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Bilinearization and Casorati Determinant Solutions to Non-autonomous $1+1$ Dimensional Discrete Soliton Equations

By

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Abstract

Some techniques of bilinearization of the non-autonomous 1+1 dimensional discrete soliton equations are discussed by taking the discrete KdV equation, the discrete Toda lattice equation, and the discrete Lotka-Volterra equation as examples. Casorati determinant solutions to those equations are also constructed explicitly.

§ 1. Introduction

The Hirota-Miwa equation, or the discrete KP equation is the bilinear difference equation of Hirota type given by

$$\begin{aligned} &a(b-c)\tau(l+1,m,n)\tau(l,m+1,n+1)+b(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1)\\ &\quad +c(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n)=0. \end{aligned} \tag{1.1}$$

Eq.(1.1) is well-known as one of the most important integrable systems[3, 11, 16]. Here, a, b, c are arbitrary constants playing a role of lattice intervals of discrete independent variables l, m, n, respectively. The Casorati determinant solution to eq.(1.1) is given by

$$\tau(l,m,n)=\begin{vmatrix} \varphi_1^{(s)}(l,m,n) & \varphi_1^{(s+1)}(l,m,n) & \cdots & \varphi_1^{(s+N-1)}(l,m,n)\\ \varphi_2^{(s)}(l,m,n) & \varphi_2^{(s+1)}(l,m,n) & \cdots & \varphi_2^{(s+N-1)}(l,m,n)\\ \vdots & \vdots & \cdots & \vdots\\ \varphi_N^{(s)}(l,m,n) & \varphi_N^{(s+1)}(l,m,n) & \cdots & \varphi_N^{(s+N-1)}(l,m,n) \end{vmatrix}, \tag{1.2}$$

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where $\varphi_r^{(s)}(l,m,n)$ $(r=1,\dots,N)$ are arbitrary functions satisfying the linear relations

$$
\begin{aligned}
\frac{\varphi_r^{(s)}(l+1,m,n)-\varphi_r^{(s)}(l,m,n)}{a} &= \varphi_r^{(s+1)}(l,m,n),\\
\frac{\varphi_r^{(s)}(k,l+1,m)-\varphi_r^{(s)}(l,m,n)}{b} &= \varphi_r^{(s+1)}(l,m,n),\\
\frac{\varphi_r^{(s)}(l,m,n+1)-\varphi_r^{(s)}(l,m,n)}{c} &= \varphi_r^{(s+1)}(l,m,n).
\end{aligned}
\tag{1.3}
$$

For example, choosing $\varphi_r^{(s)}$ to be sum of the exponential type functions as

$$
\begin{aligned}
\varphi_r^{(s)}(l,m,n) =&\alpha_r p_r^s(1+ap_r)^l(1+bp_r)^m(1+cp_r)^n\\
&+\beta_r q_r^s(1+aq_r)^l(1+bq_r)^m(1+cq_r)^n,
\end{aligned}
\tag{1.4}
$$

where α_r, β_r are arbitrary constants and p_r, q_r are parameters, then it gives the N–soliton solution.

It is known that eq.(1.1) yields various discrete and continuous soliton equations by reductions and limiting procedures. For example, let us impose the condition

$$\tau(l+1,m+1,n) \simeq \tau(l,m,n), \tag{1.5}$$

where $\simeq$ denotes the equivalence up to multiple of gauge functions which leaves the bilinear equation invariant. Then using eq.(1.5) to suppress l-dependence and taking $a=-b$, eq.(1.1) yields

$$(b-c)\ \tau_n^{m-1}\tau_{n+1}^{m+1}-(b+c)\ \tau_n^{m+1}\tau_{n+1}^{m-1}+2c\ \tau_{n+1}^m\tau_n^m=0, \tag{1.6}$$

where $\tau_n^m=\tau(l,m,n)$. Eq.(1.6) is transformed to

$$v_{n+1}^m-v_n^{m+1}=\frac{b-c}{b+c}\left(\frac{1}{v_{n+1}^{m+1}}-\frac{1}{v_n^m}\right), \tag{1.7}$$

by the dependent variable transformation

$$v_n^m=\frac{\tau_{n+1}^m\tau_n^{m+1}}{\tau_n^m\tau_{n+1}^{m+1}}. \tag{1.8}$$

Eq.(1.6) or eq.(1.7) are called the discrete KdV equation[2, 16]. The condition (1.5) is realized by choosing $q_r=-p_r$ on the level of $\varphi_r^{(s)}(l,m,n)$ in eq.(1.4). Therefore choosing the entries of the determinant as

$$\varphi_r^{(s)}(m,n)=\alpha_r p_r^s(1+bp_r)^m(1+cp_r)^n+\beta_r(-p_r)^s(1-bp_r)^m(1+cp_r)^n, \tag{1.9}$$

it gives the N–soliton solution of the discrete KdV equation. Hence, if a given equation turns out to be derived by the reduction or other procedure from the Hirota-Miwa equation, it is possible to construct wide class of solutions in this manner.

On the other hand, it has been pointed out that a generalization of the Hirota-Miwa equation is possible in such a way that the lattice intervals are arbitrary functions of the corresponding independent variables[7, 23]. Such generalization to the "inhomogeneous lattice" or to the non-autonomous equation is regarded as an important problem in the context of ultradiscretization or the box and ball systems, since it corresponds to a generalization such that the capacity of the boxes changes according to the lattice sites[14]. Moreover, many discrete soliton equations are shown to describe discrete surfaces and curves in various settings of the discrete differential geometry. In this context, such inhomogeneity of lattice corresponds to the scaling freedom of parametrization of the geometric objects and therefore it is geometrically natural[1].

The generalization to the non-autonomous equation is technically straightforward for the "generic" equation such as the Hirota-Miwa equation because of its gauge invariance. However, when we consider the reduction to $1+1$ dimensional system such as the discrete KdV equation, the reduction procedure does not work consistently because of the non-autonomous property on the level of both bilinear equation and solution. Therefore the $1+1$ dimensional non-autonomous discrete soliton equations have not been studied well.

Recently, Tsujimoto and Mukaihira have considered the non-autonomous discrete Toda lattice (1DTL) equation on semi-infinite lattice from the standpoint of R_{I} and R_{II} type bi–orthogonal functions[12, 13]. By introducing a certain auxiliary τ function which does not appear in the expression of the solution, they succeeded in bilinearization of the equation and constructing molecular type solution. Then it has been shown that the non-autonomous discrete 1DTL equation on infinite lattice also admits similar bilinearization, and the soliton type solutions have been constructed[8]. Moreover, three different bilinearizations of different origins have been presented for the non-autonomous discrete KdV equation in [9], each of which requires a certain auxiliary τ function. The techniques developed in recent researches may enable systematic studies of the non-autonomous discrete soliton equations.

The purpose of this paper is to give a review and present some new results on bilinearization of non-autonomous $1+1$ dimensional discrete soliton equations and construction of their Casorati determinant solutions. This paper is organized as follows. In Section 2, we give a brief review of the non-autonomous discrete KP hierarchy and its solutions. In Section 3, we discuss three bilinearizations of the non-autonomous discrete KdV equation. Section 4 deals with two bilinearizations of the non-autonomous discrete 1DTL equation. We discuss in Section 5 the case of discrete Lotka-Volterra equation, where the direct reduction from the discrete two-dimensional Toda lattice equation works without auxiliary τ functions.

§ 2. Non-autonomous Discrete KP Hierarchy

§ 2.1. τ Function and Bilinear Equations

We define the τ function $\tau_N(s;\boldsymbol{l},\boldsymbol{m};\boldsymbol{x},\boldsymbol{y})$ depending on infinitely many independent variables $N\in\mathbb{Z}$, s, $\boldsymbol{l}=(l_1,l_2,\ldots)$, $\boldsymbol{m}=(m_1,m_2,\ldots)$, $\boldsymbol{x}=(x_1,x_2,\ldots)$, and $\boldsymbol{y}=(y_1,y_2,\ldots)$ by

$$\tau_N(s;\boldsymbol{l},\boldsymbol{m};\boldsymbol{x},\boldsymbol{y})=\begin{vmatrix}\varphi_1^{(s)} & \varphi_1^{(s+1)} & \cdots & \varphi_1^{(s+N-1)}\\ \varphi_2^{(s)} & \varphi_2^{(s+1)} & \cdots & \varphi_2^{(s+N-1)}\\ \vdots & \vdots & \cdots & \vdots\\ \varphi_N^{(s)} & \varphi_N^{(s+1)} & \cdots & \varphi_N^{(s+N-1)}\end{vmatrix}, \tag{2.1}$$

where $\varphi_r^{(s)}=\varphi_r^{(s)}(\boldsymbol{l},\boldsymbol{m};\boldsymbol{x},\boldsymbol{y})$ $(r=1,\ldots,N)$ satisfy the linear equations

$$\frac{\varphi_r^{(s)}(l_\nu+1)-\varphi_r^{(s)}(l_\nu)}{a_\nu(l_\nu)}=\varphi_r^{(s+1)}(l_\nu), \tag{2.2}$$

$$\frac{\varphi_r^{(s)}(m_\nu+1)-\varphi_r^{(s)}(m_\nu)}{b_\nu(m_\nu)}=\varphi_r^{(s-1)}(m_\nu), \tag{2.3}$$

$$\frac{\partial}{\partial x_\nu}\varphi_r^{(s)}=\varphi_r^{(s+\nu)}, \tag{2.4}$$

$$\frac{\partial}{\partial y_\nu}\varphi_r^{(s)}=\varphi_r^{(s-\nu)}, \tag{2.5}$$

for $\nu=1,2,\ldots$. Here the lattice intervals a_ν and b_ν $(\nu=1,2,\ldots)$ are arbitrary functions with respect to the indicated variables. We note that in the following, we indicate only the relevant independent variables for notational simplicity, as in eqs.(2.2)–(2.5). For example, the N–soliton solution is obtained by choosing $\varphi_r^{(s)}$ as

$$\begin{aligned}\varphi_r^{(s)}=&\alpha_r p_r^s\prod_{\nu=1}^{\infty}\prod_{i=i_\nu}^{l_\nu-1}(1+a_\nu(i)p_r)\prod_{\mu=1}^{\infty}\prod_{i=i_\mu}^{m_\mu-1}(1+b_\mu(i)p_r^{-1})e^{\sum_{n=1}^{\infty}x_np_r^n+\sum_{n=1}^{\infty}y_ip_r^{-n}}\\&+\beta_r q_r^s\prod_{\nu=1}^{\infty}\prod_{i=i_\nu}^{l_\nu-1}(1+a_\nu(i)q_r)\prod_{\mu=1}^{\infty}\prod_{i=i_\mu}^{m_\mu-1}(1+b_\mu(i)q_r^{-1})e^{\sum_{n=1}^{\infty}x_nq_r^n+\sum_{n=1}^{\infty}y_iq_r^{-n}},\end{aligned} \tag{2.6}$$

where α_r, β_r are arbitrary constants and p_r, q_r are parameters.

It is known that $\tau_N(s;\boldsymbol{l},\boldsymbol{m};\boldsymbol{x},\boldsymbol{y})$ satisfies infinitely many difference, differential and difference-differential bilinear equations of Hirota type (for autonomous case, see for example [21]). We call this hierarchy of equations the non-autonomous discrete KP hierarchy. We give a list of some typical examples included in the hierarchy:

KP equation ($x = x_1$ $y = x_2$, $t = x_3$)

$$\left(D_x^4 - 4D_xD_t + 3D_y^2\right)\tau \cdot \tau = 0. \tag{2.7}$$

Two-dimensional Toda lattice (2DTL) equation ($x = x_1$ $y = y_1$, $n = s$)

$$\frac{1}{2}D_xD_y\ \tau_n \cdot \tau_n = \tau_n^2 - \tau_{n+1}\tau_{n-1}. \tag{2.8}$$

Non-autonomous discrete KP equation ($l = l_i$, $m = l_j$, $n = l_k$, $a_l = a_i(l_i)$, $b_m = a_j(l_j)$, $c_m = a_k(l_k)$, $\{i, j, k\} \subset \{1, 2, 3, \cdots\}$)

$$\begin{aligned} &a_l(b_m - c_n)\tau(l+1, m, n)\tau(l, m+1, n+1)\\ &+b_m(c_n - a_l)\tau(l, m+1, n)\tau(l+1, m, n+1)\\ &+c_n(a_l - b_m)\tau(l, m, n+1)\tau(l+1, m+1, n) = 0. \end{aligned} \tag{2.9}$$

Non-autonomous discrete 2DTL equation ($l = l_i$, $m = m_j$, $n = s$, $a_l = a_i(l_i)$, $b_m = b_j(m_j)$, $\{i, j\} \subset \{1, 2, \cdots\}$)

$$\begin{aligned} &(1 - a_lb_m)\tau_n(l+1, m+1)\tau_n(l, m) - \tau_n(l+1, m)\tau_n(l, m+1)\\ &+a_lb_m\tau_{n+1}(l, m+1)\tau_{n-1}(l+1, m) = 0. \end{aligned} \tag{2.10}$$

Bäcklund transformation (BT) of 2DTL equation ($x = x_1$, $m = m_i$, $n = s$, $b_m = b_i(m_i)$, $i \in \{1, 2, \cdots\}$)

$$(D_x - b_m)\,\tau_n(m) \cdot \tau_n(m+1) + b_m\tau_{n-1}(m)\tau_{n+1}(m+1) = 0. \tag{2.11}$$

BT of non-autonomous discrete KP(2DTL) equation ($l = l_i$, $m = l_j$, $n = s$, $a_l = a_i(l_i)$, $b_m = a_j(l_j)$, $\{i, j\} \subset \{1, 2, \cdots\}$)

$$\begin{aligned} &a_l\tau_{n+1}(l, m+1)\tau_n(l+1, m) - b_m\tau_{n+1}(l+1, m)\tau_n(l, m+1)\\ &-(a_l - b_m)\tau_{n+1}(l, m)\tau_n(l+1, m+1) = 0. \end{aligned} \tag{2.12}$$

§ 2.2. Casoratian Technique

In order to prove that the τ function given in the form of Casorati determinant satisfies the bilinear equations, the Casoratian technique is quite useful[16, 17]. We demonstrate the outline of the technique by taking eq.(2.11) as an example.

Under the setting of eq.(2.11), the τ function (2.1) reads

$$\tau_n(m) = \begin{vmatrix} \varphi_1^{(n)}(m) & \varphi_1^{(n+1)}(m) & \cdots & \varphi_1^{(n+N-1)}(m) \\ \varphi_2^{(n)}(m) & \varphi_2^{(n+1)}(m) & \cdots & \varphi_2^{(n+N-1)}(m) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(n)}(m) & \varphi_N^{(n+1)}(m) & \cdots & \varphi_N^{(n+N-1)}(m) \end{vmatrix}, \tag{2.13}$$

where $\varphi_k^{(n)}(m)$ $(k=1,\ldots,N)$ satisfy the linear relations

$$\partial_x \varphi_k^{(n)}(m) = \varphi_k^{(n+1)}(m), \tag{2.14}$$

$$\varphi_k^{(n)}(m) - \varphi_k^{(n)}(m-1) = b_{m-1}\ \varphi_k^{(n-1)}(m-1). \tag{2.15}$$

For instance, if we choose $\varphi_k^{(n)}(m)$ as

$$\varphi_k^{(n)}(m) = \alpha_k p_k^n \prod_{i=i_0}^{m-1}(1+b_i p_k^{-1})e^{p_k x} + \beta_k q_k^n \prod_{i=i_0}^{m-1}(1+b_i q_k^{-1})e^{q_k x}, \tag{2.16}$$

we obtain the N–soliton solution.

The bilinear equation (2.11) is reduced to the Plücker relation, which is the quadratic identity among the determinants whose columns are properly shifted. To this end, we first construct difference/differential formulas which express determinants whose columns are shifted by $\tau_n(m)$.

Lemma 2.1. *The following formulas hold.*

$$\tau_n(m) = |\ 0,\cdots,N-2,N-1\ |, \tag{2.17}$$

$$\tau_n(m-1) = |\ 0_{m-1},1,\cdots,N-2,N-1\ |, \tag{2.18}$$

$$-b_{m-1}\ \tau_n(m-1) = |\ 1_{m-1},1,\cdots,N-2,N-1\ |, \tag{2.19}$$

$$\partial_x \tau_n(m) = |\ 0,\cdots,N-2,N\ |, \tag{2.20}$$

$$(\partial_x + b_{m-1})\tau_n(m-1) = |\ 0_{m-1},\cdots,N-2,N\ |, \tag{2.21}$$

where "j_m" is the column vector

$$j_m = \begin{pmatrix} \varphi_1^{(n+j)}(m) \\ \varphi_2^{(n+j)}(m) \\ \vdots \\ \varphi_N^{(n+j)}(m) \end{pmatrix}, \tag{2.22}$$

and the subscript is shown only when m is shifted.

Proof. Eq.(2.17) follows by definition, and eq.(2.20) is derived from the differential rule of determinant. Using eq.(2.15) to the i-th column of $\tau_n(m-1)$ for $i=N,N-1,\ldots,2$, we have

$$\begin{aligned}\tau_n(m-1) &= |\ 0_{m-1},1_{m-1},\cdots,N-2_{m-1},N-1_{m-1}\ | \\ &= |\ 0_{m-1},1,\cdots,N-2,N-1\ |,\end{aligned}$$

which is eq.(2.18). Multiplying $-b_{m-1}$ to the first column of the right hand side of eq.(2.18) and using eq.(2.15) we have eq.(2.19),

$$\begin{aligned}-b_{m-1}\tau_n(m-1) &= |\ -b_{m-1}\cdot 0_{m-1},1,\cdots,N-2,N-1\ |\\ &= |\ 1_{m-1}-1,1,\cdots,N-2,N-1\ |\\ &= |\ 1_{m-1},1,\cdots,N-2,N-1\ |.\end{aligned}$$

Differentiating eq.(2.18), we have

$$\begin{aligned}\partial_x\tau_n(m-1) &= |\ 1_{m-1},1,\cdots,N-2,N-1\ | + |\ 0_{m-1},1,\cdots,N-2,N\ |\\ &= -b_{m-1}\tau_n(m-1) + |\ 0_{m-1},1,\cdots,N-2,N\ |,\end{aligned}$$

from which we obtain eq.(2.21). This completes the proof. □

Finally, eq.(2.11) is derived by applying Lemma 2.1 to the Plücker relation

$$\begin{aligned}0 &= |\ 0_{m-1},0,1,\cdots,N-2\ |\times|\ 1,\cdots,N-2,N-1,N\ |\\ &\quad -|\ 0,1,\cdots,N-2,N-1\ |\times|\ 0_{m-1},1,\cdots,N-2,N\ |\\ &\quad +|\ 0,1,\cdots,N-2,N\ |\times|\ 0_{m-1},1,\cdots,N-2,N-1\ |.\end{aligned}\tag{2.23}$$

Therefore we have shown that the τ function (2.13) actually satisfies the bilinear equation (2.11). Other equations are derived in a similar manner. We refer to [16, 17] for further details of the technique.

§ 3. Non-autonomous Discrete KdV Equation

§ 3.1. Casorati Determinant Solution

In this section we consider the following difference equation[10]

$$\begin{aligned}&\left(\frac{1}{a_m}+\frac{1}{b_{n+1}}\right)v_{n+1}^m-\left(\frac{1}{a_{m+1}}+\frac{1}{b_n}\right)v_n^{m+1}\\ &=\left(\frac{1}{a_m}-\frac{1}{b_n}\right)\frac{1}{v_n^m}-\left(\frac{1}{a_{m+1}}-\frac{1}{b_{n+1}}\right)\frac{1}{v_{n+1}^{m+1}},\end{aligned}\tag{3.1}$$

where a_m, b_n are arbitrary functions of m and n, respectively. If a_m and b_n are constants, eq.(3.1) is equivalent to the discrete KdV equation (1.7). We call eq.(3.1) the non-autonomous discrete KdV equation.

The N–soliton solutions to eq.(3.1) can be expressed by Casorati determinants as follows:

Theorem 3.1. *For each $N \in \mathbb{N}$, we define an $N \times N$ determinant τ_n^m by*

$$\tau_n^m = \begin{vmatrix} \varphi_1^{(s)}(m,n) & \varphi_1^{(s+1)}(m,n) & \cdots & \varphi_1^{(s+N-1)}(m,n) \\ \varphi_2^{(s)}(m,n) & \varphi_2^{(s+1)}(m,n) & \cdots & \varphi_2^{(s+N-1)}(m,n) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(s)}(m,n) & \varphi_N^{(s+1)}(m,n) & \cdots & \varphi_N^{(s+N-1)}(m,n) \end{vmatrix}, \tag{3.2}$$

$$\begin{aligned}\varphi_r^{(s)}(m,n) =& \alpha_r p_r^s \prod_{i=m_0}^{m-1} (1+a_i p_r) \prod_{j=n_0}^{n-1} (1+b_j p_r) \\ &+ \beta_r (-p_r)^s \prod_{i=m_0}^{m-1} (1-a_i p_r) \prod_{j=n_0}^{n-1} (1-b_j p_r).\end{aligned} \tag{3.3}$$

Then

$$v_n^m = \frac{\tau_{n+1}^m \tau_n^{m+1}}{\tau_n^m \tau_{n+1}^{m+1}}, \tag{3.4}$$

satisfies eq.(3.1).

Unlike the autonomous case, eq.(3.1) cannot be put into the bilinear equation directly in terms of a single τ function τ_n^m because of the non-autonomous property. This difficulty is overcome by introducing suitable auxiliary τ functions. In the following, we discuss three different bilinearizations.

§ 3.2. Bilinearization (I)

Proposition 3.2. *Let τ_m^n and σ_m^n be functions satisfying the bilinear equations*

$$-\epsilon(a_m - b_n)\ \tau_n^m \sigma_{n+1}^{m+1} + a_m(b_n+\epsilon)\ \tau_{n+1}^m \sigma_n^{m+1} - b_n(\epsilon + a_m)\ \tau_n^{m+1}\sigma_{n+1}^m = 0, \tag{3.5}$$

$$\epsilon(a_m - b_n)\ \sigma_n^m \tau_{n+1}^{m+1} + a_m(b_n-\epsilon)\ \tau_n^{m+1} \sigma_{n+1}^{m} + b_n(\epsilon - a_m)\ \tau_{n+1}^{m}\sigma_n^{m+1} = 0, \tag{3.6}$$

where ϵ is a constant. Then

$$\Psi_n^m = \frac{\sigma_n^m}{\tau_n^m}, \quad v_n^m = \frac{\tau_{n+1}^m \tau_n^{m+1}}{\tau_n^m \tau_{n+1}^{m+1}}, \tag{3.7}$$

satisfy

$$\left(\frac{1}{b_n} - \frac{1}{a_m}\right) \frac{1}{v_n^m}\Psi_{n+1}^{m+1} - \left(\frac{1}{\epsilon} + \frac{1}{b_n}\right) \Psi_n^{m+1} + \left(\frac{1}{\epsilon} + \frac{1}{a_m}\right) \Psi_{n+1}^m = 0, \tag{3.8}$$

$$\left(\frac{1}{b_n} - \frac{1}{a_m}\right) \frac{1}{v_n^m}\Psi_n^m + \left(\frac{1}{\epsilon} - \frac{1}{b_n}\right) \Psi_{n+1}^m + \left(\frac{1}{a_m} - \frac{1}{\epsilon}\right) \Psi_n^{m+1} = 0, \tag{3.9}$$

and the non-autonomous discrete KdV equation (3.1). In particular, eq.(3.2) and

$$\sigma_n^m = \begin{vmatrix} \widetilde{\varphi}_1^{(s)}(m,n) & \widetilde{\varphi}_1^{(s+1)}(m,n) & \cdots & \widetilde{\varphi}_1^{(s+N-1)}(m,n) \\ \widetilde{\varphi}_2^{(s)}(m,n) & \widetilde{\varphi}_2^{(s+1)}(m,n) & \cdots & \widetilde{\varphi}_2^{(s+N-1)}(m,n) \\ \vdots & \vdots & \cdots & \vdots \\ \widetilde{\varphi}_N^{(s)}(m,n) & \widetilde{\varphi}_N^{(s+1)}(m,n) & \cdots & \widetilde{\varphi}_N^{(s+N-1)}(m,n) \end{vmatrix}, \tag{3.10}$$

where

$$\begin{aligned} \widetilde{\varphi}_r^{(s)}(m,n) = {} & \alpha_r p_r^s (1+\epsilon p_r) \prod_{i=m_0}^{m-1} (1+a_i p_r) \prod_{j=n_0}^{n-1} (1+b_j p_r) \\ & + \beta_r (-p_r)^s (1-\epsilon p_r) \prod_{i=m_0}^{m-1} (1-a_i p_r) \prod_{j=n_0}^{n-1} (1-b_j p_r), \end{aligned} \tag{3.11}$$

solve the bilinear equations (3.5) and (3.6).

The bilinearization described in Proposition 3.2 is derived from the discrete KP hierarchy. The key idea is to introduce auxiliary "autonomous" independent variables (corresponding lattice intervals are constants) simultaneously, and to apply the reduction procedure through those autonomous variables. Let us take $k = l_1$, $l = l_2$, $m = l_3$ and $n = l_4$, and choose the corresponding lattice intervals as $a_1(l_1) = \delta$, $a_2(l_2) = \epsilon$, $a_3(l_3) = a_m$, $a_4(l_4) = b_n$, where δ and ϵ are constants. The variables k and l are the autonomous variables mentioned above.

We now consider the discrete KP equation (2.9) with respect to the variables (k, m, n)

$$\begin{aligned} & \delta(a_m - b_n)\, \tau(k+1,l,m,n)\tau(k,l,m+1,n+1) \\ & + a_m(b_n - \delta)\, \tau(k,l,m+1,n)\tau(k+1,l,m,n+1) \\ & + b_n(\delta - a_m)\, \tau(k,l,m,n+1)\tau(k+1,l,m+1,n) = 0, \end{aligned} \tag{3.12}$$

and the same equation with respect to the variables (l, m, n)

$$\begin{aligned} & \epsilon(a_m - b_n)\, \tau(k,l+1,m,n)\tau(k,l,m+1,n+1) \\ & + a_m(b_n - \epsilon)\, \tau(k,l,m+1,n)\tau(k,l+1,m,n+1) \\ & + b_n(\epsilon - a_m)\, \tau(k,l,m,n+1)\tau(k,l+1,m+1,n) = 0. \end{aligned} \tag{3.13}$$

Under this setting, the τ function (2.1) is written as

$$\tau(k,l,m,n) = \begin{vmatrix} \varphi_1^{(s)} & \varphi_1^{(s+1)} & \cdots & \varphi_1^{(s+N-1)} \\ \varphi_2^{(s)} & \varphi_2^{(s+1)} & \cdots & \varphi_2^{(s+N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(s)} & \varphi_N^{(s+1)} & \cdots & \varphi_N^{(s+N-1)} \end{vmatrix}, \tag{3.14}$$

$$
\begin{aligned}
\varphi_r^{(s)}(k,l,m,n) =& \alpha_r p_r^s (1+\delta p_r)^k (1+\epsilon p_r)^l \prod_{i=m_0}^{m-1}(1+a_i p_r) \prod_{j=n_0}^{n-1}(1+b_j p_r) \\
&+\beta_r q_r^s (1+\delta q_r)^k (1+\epsilon q_r)^l \prod_{i=m_0}^{m-1}(1+a_i q_r) \prod_{j=n_0}^{n-1}(1+b_j q_r).
\end{aligned} \tag{3.15}
$$

We next impose the reduction condition on the autonomous independent variables k, l as

$$
\tau(k+1,l+1,m,n) \doteqdot \tau(k,l,m,n). \tag{3.16}
$$

This is achieved by imposing the condition on $\varphi_r^{(s)}$ $(r=1,\dots,N)$ as

$$
\varphi_r^{(s)}(k+1,l+1,m,n) \doteqdot \varphi_r^{(s)}(k,l,m,n). \tag{3.17}
$$

In order to realize eq.(3.17), one may take

$$
q_r = -p_r, \quad \delta = -\epsilon, \tag{3.18}
$$

so that

$$
\begin{aligned}
\varphi_r^{(s)}(k+1,l+1,m,n) &= (1-\epsilon^2 p_r^2)\ \varphi_r^{(s)}(k,l,m,n), \\
\tau(k+1,l+1,m,n) &= \prod_{r=1}^{N}(1-\epsilon^2 p_r^2)\ \tau(k,l,m,n).
\end{aligned} \tag{3.19}
$$

Then, suppressing the k-dependence by using eq.(3.16), the bilinear equations (3.12) and (3.13) are reduced to

$$
\begin{aligned}
&-\epsilon(a_m-b_n)\ \tau(l,m,n)\tau(l+1,m+1,n+1) + a_m(b_n+\epsilon)\ \tau(l+1,m+1,n)\tau(l,m,n+1) \\
&-b_n(\epsilon+a_m)\ \tau(l+1,m,n+1)\tau(l,m+1,n) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\epsilon(a_m-b_n)\ \tau(l+1,m,n)\tau(l,m+1,n+1) + a_m(b_n-\epsilon)\ \tau(l,m+1,n)\tau(l+1,m,n+1) \\
&+b_n(\epsilon-a_m)\ \tau(l,m,n+1)\tau(l+1,m+1,n) = 0,
\end{aligned}
$$

respectively. By putting

$$
\tau_n^m = \tau(l,m,n), \quad \sigma_n^m = \tau(l+1,m,n), \tag{3.20}
$$

we obtain the bilinear equations (3.5) and (3.6). Then an easy calculation shows that Ψ_n^m and v_n^m satisfy eqs.(3.8) and (3.9).

We finally show that v_n^m satisfies eq.(3.1). Eq.(3.1) is derived from the cubic equation in terms of τ_n^m which is obtained by eliminating σ_n^m from the bilinear equations (3.5) and (3.6). However, this procedure can be done more systematically in the following manner. Introducing a vector

$$
\Phi_n^m = \begin{pmatrix} \Psi_n^{m+1} \\ \Psi_n^m \end{pmatrix}, \tag{3.21}
$$

eqs.(3.8) and (3.9) can be rewritten as the following linear system:

$$\Phi_{n+1}^m = L_n^m \Phi_n^m, \quad \Phi_n^{m+1} = M_n^m \Phi_n^m, \tag{3.22}$$

$$L_n^m = \frac{1}{\frac{1}{b_n} - \frac{1}{\epsilon}} \begin{pmatrix} \left(\frac{1}{b_n} + \frac{1}{a_m}\right) v_n^m & -\left(\frac{1}{a_m} + \frac{1}{\epsilon}\right) \\ \frac{1}{a_m} - \frac{1}{\epsilon} & \left(\frac{1}{b_n} - \frac{1}{a_m}\right) \frac{1}{v_n^m} \end{pmatrix}, \tag{3.23}$$

$$M_n^m = \frac{1}{\frac{1}{a_{m+1}} - \frac{1}{\epsilon}} \begin{pmatrix} \left(\frac{1}{b_n} + \frac{1}{a_m}\right) v_n^m - \left(\frac{1}{b_n} - \frac{1}{a_{m+1}}\right) v_n^{m+1} & -\left(\frac{1}{a_m} + \frac{1}{\epsilon}\right) \\ \frac{1}{a_{m+1}} - \frac{1}{\epsilon} & 0 \end{pmatrix}. \tag{3.24}$$

Then the compatibility condition of the linear system

$$L_n^{m+1} M_n^m = M_{n+1}^m L_n^m, \tag{3.25}$$

gives eq.(3.1). This completes the proof of Proposition 3.2.

Remark.

1. The linear system (3.22)–(3.24) is the auxiliary linear system of eq.(3.1) and the matrices L_n^m, M_n^m are the Lax pair, where the lattice interval ϵ plays a role of the spectral parameter. In this sense, the bilinearization in this section can be regarded as that for the auxiliary linear system.
2. If we eliminate v_n^m from eqs.(3.8) and (3.9), and put $w_n^m = \Psi_n^m$, we obtain the non-autonomous potential modified KdV equation

$$w_{n+1}^{m+1} = w_n^m \frac{\gamma_n^m\ w_{n+1}^m - w_n^{m+1}}{-w_{n+1}^m + \gamma_n^m\ w_n^{m+1}}, \qquad \gamma_n^m = \frac{b_n}{a_m}, \tag{3.26}$$

by taking $\epsilon \to \infty$. The solution of eq.(3.26) admits several expressions. For example, let us use the internal variable s in eq.(3.2) explicitly and write $\tau_n^m = \tau_n^m(s)$. Then it is shown that $w_n^m = \frac{\tau_n^m(s+1)}{\tau_n^m(s)}$ satisfies eq.(3.26). Similarly, writing τ_n^m with determinant size N as $\tau_n^m = \tau_n^m(N)$, then it is also shown that $w_n^m = \frac{\tau_n^m(N+1)}{\tau_n^m(N)}$ satisfies eq.(3.26).

§ 3.3. Bilinearization (II)

The non-autonomous discrete KdV equation (3.1) admits an alternate bilinearization involving an auxiliary τ function which does not appear in the expression of the solution.

Proposition 3.3. *Let τ_n^m and κ_n^m be functions satisfying the bilinear equations*

$$b_n(a_{m-1}+a_m)\kappa_{n+1}^m\tau_n^m - a_{m-1}(a_m+b_n)\tau_{n+1}^{m-1}\tau_n^{m+1} + a_m(a_{m-1}-b_n)\tau_{n+1}^{m+1}\tau_n^{m-1} = 0, \tag{3.27}$$

$$b_n(a_{m-1}-a_m)\tau_n^{m+1}\tau_{n+1}^{m-1} - a_m(a_{m-1}-b_n)\tau_{n+1}^m\kappa_n^m + a_{m-1}(a_m-b_n)\tau_n^m\kappa_{n+1}^m = 0. \tag{3.28}$$

Then v_n^m defined in eq.(3.4) satisfies eq.(3.1). In particular, τ_n^m in eq.(3.2) and

$$\kappa_n^m = \begin{vmatrix} \psi_1^{(s)}(m,n) & \psi_1^{(s+1)}(m,n) & \cdots & \psi_1^{(s+N-1)}(m,n) \\ \psi_2^{(s)}(m,n) & \psi_2^{(s+1)}(m,n) & \cdots & \psi_2^{(s+N-1)}(m,n) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_N^{(s)}(m,n) & \psi_N^{(s+1)}(m,n) & \cdots & \psi_N^{(s+N-1)}(m,n) \end{vmatrix}, \tag{3.29}$$

$$\begin{aligned} \psi_r^{(s)}(m,n) =& \alpha_r p_r^s (1+a_m p_r)\prod_{j=m_0}^{m-2}(1+a_j p_r)\prod_{k=n_0}^{n-1}(1+b_k p_r) \\ &+ \beta_r(-p_r)^s(1-a_m p_r)\prod_{j=l_0}^{m-2}(1-a_j q_r)\prod_{k=n_0}^{n-1}(1-b_k p_r), \end{aligned} \tag{3.30}$$

solve eqs. (3.27) and (3.28).

Remark. In the autonomous case, namely if a_m and b_n are constants, the auxiliary τ function κ_n^m reduces to τ_n^m, the bilinear equation (3.27) yields the equation which is equivalent to eq.(1.6), and eq.(3.28) becomes trivial, respectively.

Proposition 3.3 is proved by applying the Casoratian technique based on the linear relations among the entries of the determinants

$$\varphi_r^{(s)}(m+1,n) - \varphi_r^{(s)}(m,n) = a_m\ \varphi_r^{(s+1)}(m,n), \tag{3.31}$$

$$\varphi_r^{(s)}(m-1,n) + a_m\ \varphi_r^{(s+1)}(m-1,n) = \psi_r^{(s)}(m,n), \tag{3.32}$$

$$\psi_r^{(s)}(m,n) - a_m\ \psi_r^{(s+1)}(m,n) = (1-a_m^2p_r^2)\ \varphi_r^{(s)}(m-1,n), \tag{3.33}$$

$$\varphi_r^{(s)}(m,n+1) - \varphi_r^{(s)}(m,n) = b_n\ \varphi_r^{(s+1)}(m,n). \tag{3.34}$$

We refer to [9] for further details of the proof.

§ 3.4. Bilinearization (III)

The non-autonomous discrete KdV equation (3.1) admits the third bilinearization through the non-autonomous version of the potential discrete KdV equation[15]

$$u_{n+1}^{m+1} - u_n^m = \left(\frac{1}{a_m^2} - \frac{1}{b_n^2}\right)\frac{1}{u_n^{m+1} - u_{n+1}^m}, \tag{3.35}$$

or

$$\left[\tilde{u}_{n+1}^{m+1} - \tilde{u}_n^m - \left(\frac{1}{b_m} + \frac{1}{c_n}\right)\right] \left[\tilde{u}_n^{m+1} - \tilde{u}_{n+1}^m - \left(\frac{1}{b_m} - \frac{1}{c_n}\right)\right] = \frac{1}{b_m^2} - \frac{1}{c_n^2}, \tag{3.36}$$

where u_n^m and $\tilde{u}_n^m$ are related as

$$\tilde{u}_n^m = u_n^m + \sum_{i=m_0}^{m-1} \frac{1}{a_i} + \sum_{j=n_0}^{n-1} \frac{1}{b_j}. \tag{3.37}$$

We note that u_n^m is related to v_n^m in eq.(3.1) as

$$\left(\frac{1}{a_m} - \frac{1}{b_n}\right) \frac{1}{v_n^m} = u_{n+1}^m - u_n^{m+1}. \tag{3.38}$$

Proposition 3.4. *Let τ_n^m and ρ_n^m be functions satisfying the bilinear equations*

$$\rho_n^{m+1}\tau_{n+1}^m - \rho_{n+1}^m\tau_n^{m+1} = \left(\frac{1}{a_m} - \frac{1}{b_n}\right) \left(\tau_n^{m+1}\tau_{n+1}^m - \tau_{n+1}^{m+1}\tau_n^m\right), \tag{3.39}$$

$$\rho_{n+1}^{m+1}\tau_n^m - \rho_n^m\tau_{n+1}^{m+1} = \left(\frac{1}{a_m} + \frac{1}{b_n}\right) \left(\tau_{n+1}^{m+1}\tau_n^m - \tau_n^{m+1}\tau_{n+1}^m\right). \tag{3.40}$$

Then v_n^m defined by eq.(3.4) and

$$u_n^m = \frac{\rho_n^m}{\tau_n^m} - \sum_{i=m_0}^{m-1} \frac{1}{a_i} - \sum_{j=n_0}^{n-1} \frac{1}{b_j}, \tag{3.41}$$

satisfy eq.(3.1) and eq.(3.35), respectively. In particular, τ_n^m defined by eq.(3.2) and

$$\rho_n^m = \begin{vmatrix} \varphi_1^{(s)}(m,n) & \cdots & \varphi_1^{(s+N-2)}(m,n) & \varphi_1^{(s+N)}(m,n) \\ \varphi_2^{(s)}(m,n) & \cdots & \varphi_2^{(s+N-2)}(m,n) & \varphi_2^{(s+N)}(m,n) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(s)}(m,n) & \cdots & \varphi_N^{(s+N-2)}(m,n) & \varphi_N^{(s+N)}(m,n) \end{vmatrix}, \tag{3.42}$$

where $\varphi_r^{(s)}(m,n)$ $(r = 1, \ldots, N)$ are given by eq.(3.3) solve eqs.(3.39) and (3.40).

Proof of Proposition 3.4 is given by the Casoratian technique by using the linear relations (3.31) and (3.34)[9].

Remark.

1. Eq.(3.38) follows immediately from eq.(3.39) by dividing the both sides by $\tau_{n+1}^m\tau_n^{m+1}$.

2. If we introduce the continuous independent variables x_1, x_3, $\cdots$ through $\varphi_r^{(s)}(m,n)$ as

$$\begin{aligned}\varphi_r^{(s)}(m,n) =&\alpha_r p_r^s \prod_{j=m_0}^{m-1}(1+b_jp_r)\prod_{k=n_0}^{n-1}(1+c_kp_r)\ e^{p_rx_1+p_r^3x_3+\cdots}\\ &+\beta_r(-p_r)^s\prod_{j=m_0}^{m-1}(1-b_jp_r)\prod_{k=n_0}^{n-1}(1-c_kp_r)\ e^{-p_rx_1-p_r^3x_3+\cdots},\end{aligned} \tag{3.43}$$

then τ_n^m becomes the τ function of the KdV hierarchy. In this case, ρ_n^m and $\tilde{u}_n^m$ can be expressed as

$$\rho_n^m = \frac{\partial \tau_n^m}{\partial x_1}, \quad \tilde{u}_n^m = \frac{\partial}{\partial x_1}\log\tau_n^m, \tag{3.44}$$

respectively. Accordingly, $\tilde{u}_n^m$ satisfies (3.36) and the potential KdV equation

$$\frac{\partial \tilde{u}_n^m}{\partial x_3} - \frac{3}{2}\left(\frac{\partial \tilde{u}_n^m}{\partial x_1}\right)^2 - \frac{1}{4}\frac{\partial^3 \tilde{u}_n^m}{\partial x_1^3} = 0, \tag{3.45}$$

simultaneously. This is consistent with the fact that the potential discrete KdV equation is derived as the Bäcklund transformation of the potential KdV equation[15].

§ 4. Non-autonomous Discrete Toda Lattice Equation

§ 4.1. Casorati Determinant Solution

The non-autonomous discrete 1DTL equation is given by[4, 8, 12, 13, 22]

$$\begin{aligned}&A_n^{t+1} + B_n^{t+1} + \lambda_{t+1} = A_n^t + B_{n+1}^t + \lambda_t,\\ &A_{n-1}^{t+1}B_n^{t+1} = A_n^tB_n^t,\end{aligned} \tag{4.1}$$

where λ_t is an arbitrary function in t. The N-soliton solution is expressed by Casorati determinants as follows[8, 22]:

Theorem 4.1. *For each $N \in \mathbb{N}$, we define an $N\times N$ determinant τ_n^t by*

$$\tau_n^t = \begin{vmatrix} \varphi_1^{(n)}(t) & \varphi_1^{(n+1)}(t) & \cdots & \varphi_1^{(n+N-1)}(t)\\ \varphi_2^{(n)}(t) & \varphi_2^{(n+1)}(t) & \cdots & \varphi_2^{(n+N-1)}(t)\\ \vdots & \vdots & \cdots & \vdots\\ \varphi_N^{(n)}(t) & \varphi_N^{(n+1)}(t) & \cdots & \varphi_N^{(n+N-1)}(t)\end{vmatrix}, \tag{4.2}$$

$$\varphi_r^{(n)}(n,t) = \alpha_r p_r^n \prod_{j=t_0}^{t-1} (1 - p_r\mu_j) + \beta_r p_r^{-n} \prod_{j=t_0}^{t-1} (1 - p_r^{-1}\mu_j), \tag{4.3}$$

where λ_t *is an arbitrary function in* t, α_r, β_r *are arbitrary constants, and* p_r *are parameters* ($r = 1, \dots, N$)*. Then*

$$A_n^t = -\mu_t^{-1} \frac{\tau_n^t \tau_{n+1}^{t+1}}{\tau_{n+1}^t \tau_n^{t+1}}, \quad B_n^t = -\mu_t \frac{\tau_{n-1}^{t+1} \tau_{n+1}^t}{\tau_n^t \tau_n^{t+1}}, \quad \lambda_t = \mu_t + \mu_t^{-1}, \tag{4.4}$$

satisfy (4.1).

§ 4.2. Bilinearization (I)

Proposition 4.2. *Let* τ_n^t *and* θ_n^t *be functions satisfying bilinear equations*

$$(1 - \delta\mu_t)\ \tau_n^{t+1}\theta_n^t - \tau_n^t\theta_n^{t+1} + \delta\mu_t\ \tau_{n+1}^t\theta_{n-1}^{t+1} = 0, \tag{4.5}$$

$$\mu_t\ \tau_n^{t+1}\theta_{n+1}^t - \delta\ \tau_{n+1}^{t+1}\theta_n^t = (\mu_t - \delta)\ \tau_{n+1}^t\theta_n^{t+1}. \tag{4.6}$$

Then

$$\Psi_n^t = \frac{\theta_n^t}{\tau_n^t}, \quad A_n^t = -\mu_t^{-1} \frac{\tau_n^t \tau_{n+1}^{t+1}}{\tau_{n+1}^t \tau_n^{t+1}}, \quad B_n^t = -\mu_t \frac{\tau_{n-1}^{t+1} \tau_{n+1}^t}{\tau_n^t \tau_n^{t+1}}, \tag{4.7}$$

satisfy

$$(1 - \delta\mu_t)\ \Psi_n^t - \Psi_n^{t+1} - \delta B_n^t \Psi_{n-1}^{t+1} = 0, \tag{4.8}$$

$$\mu_t\ \Psi_{n+1}^t + \delta\mu_t\ A_n^t \Psi_n^t = (\mu_t - \delta)\ \Psi_n^{t+1}. \tag{4.9}$$

and the non-autonomous discrete 1DTL equation (4.1). In particular, eq.(4.2) and

$$\theta_n^t = \begin{vmatrix} \overline{\varphi}_1^{(n)}(t) & \overline{\varphi}_1^{(n+1)}(t) & \cdots & \overline{\varphi}_1^{(n+N-1)}(t) \\ \overline{\varphi}_2^{(n)}(t) & \overline{\varphi}_2^{(n+1)}(t) & \cdots & \overline{\varphi}_2^{(n+N-1)}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \overline{\varphi}_N^{(n)}(t) & \overline{\varphi}_N^{(n+1)}(t) & \cdots & \overline{\varphi}_N^{(n+N-1)}(t) \end{vmatrix}, \tag{4.10}$$

where

$$\overline{\varphi}_r^{(n)}(n,t) = \alpha_r p_r^n (1 - \delta p_r) \prod_{j=t_0}^{t-1} (1 - p_r\mu_j) + \beta_r p_r^{-n} \left(1 - \delta p_r^{-1}\right) \prod_{j=t_0}^{t-1} (1 - p_r^{-1}\mu_j). \tag{4.11}$$

solve the bilinear equations (4.5) and (4.6).

Proposition 4.2 can be proved by the similar technique to that in Section 3.2. Let us take $k = l_1$, $t = l_2$, $l = m_1$ and $n = s$, and choose the corresponding lattice intervals as $a_1(l_1) = -\delta$, $a_2(l_2) = -\mu_t$, $b_1(m_1) = \epsilon$, where δ and ϵ are constants. The variables k and l are the autonomous variables. We consider the discrete 2DTL equation (2.10) with respect to the variables (n, l, t)

$$\begin{aligned}(1+\epsilon\mu_t)\tau_n(k,l+1,t+1)\tau_n(k,l,t) - \tau_n(k,l,t+1)\tau_n(k,l+1,t)\\ -\epsilon\mu_t\tau_{n+1}(k,l+1,t)\tau_{n-1}(k,l,t+1) = 0,\end{aligned} \tag{4.12}$$

and its Bäcklund transformation (2.12) with respect to the variable (n, k, t)

$$\begin{aligned}\mu_t\tau_{n+1}(k+1,l,t)\tau_n(k,l,t+1) - \delta\tau_{n+1}(k,l,t+1)\tau_n(k+1,l,t)\\ -(\mu_t-\delta)\tau_{n+1}(k,l,t)\tau_n(k+1,l,t+1) = 0.\end{aligned} \tag{4.13}$$

Under this setting, τ function (2.1) is now written as

$$\tau_n(k,l,t) = \begin{vmatrix} \varphi_1^{(n)} & \varphi_1^{(n+1)} & \cdots & \varphi_1^{(n+N-1)} \\ \varphi_2^{(n)} & \varphi_2^{(n+1)} & \cdots & \varphi_2^{(n+N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(n)} & \varphi_N^{(n+1)} & \cdots & \varphi_N^{(n+N-1)} \end{vmatrix}, \tag{4.14}$$

$$\begin{aligned}\varphi_r^{(n)}(k,l,t) =& \alpha_r p_r^n (1-\delta p_r)^k (1+\epsilon p_r^{-1})^l \prod_{i=i_0}^{t-1}(1-\mu_t p_r)\\ &+ \beta_r q_r^n (1-\delta q_r)^k (1+\epsilon q_r^{-1})^l \prod_{i=i_0}^{t-1}(1-\mu_t q_r).\end{aligned} \tag{4.15}$$

We impose the reduction condition on k, l as

$$\tau_n(k+1,l+1,t) \doteqdot \tau_n(k,l,t). \tag{4.16}$$

This is realized by choosing the parameters of the solutions as

$$\epsilon = -\delta, \quad q_r = \frac{1}{p_r} \tag{4.17}$$

so that

$$\begin{aligned}\varphi_r^{(n)}(k+1,l+1,t) &= (1-\delta p_r)(1-\delta p_r^{-1})\ \varphi_r^{(n)}(k,l,t),\\ \tau_n(k+1,l+1,t) &= \prod_{r=1}^{N}(1-\delta p_r)(1-\delta p_r^{-1})\ \tau_n(k,l,t).\end{aligned} \tag{4.18}$$

Suppressing l-dependence by using eq.(4.16), and putting

$$\tau_n(k,l,t) = \tau_n^t, \quad \tau_n(k+1,l,t) = \theta_n^t, \tag{4.19}$$

the bilinear equations (4.12) and (4.13) are reduced to eqs.(4.5) and (4.6), respectively.

It is clear that the linear equations (4.8) and (4.9) follow from the bilinear equations (4.5) and (4.6), respectively, through the dependent variable transformation (4.7). In order to obtain eq.(4.1), we introduce

$$\Phi_n^t = \begin{pmatrix} \Psi_{n+1}^t \\ \Psi_n^t \end{pmatrix}. \tag{4.20}$$

After some manipulation, the linear equations (4.8) and (4.9) can be rewritten as

$$\Phi_{n+1}^t = L_n^t \Phi_n^t, \quad \Phi_n^{t+1} = M_n^t \Phi_n^t, \tag{4.21}$$

$$L_n^t = \begin{pmatrix} -\delta\left[A_{n+1}^t + B_{n+1}^t + (1-\delta\mu_t)\left(\frac{1}{\mu_t} - \frac{1}{\delta}\right)\right] & -\delta^2 A_n^t B_{n+1}^t \\ 1 & 0 \end{pmatrix}, \tag{4.22}$$

$$M_n^t = \frac{1}{\frac{1}{\mu_t} - \frac{1}{\delta}} \begin{pmatrix} B_{n+1}^t + (1-\delta\mu_t)\left(\frac{1}{\mu_t} - \frac{1}{\delta}\right) & \delta A_n^t B_{n+1}^t \\ -\frac{1}{\delta} & -A_n^t \end{pmatrix}. \tag{4.23}$$

Then the compatibility condition $L_n^{t+1} M_n^t = M_{n+1}^t L_{n+1}^t$ gives eq.(4.1). This completes the proof of Proposition 4.2.

§ 4.3. Bilinearization (II)

There is an alternate bilinearization for eq.(4.1), which is similar to the second bilinearization of the non-autonomous discrete KdV equation discussed in Section 3.3.

Proposition 4.3. *Let τ_n^t and η_n^t be functions satisfying the bilinear equations*

$$\tau_n^{t+1}\tau_n^{t-1} - \tau_n^t \eta_n^t = \mu_t \mu_{t-1}\left(\tau_{n-1}^{t+1}\tau_{n+1}^{t-1} - \tau_n^t \eta_n^t\right), \tag{4.24}$$

$$\mu_t \eta_n^t \tau_{n+1}^t - \mu_{t-1}\tau_n^t \eta_{n+1}^t = (\mu_t - \mu_{t-1})\tau_n^{t+1}\tau_{n+1}^{t-1}. \tag{4.25}$$

Then A_n^t and B_n^t defined in eq.(4.7) satisfy eq.(4.1). In particular, eq.(4.2) and

$$\eta_n^t = \begin{vmatrix} \psi_1^{(n)}(t) & \psi_1^{(n+1)}(t) & \cdots & \psi_1^{(n+N-1)}(t) \\ \psi_2^{(n)}(t) & \psi_2^{(n+1)}(t) & \cdots & \psi_2^{(n+N-1)}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_N^{(n)}(t) & \psi_N^{(n+1)}(t) & \cdots & \psi_N^{(n+N-1)}(t) \end{vmatrix}, \tag{4.26}$$

$$\psi_r^{(n)}(t) = \alpha_r p_r^n (1 - p_i\mu_t) \prod_{j=t_0}^{t-2}(1 - p_r\mu_j) + \beta_r p_r^{-n}(1 - p_i^{-1}\mu_t) \prod_{j=t_0}^{t-2}(1 - p_r^{-1}\mu_j), \tag{4.27}$$

solve eqs.(4.24) and (4.25).

Eq. (4.1) is derived from the bilinear equations (4.24) and (4.25) as follows; multiplying $(1-\frac{1}{\mu_{t-1}\mu_t})\tau_n^t\tau_{n+1}^t$ to eq.(4.25) and using eq.(4.24) we have

$$\begin{aligned}(4.28)\quad &(\tau_{n+1}^t)^2(\mu_t\tau_{n-1}^{t+1}\tau_{n+1}^{t-1}-\mu_{t-1}^{-1}\tau_n^{t-1}\tau_n^{t+1})-(\tau_n^t)^2(\mu_{t-1}\tau_{n-1}^{t+1}\tau_{n+1}^{t-1}-\mu_t^{-1}\tau_n^{t-1}\tau_n^{t+1})\\ &=(\lambda_t-\lambda_{t-1})\tau_n^t\tau_n^{t+1}\tau_{n+1}^{t-1}\tau_{n+1}^t.\end{aligned}$$

Dividing equation (4.28) by $\tau_n^t\tau_n^{t+1}\tau_{n+1}^{t-1}\tau_{n+1}^t$, we obtain the first equation of eq. (4.1). The second equation is an identity under the variable transformation (4.7).

Proposition 4.3 is proved by applying the Casoratian technique based on the linear relations among the entries of the determinant

$$(4.29)\qquad \varphi_r^{(n)}(t+1)=\varphi_r^{(n)}(t)-\mu_t\varphi_r^{(n+1)}(t),$$

$$(4.30)\qquad \psi_r^{(n)}(t)=\varphi_r^{(n)}(t-1)-\mu_t\varphi_r^{(n+1)}(t-1),$$

$$(4.31)\qquad (1-p_i\mu_t)(1-p_i^{-1}\mu_t)\varphi_r^{(n)}(t-1)=\psi_r^{(n)}(t)-\mu_t\psi_r^{(n-1)}(t).$$

We refer to [8] for further details of the proof.

Remark. Recently Tsujimoto has presented a theoretical background of appearance of the auxiliary τ function η in this section by considering the two-dimensional chain of the Darboux transformations[22].

§ 5. Non-autonomous discrete Lotka-Volterra Equation

§ 5.1. Lotka-Volterra Equation

The Lotka-Volterra equation

$$(5.1)\qquad \frac{d}{dt}\log u_n=u_{n+1}-u_{n-1},$$

can be transformed to the bilinear equation

$$(5.2)\qquad (D_t+1)\tau_{n+1}\cdot\tau_n=\tau_{n-1}\tau_{n+2},$$

through the dependent variable transformation

$$(5.3)\qquad u_n=\frac{\tau_{n-1}\tau_{n+2}}{\tau_{n+1}\tau_n}=\frac{d}{dt}\log\frac{\tau_{n+1}}{\tau_n}+1.$$

The N–soliton solution to eq.(5.1) is given by

$$(5.4)\qquad \tau_n=\begin{vmatrix}\varphi_1^{(n)} & \varphi_1^{(n+1)} & \cdots & \varphi_1^{(n+N-1)}\\ \varphi_2^{(n)} & \varphi_2^{(n+1)} & \cdots & \varphi_2^{(n+N-1)}\\ \vdots & \vdots & \cdots & \vdots\\ \varphi_N^{(n)} & \varphi_N^{(n+1)} & \cdots & \varphi_N^{(n+N-1)}\end{vmatrix},$$

$$\varphi_k^{(n)} = \alpha_k (1+r_k)^n e^{(1+r_k)t} + \beta_k \left(1+\frac{1}{r_k}\right)^n e^{(1+\frac{1}{r_k})t}. \tag{5.5}$$

where α_k, β_k are arbitrary constants and r_k are parameters $(k = 1, \ldots, N)$.

The Lotka-Volterra equation is reduced from the Bäcklund transformation of the 2DTL equation (2.11). Let us impose a reduction condition for $\tau_n(m)$ and $\varphi_k^{(n)}(m)$ given in eqs.(2.13) and (2.16), respectively:

$$\tau_n(m+1) \doteqdot \tau_{n+1}(m), \quad \varphi_k^{(n)}(m+1) \doteqdot \varphi_k^{(n+1)}(m), \tag{5.6}$$

The condition (5.6) is achieved by putting

$$b_m = -b, \quad q_k = -\frac{p_k}{1-\dfrac{p_k}{b}}, \tag{5.7}$$

or

$$p_k = b(1+r_k), \quad q_k = b\left(1+\frac{1}{r_k}\right). \tag{5.8}$$

Then eq.(2.11) is rewritten as

$$(D_x + b)\ \tau_n \cdot \tau_{n+1} = b\ \tau_{n+2}\tau_{n-1}. \tag{5.9}$$

Noticing that b can be normalized to be 1 without loss of generality, we obtain the bilinear equation (5.2) and its Casorati determinant solution (5.4) and (5.5).

§ 5.2. Non-autonomous Discrete Lotka-Volterra Equation

The discrete Lotka-Volterra equation[5, 6] can be derived by discretizing the independent variable x in the Bäcklund transformation of 2DTL equation, which implies that it can be formulated as the reduction from the discrete 2DTL equation itself. The reduction procedure works well also for the non-autonomous case without auxiliary τ functions, as shown below.

We consider the non-autonomous discrete 2DTL equation (2.10) with $l = l_1$, $m = m_1$, $n = s$, $a_l = a_1(l_1)$, $b_m = -b_1(m_1)$:

$$\begin{aligned}&(1+a_l b_m)\tau_n(l+1,m+1)\tau_n(l,m) - \tau_n(l+1,m)\tau_n(l,m+1)\\ &= a_l b_m \tau_{n+1}(l,m+1)\tau_{n-1}(l+1,m),\end{aligned} \tag{5.10}$$

where the τ function is given by

$$\tau_n^m = \begin{vmatrix} \varphi_1^{(n)} & \varphi_1^{(n+1)} & \cdots & \varphi_1^{(n+N-1)} \\ \varphi_2^{(n)} & \varphi_2^{(n+1)} & \cdots & \varphi_2^{(n+N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_N^{(n)} & \varphi_N^{(n+1)} & \cdots & \varphi_N^{(n+N-1)} \end{vmatrix},$$

$$\begin{aligned}\varphi_k^{(n)}(l,m) =&\alpha_k p_k^n \prod_{i=l_0}^{l-1}(1+a_i p_k)\prod_{j=m_0}^{m-1}\left(1-b_j p_k^{-1}\right)\\ &+\beta_k q_k^n \prod_{i=l_0}^{l-1}(1+a_i q_k)\prod_{j=m_0}^{m-1}\left(1-b_j q_k^{-1}\right).\end{aligned} \tag{5.11}$$

We impose the reduction condition,

$$\tau_n(l,m+1) \bumpeq \tau_{n+1}(l,m), \quad \varphi_k^{(n)}(l,m+1) \bumpeq \varphi_k^{(n+1)}(l,m), \tag{5.12}$$

which is achieved by putting

$$b_m = 1, \quad p_k = 1 + r_k, \quad q_k = 1 + \frac{1}{r_k}. \tag{5.13}$$

Then $\varphi_k^{(n)}$ is written as

$$\varphi_k^{(n)} = \alpha_k(1+r_k)^n \prod_{i=l_0}^{l-1}(1+a_i+a_i r_k) + \beta_k\left(1+\frac{1}{r_k}\right)^n \prod_{i=l_0}^{l-1}\left(1+a_i+\frac{a_i}{r_k}\right). \tag{5.14}$$

Now suppressing m-dependence and writing $\tau_n(l,m) = \tau_n^l$, the bilinear equation (5.10) is reduced to

$$(1+a_l)\ \tau_{n+1}^{l+1}\tau_n^l - \tau_n^{l+1}\tau_{n+1}^l = a_l\ \tau_{n+2}^l\tau_{n-1}^{l+1}, \tag{5.15}$$

from which we obtain the non-autonomous discrete Lotka-Volterra equation

$$\frac{1+a_l}{1+a_{l+1}}\ \frac{v_n^{l+1}}{v_n^l} = \frac{1+a_l v_{n+1}^l}{1+a_{l+1}v_{n-1}^{l+1}}, \tag{5.16}$$

through the dependent variable transformation

$$v_n^l = \frac{\tau_{n+2}^l\tau_{n-1}^{l+1}}{\tau_n^{l+1}\tau_{n+1}^l}. \tag{5.17}$$

Eq.(5.16) is equivalent to the generalization of the discrete Lotka-Volterra equation in [4, 18, 19, 20]. Further generalization of eq.(5.16) is proposed in [18] which corresponds to de-autonomization of n. This equation is derived from the non-autonomous discrete potential KdV equation (3.35) through the change of coordinates and the Miura transformation.

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