(\infty)}(x)$ for the solution III of q-P_{VI}. We denote $(a,b;q)_\infty=(a;q)_\infty(b;q)_\infty$.

Theorem 4.3. *The connection matrix of* (2.1) *for the solution III between* $x=0$ *and* $x=\infty$ *is given by*

$$P(x)=\begin{pmatrix} p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix},$$

where

$$\begin{aligned} p_{11} &= \frac{\left(\frac{a_4b_1}{a_1b_4},\frac{a_3b_1}{a_1b_4};q\right)_\infty\left(\frac{b_1x}{a_1b_3q},\frac{a_1b_3q^2}{b_1x};q\right)_\infty}{C_{11}\left(\frac{a_2b_1}{a_1b_2},\frac{b_3q}{b_4};q\right)_\infty\left(\frac{x}{a_3},\frac{a_3q}{x};q\right)_\infty}x^{\mathrm{lq}\,(a_2b_4/a_4b_2)},\\ p_{12} &= \frac{b_3qw(0)}{b_3-b_4}\cdot\frac{\left(\frac{a_4b_1}{a_1b_3},\frac{a_3b_1}{a_1b_3};q\right)_\infty\left(\frac{b_1x}{a_1b_4q},\frac{a_1b_4q^2}{b_1x};q\right)_\infty}{C_{11}\left(\frac{a_2b_1}{a_1b_2},\frac{b_4q}{b_3};q\right)_\infty\left(\frac{x}{a_3},\frac{a_3q}{x};q\right)_\infty}x^{\mathrm{lq}\,(a_3b_1/a_1b_4q)},\\ p_{21} &= \frac{\left(\frac{a_4b_2}{a_2b_4},\frac{a_3b_2}{a_2b_4};q\right)_\infty\left(\frac{b_2x}{a_2b_3q},\frac{a_2b_3q^2}{b_2x};q\right)_\infty}{C_{12}\left(\frac{a_1b_2}{a_2b_1},\frac{b_3q}{b_4};q\right)_\infty\left(\frac{x}{a_3},\frac{a_3q}{x};q\right)_\infty}x^{\mathrm{lq}\,(a_1b_4/a_4b_1)},\\ p_{22} &= \frac{b_3qw(0)}{b_3-b_4}\cdot\frac{\left(\frac{a_4b_2}{a_2b_3},\frac{a_3b_2}{a_2b_3};q\right)_\infty\left(\frac{b_2x}{a_2b_4q},\frac{a_2b_4q^2}{b_2x};q\right)_\infty}{C_{12}\left(\frac{a_1b_2}{a_1b_1},\frac{b_4q}{b_3};q\right)_\infty\left(\frac{x}{a_3},\frac{a_3q}{x};q\right)_\infty}x^{\mathrm{lq}\,(a_3b_2/a_2b_4q)}. \end{aligned}$$

We remark that P is pseudo-constant expressed by theta functions.

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