

# Constructing two-dimensional integrable mappings that possess invariants of high degree

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## Abstract

We propose a method for constructing two-dimensional integrable mappings that possess invariants with degree higher than two. Such integrable mappings are obtained by making a composition of a QRT mapping and a mapping that preserves the invariant curve of the QRT mapping except for changing the integration constant involved. We show several concrete examples whose integration constants change with period 2 and 3.

## § 1. Introduction

Integrable mappings have attracted much attention, and many studies have examined them from various viewpoints, such as integrability criteria (singularity confinement property [1], algebraic entropy [2]) and geometric or algebraic description of the equations [3–8].

In particular, second-order integrable mappings have been extensively studied, and a number of significant properties have been obtained.

The QRT mapping introduced by Quispel, Roberts and Thompson in 1989 [9, 10] is the 12-parameter family of second-order integrable mappings and is given by

$$(1.1) \quad x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_2(x_n) - x_{n-1}f_3(x_n)},$$

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where  $f_j(x)$  is defined by

$$(1.2) \quad \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = A_0 \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times A_1 \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

with arbitrary symmetric  $3 \times 3$  matrices

$$(1.3) \quad A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \beta_i & \epsilon_i & \zeta_i \\ \gamma_i & \zeta_i & \mu_i \end{pmatrix}, \quad i = 0, 1.$$

Mapping (1.1) possesses an invariant  $K = K(x_{n-1}, x_n)$  satisfying  $K(x_{n-1}, x_n) = K(x_n, x_{n+1})$ , where  $K$  is a ratio of biquadratic polynomials of the form

$$(1.4) \quad K(x, y) = \frac{\alpha_0 x^2 y^2 + \beta_0 x y (x + y) + \gamma_0 (x^2 + y^2) + \epsilon_0 x y + \zeta_0 (x + y) + \mu_0}{\alpha_1 x^2 y^2 + \beta_1 x y (x + y) + \gamma_1 (x^2 + y^2) + \epsilon_1 x y + \zeta_1 (x + y) + \mu_1}.$$

A generalization of mapping (1.1) has been proposed by Quispel, Roberts and Thompson under the name ‘‘asymmetric.’’ It is an 18-parameter family of two-dimensional mappings of the forms

$$(1.5) \quad x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}, \quad y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})},$$

where  $f_j(x), g_j(x)$  are defined by

$$(1.6) \quad \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = A \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times B \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix},$$

$$(1.7) \quad \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = {}^t A \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times {}^t B \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix},$$

with arbitrary  $3 \times 3$  matrices  $A$  and  $B$ . Mapping (1.5) possesses an invariant  $K = K(x_{n-1}, y_{n-1})$  satisfying  $K(x_{n-1}, y_{n-1}) = K(x_n, y_{n-1}) = K(x_n, y_n)$ , where  $K$  is a

ratio of biquadratic polynomials of the form

(1.8)

$$K(x, y) = \frac{\begin{pmatrix} x^2 & x & 1 \end{pmatrix} A \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}}{\begin{pmatrix} x^2 & x & 1 \end{pmatrix} B \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}}$$

$$(1.9) \quad = \frac{a_{00}x^2y^2 + a_{01}x^2y + a_{02}x^2 + a_{10}xy^2 + a_{11}xy + a_{12}x + a_{20}y^2 + a_{21}y + a_{22}}{b_{00}x^2y^2 + b_{01}x^2y + b_{02}x^2 + b_{10}xy^2 + b_{11}xy + b_{12}x + b_{20}y^2 + b_{21}y + b_{22}}.$$

Since both symmetric and asymmetric QRT mappings constitute a multi-parameter family of integrable systems and meet the criteria for integrability, such as singularity confinement and nonzero algebraic entropy, they are the prototypical second-order integrable mappings. However, whether the QRT mapping is the most general second-order integrable mapping has not yet been clarified.

In [11], Hirota et al. investigated the integrability of third-order mappings and proposed nine integrable cases. In a recent work [12], most of them turned out to be compositions of two particular QRT mappings.

While investigating third-order mappings in [11], Hirota et al. have found a new second-order integrable mapping [13]

$$(1.10) \quad (x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n^2 - 1)}{p^2 x_n^2 - 1},$$

which possesses a biquartic invariant

(1.11)

$$K(x_{n-1}, x_n) = \frac{((x_n - x_{n-1})^2 - p^2(x_n x_{n-1} - 1)^2)((x_n + x_{n-1} - a - 1/a)^2 - p^2(x_n x_{n-1} - 1)^2)}{(x_n x_{n-1} - 1)^2}.$$

As the invariant is not biquadratic but biquartic, this is not a QRT-type mapping. After this discovery, Joshi et al. derived this type of mapping as reductions from integrable lattice equations called ABS lattices and revealed that biquadratic invariants of QRT type are in fact building blocks of the biquartic invariants [14]. Integrable mappings that possess invariants of much higher degree have also been obtained [15].

## § 2. A discrete system that possesses a biquartic invariant

In this section, we start with the following second-order mapping:

$$(2.1) \quad x_{n+1} = \frac{1 - x_n^2}{x_{n-1}}.$$

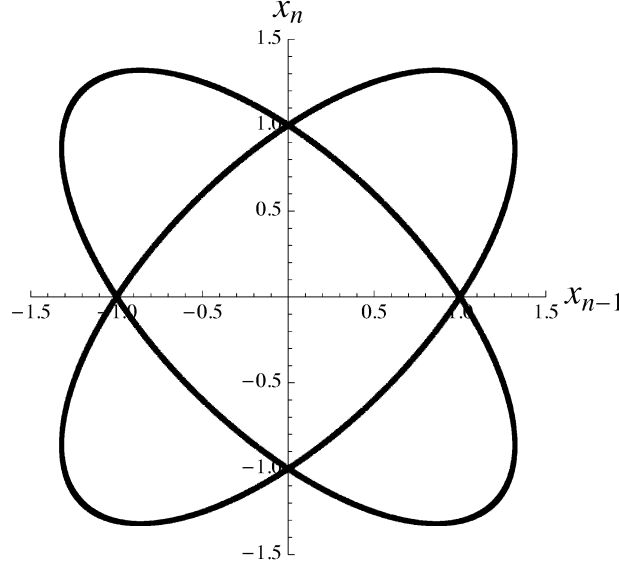


Figure 1. A trajectory of a solution of Eq. (2.1) for  $x_0 = 1.2$ ,  $x_1 = 1.2$

Fig. 1 shows a solution of (2.1) in the phase plane. From this figure, we expect that mapping (2.1) possesses an invariant that is the product of two invariants of certain mappings. In fact, this is true. The invariant of (2.1) is given by

$$(2.2) \quad L(x_{n-1}, x_n) = \left( \frac{x_{n-1}^2 + x_n^2 - 1}{x_{n-1}x_n} \right)^2,$$

which is biquartic and is also the square of a biquadratic,

$$(2.3) \quad K(x_{n-1}, x_n) = \frac{x_{n-1}^2 + x_n^2 - 1}{x_{n-1}x_n}.$$

Two closed smooth curves in the phase plane are given by  $K(x_{n-1}, x_n) = \kappa$  and  $K(x_{n-1}, x_n) = -\kappa$ , where  $\kappa$  is a constant determined by an initial condition of (2.1). From the conservation of  $K(x_{n-1}, x_n) = K(x_n, x_{n+1})$ , we find the QRT mapping

$$(2.4) \quad x_{n+1} = \frac{-1 + x_n^2}{x_{n-1}}.$$

Also, from  $K(x_{n-1}, x_n) = -K(x_n, x_{n+1})$ , we find (2.1). This kind of conservation has been studied by Joshi et al. in [14]. They have found that integrable mappings possessing invariants of high degree are derived from QRT invariants using this kind of conservation, although explicit direct relations between the newly derived mappings and the QRT mappings were not given.

Comparing (2.1) with (2.4), we notice a similarity between the two systems, which reflects a symmetry of  $K(x_{n-1}, x_n)$ .

Let us consider the involution  $\sigma : (x, y) \mapsto (-x, y)$  and the QRT mapping  $\varphi : (x, y) \mapsto \left(y, \frac{-1 + y^2}{x}\right)$ . From the composition of  $\sigma$  and  $\varphi$ , we obtain

$$(2.5) \quad \varphi \circ \sigma : (x, y) \mapsto \left(y, \frac{1 - y^2}{x}\right),$$

which is identical to (2.1). By the involution  $\sigma$ ,  $K(x, y)$  is transformed as follows.

$$(2.6) \quad K(\sigma(x, y)) = -K(x, y),$$

which shows the symmetry of the invariant of  $L(x, y)$ . From the fact above, we obtain

$$(2.7) \quad K(x_n, x_{n+1}) = K(\varphi \circ \sigma(x_{n-1}, x_n)) = K(\sigma(x_{n-1}, x_n)) = -K(x_{n-1}, x_n),$$

which is the conservation of (2.1) where we use the conservation of the QRT mapping  $K(\varphi(x, y)) = K(x, y)$ .

To summarize, the mapping (2.1) is a composition of the QRT mapping (2.4) and the involution  $\sigma$ , which transforms the invariant of the QRT mapping to itself, changing the integration constant.

On the basis of what we explained above, we propose a method for constructing two-dimensional integrable mappings that possess invariants with degree higher than two in the next section.

### § 3. Constructing two-dimensional integrable mappings that possess invariants of high degree

Let us start with invariant curves of QRT mappings

$$(3.1) \quad {}^t\mathbf{x}(A + \kappa B)\mathbf{y} = (a_{00} + \kappa b_{00})x^2y^2 + (a_{01} + \kappa b_{01})x^2y + (a_{02} + \kappa b_{02})x^2 + (a_{10} + \kappa b_{10})xy^2 + (a_{11} + \kappa b_{11})xy + (a_{12} + \kappa b_{12})x + (a_{20} + \kappa b_{20})y^2 + (a_{21} + \kappa b_{21})y + (a_{22} + \kappa b_{22}) = 0,$$

where  $\mathbf{x} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$ ,  $\mathbf{y} = \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}$  and  $\kappa$  is an integration constant. We define the mapping  $\varphi, \psi$  by

$$(3.2) \quad \varphi : (x, y) \mapsto \left( \frac{f_1(y) - xf_2(y)}{f_2(y) - xf_3(y)}, y \right),$$

$$(3.3) \quad \psi : (x, y) \mapsto \left( x, \frac{g_1(x) - yg_2(x)}{g_2(x) - yg_3(x)} \right),$$

where  $f_j(x)$ ,  $g_j(x)$  are given by (1.6) and (1.7).

Next we introduce a mapping  $\sigma : (x, y) \mapsto (u(x, y), v(x, y))$  and require that relations

$$(3.4) \quad \sigma({}^t\mathbf{x}(A + \kappa B)\mathbf{y}) = \sigma({}^t\mathbf{x})(A + \kappa B)\sigma(\mathbf{y}) = \alpha(x, y, \kappa){}^t\mathbf{x}(A + \beta(\kappa)B)\mathbf{y},$$

$$(3.5) \quad \beta^n(\kappa) = \underbrace{\beta \circ \cdots \circ \beta}_n(\kappa) = \kappa,$$

hold with certain matrices  $A$  and  $B$ , a factor  $\alpha(x, y, \kappa)$ , and an  $n$ -unipotent function  $\beta(\kappa)$ . The mapping  $\sigma$  transforms the invariant curve (3.1) into itself, changing the integration constant  $\kappa$  into  $\beta(\kappa)$ .

By constructing the compositions of mappings  $\varphi$ ,  $\psi$ , and  $\sigma$ , we obtain a new mapping

$$(3.6) \quad \psi \circ \varphi \circ \sigma : (x, y) \mapsto (\bar{x}, \bar{y}),$$

$$(3.7) \quad \bar{x} = \frac{\sigma(f_1(y)) - \sigma(xf_2(y))}{\sigma(f_2(y)) - \sigma(xf_3(y))},$$

$$(3.8) \quad \bar{y} = \frac{g_1(\bar{x}) - \sigma(y)g_2(\bar{x})}{g_2(\bar{x}) - \sigma(y)g_3(\bar{x})},$$

which possesses the invariant curve given by

$$(3.9) \quad ({}^t\mathbf{x}(A + \kappa B)\mathbf{y})({}^t\mathbf{x}(A + \beta(\kappa)B)\mathbf{y}) \cdots ({}^t\mathbf{x}(A + \beta^{n-1}(\kappa)B)\mathbf{y}) = 0.$$

In the next section, we show some examples of such mappings using the procedure we have described above.

## § 4. examples

### Example 4.1.

Let us take a mapping  $\sigma$  and matrices  $A, B$  as follows.

$$(4.1) \quad \sigma : (x, y) \mapsto (-x, y),$$

$$(4.2) \quad A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ 0 & 0 & 0 \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 \end{pmatrix}.$$

The invariant curve of the QRT mapping is given by

$$(4.3) \quad a_{00}x^2y^2 + a_{01}x^2y + a_{02}x^2 + a_{20}y^2 + a_{21}y + a_{22} + \kappa(x(b_{10}y^2 + b_{11}y + b_{12})) = 0.$$

It is easy to see that  $\beta(\kappa) = -\kappa$  and  $\beta^2(\kappa) = \kappa$ . The form of the composed mapping  $\psi \circ \varphi \circ \sigma$  is given by

(4.4)

$$\psi \circ \varphi \circ \sigma : (x, y) \mapsto (\bar{x}, \bar{y}),$$

(4.5)

$$\bar{x} = -\frac{a_{20}y^2 + a_{21}y + a_{22}}{x(a_{00}y^2 + a_{01}y + a_{02})},$$

(4.6)

$$\bar{y} = \frac{b_{12}(a_{01}\bar{x}^2 + a_{21}) - b_{11}(a_{02}\bar{x}^2 + a_{22}) - y(b_{10}(a_{02}\bar{x}^2 + a_{22}) - b_{12}(a_{00}\bar{x}^2 + a_{20}))}{b_{10}(a_{02}\bar{x}^2 + a_{22}) - b_{12}(a_{00}\bar{x}^2 + a_{20}) - y(b_{11}(a_{00}\bar{x}^2 + a_{20}) - b_{10}(a_{01}\bar{x}^2 + a_{21}))}.$$

Fig. 2 shows a solution of (4.5) and (4.6) in the phase plane, where

$$(4.7) \quad A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (x_0, y_0) = (1.4, 1.4).$$

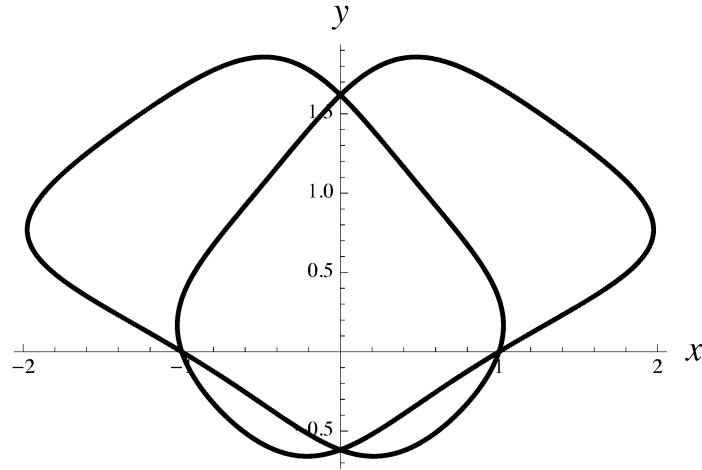


Figure 2.

**Example 4.2.**

Let us take a mapping  $\sigma$  and matrices  $A, B$  as follows.

(4.8)

$$\sigma : (x, y) \mapsto \left( -\frac{1+x}{x}, -\frac{1+y}{y} \right),$$

(4.9)

$$A = \begin{pmatrix} -b_{10} + \frac{1}{2}b_{11} + b_{20} - b_{21}, & -b_{10} - \frac{1}{2}b_{11} + b_{12} + 2b_{20} - 2b_{22}, & -\frac{1}{2}b_{11} + b_{21} - b_{22} \\ -b_{10} + b_{12} - 2b_{22}, & -2b_{11} + 2b_{12} + 2b_{21} - 4b_{22}, & b_{10} - b_{11} - 2b_{20} + 2b_{21} - 2b_{22} \\ -\frac{1}{2}b_{11} + b_{12} - b_{22}, & b_{10} - \frac{3}{2}b_{11} + b_{12} + b_{21} - 2b_{22}, & b_{10} - \frac{1}{2}b_{11} - b_{20} + b_{21} - b_{22} \end{pmatrix},$$

(4.10)

$$B = \begin{pmatrix} b_{10} - \frac{1}{2}b_{11} - b_{20} + b_{21} - b_{22}, & b_{10} + \frac{1}{2}b_{11} - b_{12} - 2b_{20} + b_{21}, & \frac{1}{2}b_{11} - b_{20} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix}.$$

A calculation shows that  $\beta(\kappa) = \frac{1}{1-\kappa}$  and  $\beta^3(\kappa) = \kappa$ .

In the case of

$$(4.11) \quad A = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

the mapping  $\psi \circ \varphi \circ \sigma$  is given by

$$(4.12) \quad \bar{x} = \frac{y^4 + 6y^3 + 8y^2 + y - 2 - x(y^3 - 4y^2 - 2y + 1)}{y^4 + 3y^3 - y^2 + 2 + xy(2y^3 + 9y^2 + 7y + 1)},$$

$$(4.13) \quad \bar{y} = \frac{-2\bar{x}^4 - 9\bar{x}^3 - 7\bar{x}^2 - \bar{x} - y(\bar{x}^4 + 6\bar{x}^3 + 8\bar{x}^2 + \bar{x} - 2)}{2\bar{x}^4 + 8\bar{x}^3 + 11\bar{x}^2 + 3\bar{x} - 1 - y(\bar{x}^3 - 4\bar{x}^2 - 2\bar{x} + 1)},$$

and a solution for  $(x_0, y_0) = (-0.2611, -0.2611)$  is shown in Fig. 3.

The invariant curve is given by the union of three QRT invariant curves,

$$(4.14) \quad \left( {}^t\mathbf{x}(A + \kappa B)\mathbf{y} \right) \left( {}^t\mathbf{x} \left( A + \frac{1}{1-\kappa}B \right) \mathbf{y} \right) \left( {}^t\mathbf{x} \left( A + \frac{-1+\kappa}{\kappa}B \right) \mathbf{y} \right) = 0,$$

which is illustrated in Fig. 3.

A composition  $\psi \circ \varphi$  gives the QRT mapping

$$(4.15) \quad \bar{x} = \frac{-y^4 - 3y^3 + y^2 - 2 - x(2y^3 + 9y^2 + 7y + 1)}{y(2y^3 + 9y^2 + 7y + 1) + x(2y^4 + 8y^3 + 11y^2 + 3y - 1)},$$

$$(4.16) \quad \bar{y} = \frac{-\bar{x}^4 - 3\bar{x}^3 + \bar{x}^2 - 2 - y(2\bar{x}^3 + 9\bar{x}^2 + 7\bar{x} + 1)}{\bar{x}(2\bar{x}^3 + 9\bar{x}^2 + 7\bar{x} + 1) + y(2\bar{x}^4 + 8\bar{x}^3 + 11\bar{x}^2 + 3\bar{x} - 1)},$$

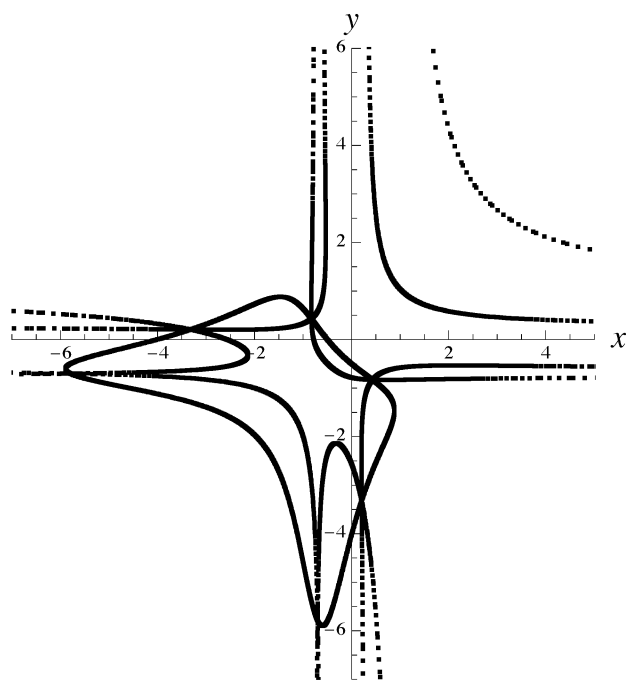


Figure 3.

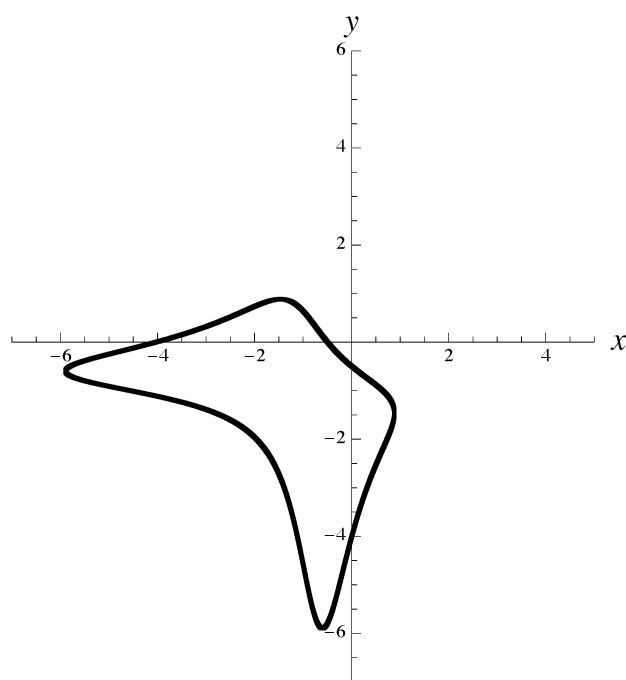


Figure 4.

and its trajectory with initial values  $(x_0, y_0) = (-0.2611, -0.2611)$  is illustrated in Fig. 4, which is identical to the invariant curve (3.1).

## § 5. conclusion

In this paper, we have proposed a method for constructing two-dimensional integrable mappings that possess invariants with degree higher than two. By constructing the composition of the QRT mapping and the mapping that transforms the invariant of the QRT mapping to itself, changing an integration constant, we can obtain integrable mappings with invariants of higher degree. We have shown several concrete examples of cases in which integration constants changed by mappings with period 2 and 3.

Once we have a mapping  $\sigma$  and QRT matrices  $A, B$  that satisfy relation (3.4), we can construct a mapping with invariants of higher degree in general. However, we haven't investigated all possible cases. Since the form of  $\sigma$  reflects the symmetries of the QRT invariants, we may need some algebraic theories to classify them. We hope to return to this problem in near future.

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