RIMS Kôkyûroku Bessatsu
B13 (2009), 95–124

Ultradiscrete Analogue of the Identity of Pfaffians

By

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Contents

Abstract

We present an algebraic identity of ultradiscretized hafnians, which is an ultradiscrete form of the identity of pfaffians. The identity stems from a decomposition of a product of the hafnians.

Received December 25, 2008. Accepted March 17, 2009.
2000 Mathematics Subject Classification(s): 15A15, 35Q51, 37B15
Key Words: *ultradiscrete, permanent, pfaffian, hafnian, bilinear equation*
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§ 1. Introduction

Recent progress in the direct method in soliton theory reveals that a soliton equation which exhibits multi-soliton solution is reduced to an identity of pfaffians. The identity of determinants such as the Plücker relation and Jacobi's identity of determinants are special cases of the identity of the pfaffians [1]. Takahashi and the author of the present article have shown that soliton solutions to the box and ball system follow a form of ultradiscretized *permanent* [2]. A permanent is a signature free determinant. Nagai has shown that soliton solutions to the ultradiscrete Toda equation are expressed also by the ultradiscretized permanents [3]. These facts suggest that there must be an identity of ultradiscretized permanents instead of determinants. More generally we expect an identity of ultradiscretized hafnians instead of pfaffians. A hafnian is a signature free pfaffian introduced by Caieniello [4].

We look for the ultradiscretization of the following simple identity of determinants

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0, \tag{1.1}$$

which is one of the Plücker relations.

Let each term in Eq.(1.1) be p_1, p_2 and p_3,namely

$$p_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} = a_1a_3b_2b_4 - a_1a_4b_2b_3 - a_2a_3b_1b_4 + a_2a_4b_1b_3, \tag{1.2}$$

$$p_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} = a_1a_2b_3b_4 - a_1a_4b_2b_3 - a_2a_3b_1b_4 + a_3a_4b_1b_2, \tag{1.3}$$

$$p_3 = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_1a_2b_3b_4 - a_1a_3b_2b_4 - a_2a_4b_1b_3 + a_3a_4b_1b_2. \tag{1.4}$$

Then the Plücker relation is written as

$$p_1 - p_2 + p_3 = 0, \tag{1.5}$$

which cannot be ultradiscretized because of negative terms in p_1, p_2 and p_3.

Now we replace the determinants by the corresponding permanents

$$q_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}_+ \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}_+ = a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3, \tag{1.6}$$

$$q_2 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}_+ = a_1a_2b_3b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_3a_4b_1b_2, \tag{1.7}$$

$$q_3 = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}_+ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}_+ = a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2, \tag{1.8}$$

where q_1, q_2 and q_3 have no negative terms and can be ultradiscretized. However the corresponding Plücker relation does not hold,

$$q_1 - q_2 + q_3 = 2(a_1a_3b_2b_4 + a_2a_4b_1b_3) \neq 0, \tag{1.9}$$

which indicates that ultradiscrete analogue of the Plücker relation is not easy to find.

We notice comparing the expressions for the products of the determinants, p_1, p_2 and p_3 that they are decomposed into a sum of common terms p_{12}, p_{13} and p_{23} where p_{ij} is the common term of p_i and p_j for $i, j = 1, 2, 3$,

$$p_1 = -p_{12} + p_{13}, \quad p_2 = -p_{12} + p_{23}, \quad p_3 = -p_{13} + p_{23}, \tag{1.10}$$

where

$$p_{12} = a_1a_4b_2b_3 + a_2a_3b_1b_4, \tag{1.11}$$

$$p_{13} = a_1a_3b_2b_4 + a_2a_4b_1b_3, \tag{1.12}$$

$$p_{23} = a_1a_2b_3b_4 + a_3a_4b_1b_2. \tag{1.13}$$

The permanents have the same terms as the determinants except the signature. Accordingly we find that the products of the permanents, q_1, q_2 and q_3 are decomposed into a sum of common terms q_{12}, q_{13} and q_{23},where q_{ij} is the common term of q_i and q_j for $i, j = 1, 2, 3$,

$$q_1 = q_{12} + q_{13}, \quad q_2 = q_{12} + q_{23}, \quad q_3 = q_{13} + q_{23}, \tag{1.14}$$

where

$$q_{12} = a_1a_4b_2b_3 + a_2a_3b_1b_4, \tag{1.15}$$

$$q_{13} = a_1a_3b_2b_4 + a_2a_4b_1b_3, \tag{1.16}$$

$$q_{23} = a_1a_2b_3b_4 + a_3a_4b_1b_2. \tag{1.17}$$

The Plücker relation Eq.(1.1) is comfirmed by the decomposition of products of determinanats,

$$p_1 - p_2 + p_3 = -p_{12} + p_{13} - (-p_{12} + p_{23}) - p_{13} + p_{23} = 0. \tag{1.18}$$

An ultradiscrete analogue of the Plücker relation is obtained as follows.

Replacing the determinants by the correspoding permanents we have

$$q_1 + q_3 = q_2. \tag{1.19}$$

Let

$$q_i = \exp(Q_i/\epsilon) \text{ for } i = 1, 2, 3, \tag{1.20}$$

$$q_{ij} = \exp(Q_{ij}/\epsilon) \text{ for } i, j = 1, 2, 3, \tag{1.21}$$

where ϵ is the ultradiscrete parameter [5]. In the small limit of ϵ we have an ultradiscrete analogue of the Plücker relation , Eq.(1.19),

$$Q_2 = \max(Q_1, Q_3), \tag{1.22}$$

which does not hold in general.

We investigate under what conditions on Q_1, Q_2 and Q_3 Eq.(1.22) does hold. The ultradiscrete form of Eq.(1.14) are

$$\begin{aligned} Q_1 &= \max(Q_{12}, Q_{13}), \\ Q_2 &= \max(Q_{12}, Q_{23}), \\ Q_3 &= \max(Q_{13}, Q_{23}). \end{aligned} \tag{1.23}$$

Substituting these expressions into Eq.(1.22) we obtain

$$\max(Q_{12}, Q_{23}) = \max(Q_{12}, Q_{13}, Q_{23}). \tag{1.24}$$

Obviously Eq.(1.24) does hold if $Q_{13} \leq \max(Q_{12}, Q_{23})$. But it does not hold if $Q_{13} > \max(Q_{12}, Q_{23})$.

However if $Q_{13} > \max(Q_{12}, Q_{23})$ we find, using Eq.(1.23)

$$Q_1 = Q_3. \tag{1.25}$$

Hence we obtain the following algebraic identity of the ultradiscretized permanents,

$$[Q_2 - \max(Q_1, Q_3)](Q_1 - Q_3) = 0, \tag{1.26}$$

which we call "ultradiscrete analogue of the Plücker relation".

The contents of this article are as follows. In §2,we describe the fundamental properties of determinants,pfaffians,permanents and hafnians. In §3 we prove by induction a decomposition of a product of hafnians into a sum of common terms, which will be used in §4 to obtain an algebraic identity of ultradiscretized hafnians that is a ultradiscrete analogue of the identity of pfaffians.

§ 2. Preliminaries

In this section we describe elementary properties of determinants, permanents, pfaffians and hafnians.

§ 2.1. Determinants, pfaffians, permanents and hafnians

Consider an $n \times n$ matrix $\boldsymbol{D} = (x_{ij})$ whose determinant is D. We write the nth-order determinant D as

$$D = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_{-} . \tag{2.1}$$

An nth-order pfaffian denoted by $\mathrm{pf}(1, 2, \cdots, 2n)$ is anti-symmetric with respect to the indices, $1, 2, \cdots, 2n$, namely

$$\mathrm{pf}(1, 2, \cdots, i, \cdots, j, \cdots, 2n) = -\mathrm{pf}(1, 2, \cdots, j, \cdots, i, \cdots, 2n) \tag{2.2}$$

and is defined by the expansion formula

$$\mathrm{pf}(1, 2, \cdots, 2n) = \sum_{j=2}^{2n} \mathrm{pf}(1, j)(-1)^j \mathrm{pf}(2, 3, \cdots, \hat{j}, \cdots, 2n), \tag{2.3}$$

where $\hat{j}$ means that index j is omitted.

We have, for example,

$$\mathrm{pf}(1, 2, 3, 4) = \mathrm{pf}(1, 2)\mathrm{pf}(3, 4) - \mathrm{pf}(1, 3)\mathrm{pf}(2, 4) + \mathrm{pf}(1, 4)\mathrm{pf}(2, 3). \tag{2.4}$$

An nth-order determinant D is expressed by an nth-order pfaffian as follows

$$D = \mathrm{pf}(x_1, x_2, \cdots, x_n, n, n-1, \cdots, 2, 1) \tag{2.5}$$

$$= (-1)^{n(n-1)/2}\mathrm{pf}(x_1, x_2, \cdots, x_n, 1, 2, \cdots, n), \tag{2.6}$$

where the pfaffian entries are defined by

$$\mathrm{pf}(x_i, x_j) = 0, \quad \mathrm{pf}(x_i, j) = x_{ij}, \quad \text{for } i, j = 1, 2, \cdots, n. \tag{2.7}$$

A permanent is a signature free determinant, that is the signatures of permutations are not taken into account in its definition.

The permanent of an $n \times n$ matrix $\boldsymbol{D} = (x_{ij})$ is defined as

$$perm(D) = \sum_{\sigma \in S} \prod_{i=1}^{n} x_{i,\sigma(i)}, \tag{2.8}$$

where the sum extends over all permutations of number $1, 2, \cdots, n$. We write the nth-order permanent P as

$$P = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_{+} . \tag{2.9}$$

An nth-order hafnian denoted by $\mathrm{hf}(1,2,\cdots,2n)$ is a signature free pfaffian, which is symmetric with respect to the indices, $1,2,\cdots,2n$ and is defined by the expansion formula

$$\mathrm{hf}(1,2,\cdots,2n)=\sum_{j=2}^{2n}\mathrm{hf}(1,j)\mathrm{hf}(2,3,\cdots,\hat{j},\cdots,2n). \tag{2.10}$$

We have, for example,

$$\mathrm{hf}(1,2,3,4)=\mathrm{hf}(1,2)\mathrm{hf}(3,4)+\mathrm{hf}(1,3)\mathrm{hf}(2,4)+\mathrm{hf}(1,4)\mathrm{hf}(2,3). \tag{2.11}$$

The nth-order permanent P is expressed by the nth-order hafnian.

$$P=\mathrm{hf}(x_1,x_2,\cdots,x_n,1,2,\cdots,n), \tag{2.12}$$

if one define the hafnian entries by

$$\mathrm{hf}(x_i,x_j)=0,\quad \mathrm{hf}(x_i,j)=x_{ij},\quad \text{for } i,j=1,2,\cdots,n. \tag{2.13}$$

§ 2.2. Laplace expansion of permanents

Consider the permanent P_4 of an 4×4 matrix $\boldsymbol{D}=(a_{ij})$,

$$P_4=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{vmatrix}_+ . \tag{2.14}$$

We have the Laplace expansion formula of P_4,

$$\begin{aligned} P_4 = &\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}_+ \begin{vmatrix} a_{33} & a_{34}\\ a_{43} & a_{44}\end{vmatrix}_+ + \begin{vmatrix} a_{13} & a_{14}\\ a_{23} & a_{24}\end{vmatrix}_+ \begin{vmatrix} a_{31} & a_{32}\\ a_{41} & a_{42}\end{vmatrix}_+ \\ &+ \begin{vmatrix} a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}_+ \begin{vmatrix} a_{32} & a_{34}\\ a_{42} & a_{44}\end{vmatrix}_+ + \begin{vmatrix} a_{12} & a_{14}\\ a_{22} & a_{24}\end{vmatrix}_+ \begin{vmatrix} a_{31} & a_{33}\\ a_{41} & a_{43}\end{vmatrix}_+ \\ &+ \begin{vmatrix} a_{11} & a_{14}\\ a_{21} & a_{24}\end{vmatrix}_+ \begin{vmatrix} a_{32} & a_{33}\\ a_{42} & a_{43}\end{vmatrix}_+ + \begin{vmatrix} a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}_+ \begin{vmatrix} a_{31} & a_{34}\\ a_{41} & a_{44}\end{vmatrix}_+ , \end{aligned} \tag{2.15}$$

which is expressed by the 4th-order hafnians with the entries, $\mathrm{hf}(y_i,y_j)=0$ and $\mathrm{hf}(y_i,j)=a_{i,j}$ for $i,j=1,2,3,4$,

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1,y_2,y_3,y_4,1,2,3,4)\\ &= \mathrm{hf}(y_1,y_2,1,2)\mathrm{hf}(y_3,y_4,3,4)+\mathrm{hf}(y_1,y_2,3,4)\mathrm{hf}(y_3,y_4,1,2)\\ &\quad +\mathrm{hf}(y_1,y_2,1,3)\mathrm{hf}(y_3,y_4,2,4)+\mathrm{hf}(y_1,y_2,2,4)\mathrm{hf}(y_3,y_4,1,3)\\ &\quad +\mathrm{hf}(y_1,y_2,1,4)\mathrm{hf}(y_3,y_4,2,3)+\mathrm{hf}(y_1,y_2,2,3)\mathrm{hf}(y_3,y_4,1,4). \end{aligned} \tag{2.16}$$

Interchanging the index y_2 with the index y_3 in (2.16) we obtain

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_3, y_2, y_4, 1, 2, 3, 4) \\ &= \mathrm{hf}(y_1, y_3, 1, 2)\mathrm{hf}(y_2, y_4, 3, 4) + \mathrm{hf}(y_1, y_3, 3, 4)\mathrm{hf}(y_2, y_4, 1, 2) \\ &\quad + \mathrm{hf}(y_1, y_3, 1, 3)\mathrm{hf}(y_2, y_4, 2, 4) + \mathrm{hf}(y_1, y_3, 2, 4)\mathrm{hf}(y_2, y_4, 1, 3) \\ &\quad + \mathrm{hf}(y_1, y_3, 1, 4)\mathrm{hf}(y_2, y_4, 2, 3) + \mathrm{hf}(y_1, y_3, 2, 3)\mathrm{hf}(y_2, y_4, 1, 4). \end{aligned} \tag{2.17}$$

Similarly interchanging the index y_2 with the index y_4 in (2.16) we obtain

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_4, 1, 2)\mathrm{hf}(y_2, y_3, 3, 4) + \mathrm{hf}(y_1, y_4, 3, 4)\mathrm{hf}(y_2, y_3, 1, 2) \\ &\quad + \mathrm{hf}(y_1, y_4, 1, 3)\mathrm{hf}(y_2, y_3, 2, 4) + \mathrm{hf}(y_1, y_4, 2, 4)\mathrm{hf}(y_2, y_3, 1, 3) \\ &\quad + \mathrm{hf}(y_1, y_4, 1, 4)\mathrm{hf}(y_2, y_3, 2, 3) + \mathrm{hf}(y_1, y_4, 2, 3)\mathrm{hf}(y_2, y_3, 1, 4). \end{aligned} \tag{2.18}$$

Permanent P_4 is invariant under changing the rows and columns,

$$P_4 = \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}_+ . \tag{2.19}$$

Then the Laplace expansion formula of P_4 becomes

$$\begin{aligned} P_4 = &\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}_+ \begin{vmatrix} a_{33} & a_{43} \\ a_{34} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{31} & a_{41} \\ a_{32} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{23} \\ a_{14} & a_{24} \end{vmatrix}_+ \\ &+ \begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{vmatrix}_+ \begin{vmatrix} a_{23} & a_{43} \\ a_{24} & a_{44} \end{vmatrix}_+ + \begin{vmatrix} a_{21} & a_{41} \\ a_{22} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{33} \\ a_{14} & a_{34} \end{vmatrix}_+ \\ &+ \begin{vmatrix} a_{11} & a_{41} \\ a_{12} & a_{42} \end{vmatrix}_+ \begin{vmatrix} a_{23} & a_{33} \\ a_{24} & a_{34} \end{vmatrix}_+ + \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}_+ \begin{vmatrix} a_{13} & a_{43} \\ a_{14} & a_{44} \end{vmatrix}_+ , \end{aligned} \tag{2.20}$$

which is expressed by the 4th-order hafnians with the entries, $\mathrm{hf}(y_i, y_j) = 0$ and $\mathrm{hf}(y_i, j) = a_{ij}$ for $i, j = 1, 2, 3, 4$,

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_2, y_3, y_4, 1, 2, 3, 4) \\ &= \mathrm{hf}(y_1, y_2, 1, 2)\mathrm{hf}(y_3, y_4, 3, 4) + \mathrm{hf}(y_1, y_2, 3, 4)\mathrm{hf}(y_3, y_4, 1, 2) \\ &\quad + \mathrm{hf}(y_1, y_3, 1, 2)\mathrm{hf}(y_2, y_4, 3, 4) + \mathrm{hf}(y_1, y_3, 3, 4)\mathrm{hf}(y_2, y_4, 1, 2) \\ &\quad + \mathrm{hf}(y_1, y_4, 1, 2)\mathrm{hf}(y_2, y_3, 3, 4) + \mathrm{hf}(y_1, y_4, 3, 4)\mathrm{hf}(y_2, y_3, 1, 2). \end{aligned} \tag{2.21}$$

Interchanging the index 2 with the index 4 in (2.21) we obtain

$$\begin{aligned} P_4 &= \mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4) \\ &\quad + \mathrm{hf}(y_1, y_3, 1, 4)\mathrm{hf}(y_2, y_4, 2, 3) + \mathrm{hf}(y_1, y_3, 2, 3)\mathrm{hf}(y_2, y_4, 1, 4) \\ &\quad + \mathrm{hf}(y_1, y_4, 1, 4)\mathrm{hf}(y_2, y_3, 2, 3) + \mathrm{hf}(y_1, y_4, 2, 3)\mathrm{hf}(y_2, y_3, 1, 4). \end{aligned} \tag{2.22}$$

Subtracting the terms, $\mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4)$ from Eqs.(2.16) and (2.22) we obtain an identity of the hafnians,

$$\begin{aligned} &\mathrm{hf}(y_1, y_2, 1, 2)\mathrm{hf}(y_3, y_4, 3, 4) + \mathrm{hf}(y_1, y_2, 3, 4)\mathrm{hf}(y_3, y_4, 1, 2) \\ &+\mathrm{hf}(y_1, y_2, 1, 3)\mathrm{hf}(y_3, y_4, 2, 4) + \mathrm{hf}(y_1, y_2, 2, 4)\mathrm{hf}(y_3, y_4, 1, 3) \\ &= \mathrm{hf}(y_1, y_3, 1, 4)\mathrm{hf}(y_2, y_4, 2, 3) + \mathrm{hf}(y_1, y_3, 2, 3)\mathrm{hf}(y_2, y_4, 1, 4) \\ &+\mathrm{hf}(y_1, y_4, 1, 4)\mathrm{hf}(y_2, y_3, 2, 3) + \mathrm{hf}(y_1, y_4, 2, 3)\mathrm{hf}(y_2, y_3, 1, 4). \end{aligned} \tag{2.23}$$

Interchanging the index 3 with the index 4 in (2.23) we obtain an identity,

$$\begin{aligned} &\mathrm{hf}(y_1, y_2, 1, 2)\mathrm{hf}(y_3, y_4, 3, 4) + \mathrm{hf}(y_1, y_2, 3, 4)\mathrm{hf}(y_3, y_4, 1, 2) \\ &+\mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4) \\ &= \mathrm{hf}(y_1, y_3, 1, 3)\mathrm{hf}(y_2, y_4, 2, 4) + \mathrm{hf}(y_1, y_3, 2, 4)\mathrm{hf}(y_2, y_4, 1, 3) \\ &+\mathrm{hf}(y_1, y_4, 1, 3)\mathrm{hf}(y_2, y_3, 2, 4) + \mathrm{hf}(y_1, y_4, 2, 4)\mathrm{hf}(y_2, y_3, 1, 3). \end{aligned} \tag{2.24}$$

Similarly interchanging the index 2 with the index 3 in (2.24) we obtain an identity,

$$\begin{aligned} &\mathrm{hf}(y_1, y_2, 1, 3)\mathrm{hf}(y_3, y_4, 2, 4) + \mathrm{hf}(y_1, y_2, 2, 4)\mathrm{hf}(y_3, y_4, 1, 3) \\ &+\mathrm{hf}(y_1, y_2, 1, 4)\mathrm{hf}(y_3, y_4, 2, 3) + \mathrm{hf}(y_1, y_2, 2, 3)\mathrm{hf}(y_3, y_4, 1, 4) \\ &= \mathrm{hf}(y_1, y_3, 1, 2)\mathrm{hf}(y_2, y_4, 3, 4) + \mathrm{hf}(y_1, y_3, 3, 4)\mathrm{hf}(y_2, y_4, 1, 2) \\ &+\mathrm{hf}(y_1, y_4, 1, 2)\mathrm{hf}(y_2, y_3, 3, 4) + \mathrm{hf}(y_1, y_4, 3, 4)\mathrm{hf}(y_2, y_3, 1, 2). \end{aligned} \tag{2.25}$$

These identities will be used in the section 3 in order to decompose $\hat{h}_1$,$\hat{h}_2$ and $\hat{h}_3$.

§ 2.3. Plücker relations

Let a_i for $i = 1, 2, \cdots, N-1$ and b_i for $i = 1, 2, \cdots, n$ be N-dimensional vectors and $|\cdots|$ express determinants. The Plücker relations are expressed in general by

$$\begin{aligned} &\sum_{i=1}^{n}(-1)^i |a_1, a_2, \cdots, a_{N-1}, b_i| \\ &\qquad \times |a_1, a_2, \cdots, a_{N-n+1}, b_n, b_{n-1}, \cdots, \hat{b}_i, \cdots, b_1| = 0, \end{aligned} \tag{2.26}$$

where $\hat{b}$ means that symbol b is omitted.

We consider the simplest case, namely $n = 3$ and write the Plücker relation as,

$$D_{ab}D_{cd} - D_{ac}D_{bd} + D_{ad}D_{bc} = 0, \tag{2.27}$$

where $D_{ab}, D_{cd}, D_{ac}, D_{bd}, D_{ad}$ and D_{bc} are nth-order determinants defined by

$$D_{ab} = |a, b, x_3, x_4, \cdots, x_n|, \tag{2.28}$$

$$D_{ac} = |a, c, x_3, x_4, \cdots, x_n|, \tag{2.29}$$

$$D_{ad} = |a, d, x_3, x_4, \cdots, x_n|, \tag{2.30}$$

$$D_{bc} = |b, c, x_3, x_4, \cdots, x_n|, \tag{2.31}$$

$$D_{bd} = |b, d, x_3, x_4, \cdots, x_n|, \tag{2.32}$$

$$D_{cd} = |c, d, x_3, x_4, \cdots, x_n|, \tag{2.33}$$

where a, b, c, d and $x_i (i = 3, 4, \cdots, n)$ are n-dimensional vectors.

§ 2.4. Identity of pfaffians

We have the following identity of pfaffians,

$$\begin{aligned} \mathrm{pf}(1, 2, \cdots, 2n)\mathrm{pf}(5, 6, \cdots, 2n) = {} & \mathrm{pf}(1, 2, 5, 6, \cdots, 2n)\mathrm{pf}(3, 4, 5, 6, \cdots, 2n) \\ & - \mathrm{pf}(1, 3, 5, 6, \cdots, 2n)\mathrm{pf}(2, 4, 5, 6, \cdots, 2n) \\ & + \mathrm{pf}(1, 4, 5, 6, \cdots, 2n)\mathrm{pf}(2, 3, 5, 6, \cdots, 2n), \end{aligned} \tag{2.34}$$

whose special cases are the Plücker relation and Jacobi's identity.

§ 2.5. Expansion fomulae of pfaffians and hafnians

We have the expansion formulae of pfaffians,

$$\begin{aligned} & \mathrm{pf}(x_1, x_2, 5, 6, \cdots, 2n) \\ & = \sum_{5 \le i < j \le 2n} (-1)^{i+j-1} \mathrm{pf}(x_1, x_2, i, j)\mathrm{pf}(5, 6, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} & \mathrm{pf}(x_1, x_2, x_3, x_4, 5, 6, \cdots, 2n) \\ & = \sum_{5 \le i < j \le 2n} (-1)^{i+j-1} \mathrm{pf}(x_1, x_2, i, j)\mathrm{pf}(x_3, x_4, 5, 6, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n), \end{aligned} \tag{2.36}$$

where $\mathrm{pf}(x_j, x_k) = 0$ for $j, k = 1, 2, 3, 4$.

Similar expansion formulae hold for hafnians,

$$\begin{aligned} & \mathrm{hf}(x_1, x_2, 5, 6, \cdots, 2n) \\ & = \sum_{5 \le i < j \le 2n} \mathrm{hf}(x_1, x_2, i, j)\mathrm{hf}(5, 6, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n) \end{aligned} \tag{2.37}$$

and

$$
\begin{aligned}
&\mathrm{hf}(x_1,x_2,x_3,x_4,5,6,\cdots,2n)\\
&\quad=\sum_{5\le i<j\le 2n}\mathrm{hf}(x_1,x_2,i,j)\mathrm{hf}(x_3,x_4,5,6,\cdots,\hat{i},\cdots,\hat{j},\cdots,2n),\\
&\quad=\sum_{5\le i<j\le 2n}\sum_{5\le k<l\le 2n}\mathrm{hf}(x_1,x_2,i,j)\mathrm{hf}(x_3,x_4,k,l)\\
&\qquad\qquad\times\mathrm{hf}(5,6,\cdots,\hat{i},\cdots,\hat{j},\cdots,\hat{k},\cdots,\hat{l},\cdots,2n),\\
&\quad=\sum_{5\le i<j<k<l\le 2n}[\mathrm{hf}(x_1,x_2,i,j)\mathrm{hf}(x_3,x_4,k,l)+\mathrm{hf}(x_1,x_2,k,l)\mathrm{hf}(x_3,x_4,i,j)\\
&\qquad\qquad+\mathrm{hf}(x_1,x_2,i,k)\mathrm{hf}(x_3,x_4,j,l)+\mathrm{hf}(x_1,x_2,j,l)\mathrm{hf}(x_3,x_4,i,k)\\
&\qquad\qquad+\mathrm{hf}(x_1,x_2,i,l)\mathrm{hf}(x_3,x_4,j,k)+\mathrm{hf}(x_1,x_2,j,k)\mathrm{hf}(x_3,x_4,i,l)]\\
&\qquad\qquad\times\mathrm{hf}(5,6,\cdots,\hat{i},\cdots,\hat{j},\cdots,\hat{k},\cdots,\hat{l},\cdots,2n), \qquad (2.38)\\
&\quad=\sum_{5\le i<j<k<l\le 2n}\mathrm{hf}(x_1,x_2,x_3,x_4,i,j,k,l)\\
&\qquad\qquad\times\mathrm{hf}(5,6,\cdots,\hat{i},\cdots,\hat{j},\cdots,\hat{k},\cdots,\hat{l},\cdots,2n), \qquad (2.39)
\end{aligned}
$$

where $\mathrm{hf}(x_i,x_j)=0$ for $i,j=1,2,3,4$.

Hereafter we write, for simplicity, nth-order, $(n-1)$th-oder and $(n-2)$th-order hafnians, respectively as

$$\mathrm{hf}(1,2,\cdots,2n)=\mathrm{hf}(\cdots)_n, \tag{2.40}$$

$$\mathrm{hf}(1,2,\cdots,\hat{j},\cdots,\hat{k},\cdots,2n)=\mathrm{hf}(\hat{j},\hat{k})_n, \tag{2.41}$$

$$\mathrm{hf}(1,2,\cdots,\hat{i},\cdots,\hat{j},\cdots,\hat{k},\cdots,\hat{l},\cdots,2n)=\mathrm{hf}(\hat{i},\hat{j},\hat{k},\hat{l})_n. \tag{2.42}$$

Let

$$\mathrm{hf}(\cdots)_{n-2}=\mathrm{hf}(5,6,\cdots,2n). \tag{2.43}$$

Then the expansion formulae of the hafnians are expressed by

$$\mathrm{hf}(x_1,x_2,5,6,\cdots,2n)=\sum_{5\le i<j\le 2n}\mathrm{hf}(x_1,x_2,i,j)\mathrm{hf}(\hat{i},\hat{j})_{n-2} \tag{2.44}$$

and

$$
\begin{aligned}
&\mathrm{hf}(x_1,x_2,x_3,x_4,5,6,\cdots,2n)\\
&\quad=\sum_{5\le i<j<k<l\le 2n}\mathrm{hf}(x_1,x_2,x_3,x_4,i,j,k,l)\mathrm{hf}(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}.
\end{aligned} \tag{2.45}
$$

A product of the hafnians is then expressed by

$$
\begin{aligned}
&\mathrm{hf}(x_1,x_2,5,6,\cdots,2n)\mathrm{hf}(x_3,x_4,5,6,\cdots,2n)\\
&\quad=\sum_{5\le i<j\le 2n}\sum_{5\le k<l\le 2n}\mathrm{hf}(x_1,x_2,i,j)\mathrm{hf}(x_3,x_4,k,l)\mathrm{hf}(\hat{i},\hat{j})_{n-2}\mathrm{hf}(\hat{k},\hat{l})_{n-2}.
\end{aligned} \tag{2.46}
$$

Hereafter we omit the character hf denoting hafnians for short. Then the above expansion formula is written simply as

$$(x_1,x_2,5,6,\cdots,2n)(x_3,x_4,5,6,\cdots,2n) = \sum_{5\le i<j\le 2n}\sum_{5\le k<l\le 2n}(x_1,x_2,i,j)(x_3,x_4,k,l)(\hat{i},\hat{j})_{n-2}(\hat{k},\hat{l})_{n-2}. \tag{2.47}$$

§ 3. Decomposition of products of the hafnians

Let us consider the following products of the hafnians,

$$f_0 = (1,2,3,4,5,\cdots,2n)(5,\cdots,2n), \tag{3.1}$$

$$f_1 = (1,2,5,\cdots,2n)(3,4,5,\cdots,2n), \tag{3.2}$$

$$f_2 = (1,3,5,\cdots,2n)(2,4,5,\cdots,2n), \tag{3.3}$$

$$f_3 = (1,4,5,\cdots,2n)(2,3,5,\cdots,2n). \tag{3.4}$$

We shall prove by induction that f_0, f_1, f_2 and f_3 are decomposed into the following forms,

$$f_0 = f_{01}+f_{02}+f_{03}, \tag{3.5}$$

$$f_1 = f_{01}+f_{12}+f_{13}, \tag{3.6}$$

$$f_2 = f_{02}+f_{12}+f_{23}, \tag{3.7}$$

$$f_3 = f_{03}+f_{13}+f_{23}. \tag{3.8}$$

In the above expressions, if one expands f_i and f_j according to the definition(2.10), these two expressions possess common terms,which we denote $f_{i,j}$.

The proof consists of three steps. First we introduce new indices, x_1, x_2, x_3, x_4 in place of the indices $1, 2, 3, 4$ and express f_0, f_1, f_2 and f_3 by the following forms,

$$f_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3 + h_0, \tag{3.9}$$

$$f_1 = \overline{f}_1 + g_1, \tag{3.10}$$

$$f_2 = \overline{f}_2 + g_2, \tag{3.11}$$

$$f_3 = \overline{f}_3 + g_3, \tag{3.12}$$

where $\overline{f}_j$ is defined by Eqs.(3.67), (3.69) and (3.71) for $j = 1, 2, 3$ respectively, and

$$h_0 = (x_1,x_2,x_3,x_4,5,\cdots,2n)(5,\cdots,2n), \tag{3.13}$$

$$g_1 = (x_1,x_2,5,\cdots,2n)(x_3,x_4,5,\cdots,2n), \tag{3.14}$$

$$g_2 = (x_1,x_3,5,\cdots,2n)(x_2,x_4,5,\cdots,2n), \tag{3.15}$$

$$g_3 = (x_1,x_4,5,\cdots,2n)(x_2,x_3,5,\cdots,2n), \tag{3.16}$$

and $(x_i, x_j) = 0$ for $i, j = 1, 2, 3, 4$.

Second we introduce new indices,y_1, y_2, y_3, y_4 in place of the indices x_1, x_2, x_3, x_4 and express g_1, g_2 and g_3 by the following forms,

$$g_1 = \overline{g}_{12} + \overline{g}_{13} + h_1, \tag{3.17}$$

$$g_2 = \overline{g}_{12} + \overline{g}_{23} + h_2, \tag{3.18}$$

$$g_3 = \overline{g}_{23} + \overline{g}_{23} + h_3, \tag{3.19}$$

where

$$h_1 = (y_1, y_2, 5, \cdots, 2n)(y_3, y_4, 5, \cdots, 2n), \tag{3.20}$$

$$h_2 = (y_1, y_3, 5, \cdots, 2n)(y_2, y_4, 5, \cdots, 2n), \tag{3.21}$$

$$h_3 = (y_1, y_4, 5, \cdots, 2n)(y_2, y_3, 5, \cdots, 2n), \tag{3.22}$$

where $(y_i, y_j) = 0$ for $i, j = 1, 2, 3, 4$ and

$$(y_i, j) = a_{i,j} \quad \text{and} \quad a_{i,k} a_{j,k} = 0, \tag{3.23}$$

for $i, j = 1, 2, 3, 4$ and for $k = 4, 5, \cdots, 2n$.

Finally we prove, by induction, decomposition of h_0, h_1, h_2 and h_3, which gives immediately the decomposition of f_0, f_1, f_2 and f_3.

§ 3.1. Decomposition of products of $(n-2)$th-oder hafnians

In order to use induction we consider the decomposition of f_0, f_1, f_2 and f_3 for $(n-2)$th-oder hafnians such as

$$\begin{aligned} f_0^{n-2} &= (5, 6, 7, 8, 9, 10, \cdots, 2n)(9, 10, \cdots, 2n), \\ &= f_{01}^{n-2} + f_{02}^{n-2} + f_{03}^{n-2}. \end{aligned} \tag{3.24}$$

$$\begin{aligned} f_1^{n-2} &= (5, 6, 9, 10, \cdots, 2n)(7, 8, 9, 10, \cdots, 2n), \\ &= f_{01}^{n-2} + f_{12}^{n-2} + f_{13}^{n-2}. \end{aligned} \tag{3.25}$$

$$\begin{aligned} f_2^{n-2} &= (5, 7, 9, 10, \cdots, 2n)(6, 8, 9, 10, \cdots, 2n), \\ &= f_{02}^{n-2} + f_{12}^{n-2} + f_{23}^{n-2}. \end{aligned} \tag{3.26}$$

$$\begin{aligned} f_3^{n-2} &= (5, 8, 9, 10, \cdots, 2n)(6, 7, 9, 10, \cdots, 2n), \\ &= f_{03}^{n-2} + f_{13}^{n-2} + f_{23}^{n-2}, \end{aligned} \tag{3.27}$$

which may be written, using

$$(\cdots)_{n-2} = (5, 6, 7, 8, 9, 10, \cdots, 2n), \tag{3.28}$$

$$(\hat{5}, \hat{6}, \hat{7}, \hat{8})_{n-2} = (9, 10, \cdots, 2n), \tag{3.29}$$

$$f_{ij}(\hat{5}, \hat{6}, \hat{7}, \hat{8}) = f_{ij}^{n-2}, \text{ for } i, j = 0, 1, 2, 3, \tag{3.30}$$

as

$$
\begin{aligned}
f_0^{n-2} &= (\hat{5},\hat{6},\hat{7},\hat{8})_{n-2}(\cdots)_{n-2},\\
&= f_{01}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{02}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{03}(\hat{5},\hat{6},\hat{7},\hat{8}), \qquad (3.31)\\
f_1^{n-2} &= (\hat{5},\hat{6})_{n-2}(\hat{7},\hat{8})_{n-2},\\
&= f_{01}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{12}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{13}(\hat{5},\hat{6},\hat{7},\hat{8}), \qquad (3.32)\\
f_2^{n-2} &= (\hat{5},\hat{7})_{n-2}(\hat{6},\hat{8})_{n-2},\\
&= f_{02}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{12}(\hat{5},\hat{6},\hat{7},\hat{8}) + f_{23}(\hat{5},\hat{6},\hat{7},\hat{8}), \qquad (3.33)\\
f_3^{n-2} &= (\hat{5},\hat{8})_{n-2}(\hat{6},\hat{7})_{n-2},\\
&= f_{03}(\hat{5},\hat{8},\hat{6},\hat{7}) + f_{13}(\hat{5},\hat{8},\hat{6},\hat{7}) + f_{23}(\hat{5},\hat{8},\hat{6},\hat{7}). \qquad (3.34)
\end{aligned}
$$

In general the decomposition of products of $(n-2)$th-oder hafnians are expressed by

$$
\begin{aligned}
f_0^{n-2} &= (\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2},\\
&= f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}), \qquad (3.35)\\
f_1^{n-2} &= (\hat{i},\hat{j})_{n-2}(\hat{k},\hat{l})_{n-2},\\
&= f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{13}(\hat{i},\hat{j},\hat{k},\hat{l}), \qquad (3.36)\\
f_2^{n-2} &= (\hat{i},\hat{k})_{n-2}(\hat{j},\hat{l})_{n-2},\\
&= f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}), \qquad (3.37)\\
f_3^{n-2} &= (\hat{i},\hat{l})_{n-2}(\hat{j},\hat{k})_{n-2},\\
&= f_{03}(\hat{i},\hat{l},\hat{j},\hat{k}) + f_{13}(\hat{i},\hat{l},\hat{j},\hat{k}) + f_{23}(\hat{i},\hat{l},\hat{j},\hat{k}), \qquad (3.38)
\end{aligned}
$$

where indices i, j, k, l are members of a list $\{9, 10, \cdots, 2n\}$.

We assume the decomposition of products of $(n-2)$th-order hafnians of the forms,(3.35), (3.36), (3.37) and (3.38) for the use of induction.

§ 3.2. Expressions for f_0, f_1, f_2 and f_3

Consider an entry (j, k) of the hafnian $(1, 2, \cdots, 2n)$. Let an index j be a sum of indices $\overline{j}$ and x_j for $j = 1, 2, 3, 4$,namely

$$j = \overline{j} + x_j, \quad \text{for } j = 1, 2, 3, 4 \qquad (3.39)$$

and introduce new entries of the hafnians,

$$(\bar{j},\bar{k}) = a_{jk}, \tag{3.40}$$

$$(x_j, x_k) = 0, \tag{3.41}$$

$$(x_j, \bar{k}) = 0, \quad \text{for } j,k = 1,2,3,4, \tag{3.42}$$

$$(\bar{j}, k) = 0, \tag{3.43}$$

$$(x_j, k) = a_{jk}, \quad \text{for } j = 1,2,3,4 \text{ and } k = 5,6,\cdots,2n, \tag{3.44}$$

$$(j, k) = a_{jk}, \quad \text{for } j,k = 5,6,\cdots,2n, \tag{3.45}$$

so that

$$(j,k) = (\bar{j}+x_j, \bar{k}+x_k) = (\bar{j},\bar{k}) + (\bar{j},x_k) + (x_j,\bar{k}) + (x_j,x_k) \tag{3.46}$$

$$= a_{j,k}, \quad \text{for } j,k = 1,2,3,4, \tag{3.47}$$

and

$$(j,k) = (\bar{j}+x_j, k) = (\bar{j},k) + (x_j,k) \tag{3.48}$$

$$= a_{jk}, \quad \text{for } j = 1,2,3,4 \text{ and } k = 5,6,\cdots,2n. \tag{3.49}$$

Accordingly we have, for example,

$$(1,2,5,\cdots,2n) = (\bar{1}+x_1, \bar{2}+x_2, 5,6,\cdots,2n) \tag{3.50}$$

$$= (\bar{1},\bar{2},5,6,\cdots,2n) + (\bar{1},x_2,5,6,\cdots,2n) \tag{3.51}$$

$$+(x_1,\bar{2},5,6,\cdots,2n) + (x_1,x_2,5,6,\cdots,2n) \tag{3.52}$$

$$= a_{12}(5,6,\cdots,2n) + (x_1,x_2,5,6,\cdots,2n), \tag{3.53}$$

Then hafnians of order $(n-1)$ and of order n are written as sums of terms of special form, respectively, as follows

$$(1,2,5,\cdots,2n) = a_{12}(5,\cdots,2n) + (x_1,x_2,5,\cdots,2n), \tag{3.54}$$

$$(3,4,5,\cdots,2n) = a_{34}(5,\cdots,2n) + (x_3,x_4,5,\cdots,2n), \tag{3.55}$$

$$(1,3,5,\cdots,2n) = a_{13}(5,\cdots,2n) + (x_1,x_3,5,\cdots,2n), \tag{3.56}$$

$$(2,4,5,\cdots,2n) = a_{24}(5,\cdots,2n) + (x_2,x_4,5,\cdots,2n), \tag{3.57}$$

$$(1,4,5,\cdots,2n) = a_{14}(5,\cdots,2n) + (x_1,x_4,5,\cdots,2n), \tag{3.58}$$

$$(2,3,5,\cdots,2n) = a_{23}(5,\cdots,2n) + (x_2,x_3,5,\cdots,2n), \tag{3.59}$$

$$\begin{aligned}(1,2,3,4,5,\cdots,2n) &= (a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})(5,6,\cdots,2n)\\ &\quad +a_{12}(x_3,x_4,5,\cdots,2n) + a_{34}(x_1,x_2,5,\cdots,2n)\\ &\quad +a_{13}(x_2,x_4,5,\cdots,2n) + a_{24}(x_1,x_3,5,\cdots,2n)\\ &\quad +a_{14}(x_2,x_3,5,\cdots,2n) + a_{23}(x_1,x_4,5,\cdots,2n)\\ &\quad +(x_1,x_2,x_3,x_4,5,\cdots,2n).\end{aligned} \tag{3.60}$$

Then f_0, f_1, f_2, f_3 are written, using the above relations, as

$$f_0 = \overline{f}_0 + h_0, \tag{3.61}$$

$$f_1 = \overline{f}_1 + g_1, \tag{3.62}$$

$$f_2 = \overline{f}_2 + g_2, \tag{3.63}$$

$$f_3 = \overline{f}_3 + g_3, \tag{3.64}$$

where we have used h_0 instead of g_0 for later convenience, and

$$\begin{aligned}\overline{f}_0 = [&(a_{12}a_{34} + a_{13}a_{24} + a_{14}a_{23})(5, 6, \cdots, 2n)\\ &+a_{12}(x_3, x_4, 5, \cdots, 2n) + a_{34}(x_1, x_2, 5, \cdots, 2n)\\ &+a_{13}(x_2, x_4, 5, \cdots, 2n) + a_{24}(x_1, x_3, 5, \cdots, 2n)\\ &+a_{14}(x_2, x_3, 5, \cdots, 2n) + a_{23}(x_1, x_4, 5, \cdots, 2n)]\\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.65}$$

$$h_0 = (x_1, x_2, x_3, x_4, 5, \cdots, 2n)(5, \cdots, 2n), \tag{3.66}$$

$$\begin{aligned}\overline{f}_1 = [&a_{12}a_{34}(5, 6, \cdots, 2n)\\ &+a_{12}(x_3, x_4, 5, \cdots, 2n) + a_{34}(x_1, x_2, 5, \cdots, 2n)]\\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.67}$$

$$g_1 = (x_1, x_2, 5, \cdots, 2n)(x_3, x_4, 5, \cdots, 2n), \tag{3.68}$$

$$\begin{aligned}\overline{f}_2 = [&a_{13}a_{24}(5, 6, \cdots, 2n)\\ &+a_{13}(x_2, x_4, 5, \cdots, 2n) + a_{24}(x_1, x_3, 5, \cdots, 2n)]\\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.69}$$

$$g_2 = (x_1, x_3, 5, \cdots, 2n)(x_2, x_4, 5, \cdots, 2n), \tag{3.70}$$

$$\begin{aligned}\overline{f}_3 = [&a_{14}a_{23}(5, 6, \cdots, 2n)\\ &+a_{14}(x_2, x_3, 5, \cdots, 2n) + a_{23}(x_1, x_4, 5, \cdots, 2n)]\\ &\times(5, 6, \cdots, 2n),\end{aligned} \tag{3.71}$$

$$g_3 = (x_1, x_4, 5, \cdots, 2n)(x_2, x_3, 5, \cdots, 2n). \tag{3.72}$$

Then we find a decomposition of $\overline{f}_0$,

$$\overline{f}_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3. \tag{3.73}$$

Accordingly f_0, f_1, f_2, f_3 are expressed by

$$f_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3 + h_0, \tag{3.74}$$

$$f_1 = \overline{f}_1 + g_1, \tag{3.75}$$

$$f_2 = \overline{f}_2 + g_2, \tag{3.76}$$

$$f_3 = \overline{f}_3 + g_3. \tag{3.77}$$

§ 3.3. Expressions for g_1, g_2 and g_3

Let $\delta_{i,j}$ be Kronecker's delta function,

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} . \tag{3.78}$$

We have an identity, for $i < j$ and $k < l$,

$$\begin{aligned} 1 = &\delta_{i,k}\delta_{j,l} + (\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l})(1 - \delta_{i,k}\delta_{j,l}) \\ &+(1 - \delta_{i,k})(1 - \delta_{i,l})(1 - \delta_{j,k})(1 - \delta_{j,l}). \end{aligned} \tag{3.79}$$

Accordingly a sum of a product of arbitrary functions, $v(i, j)$ and $w(k, l)$ are written as

$$\begin{aligned} \sum_{i<j}\sum_{k<l} v(i,j)w(k,l) &= \sum_{i<j}\sum_{k<l} \delta_{i,k}\delta_{j,l} v(i,j)w(k,l) \\ &+ \sum_{i<j}\sum_{k<l} (\delta_{i,k} + \delta_{i,l} + \delta_{j,k} + \delta_{j,l})(1 - \delta_{i,k}\delta_{j,l}) v(i,j)w(k,l) \\ &+ \sum_{i<j}\sum_{k<l} (1 - \delta_{i,k})(1 - \delta_{i,l})(1 - \delta_{j,k})(1 - \delta_{j,l}) v(i,j)w(k,l), \end{aligned} \tag{3.80}$$

which is rewritten as

$$\begin{aligned} \sum_{i<j}\sum_{k<l} v(i,j)w(k,l) &= \sum_{i<j} v(i,j)w(i,j) + \sum_{i<j<k} (v(i,j)w(i,k) + v(i,k)w(i,j)) \\ &+ \sum_{i<k<j} (v(i,j)w(k,j) + v(k,j)w(i,j)) + \sum_{i<j<k} (v(i,j)w(j,k) + v(j,k)w(i,j)) \\ &+ \sum_{i<j<k<l} (v(i,j)w(k,l) + v(i,k)w(j,l) + v(i,l)w(j,k) \\ &\quad +v(k,l)w(i,j) + v(j,l)w(i,k) + v(j,k)w(i,l)). \end{aligned} \tag{3.81}$$

We expand g_1,

$$g_1 = (x_1, x_2, 5, 6, \cdots, 2n)(x_3, x_4, 5, 6, \cdots, 2n) \tag{3.82}$$

$$= \sum_{5 \le i<j \le 2n} \sum_{5 \le k<l \le 2n} (x_1, x_2, i, j)(x_3, x_4, k, l)(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}. \tag{3.83}$$

Let

$$u(i,j) = (x_1, x_2, i, j)(\hat{i}, \hat{j}), \quad v(i,j) = (x_3, x_4, i, j)(\hat{i}, \hat{j}). \tag{3.84}$$

We obtain, using (3.81),

$$
\begin{aligned}
g_1 = &\sum_{5\le i<j\le 2n} (x_1,x_2,i,j)(x_3,x_4,i,j)[(\hat{i},\hat{j})_{n-2}]^2\\
&+\sum_{5\le i<j<k\le 2n} [(x_1,x_2,i,j)(x_3,x_4,i,k)+(x_1,x_2,i,k)(x_3,x_4,i,j)]\\
&\qquad\qquad \times(\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2}\\
&+\sum_{5\le i<k<j\le 2n} [(x_1,x_2,i,j)(x_3,x_4,k,j)+(x_1,x_2,k,j)(x_3,x_4,i,j)]\\
&\qquad\qquad \times(\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2}\\
&+\sum_{5\le i<j<k\le 2n} [(x_1,x_2,i,j)(x_3,x_4,j,k)+(x_1,x_2,j,k)(x_3,x_4,i,j)]\\
&\qquad\qquad \times(\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2}\\
&+\sum_{5\le i<j<k<l\le 2n} [(x_1,x_2,i,j)(x_3,x_4,k,l)+(x_1,x_2,k,l)(x_3,x_4,i,j)]\\
&\qquad\qquad \times(\hat{i},\hat{j})_{n-2}(\hat{k},\hat{l})_{n-2}.\\
&\qquad +[(x_1,x_2,i,k)(x_3,x_4,j,l)+(x_1,x_2,j,l)(x_3,x_4,i,k)]\\
&\qquad\qquad \times(\hat{i},\hat{k})_{n-2}(\hat{j},\hat{l})_{n-2}.\\
&\qquad +[(x_1,x_2,i,l)(x_3,x_4,j,k)+(x_1,x_2,j,k)(x_3,x_4,i,l)]\\
&\qquad\qquad \times(\hat{i},\hat{l})_{n-2}(\hat{j},\hat{k})_{n-2}.
\end{aligned}
\tag{3.85}
$$

On the other hand we have the following expressions,

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,i,j)\\
&\quad = a_{1i}a_{3i}a_{2j}a_{4j}+a_{1j}a_{3j}a_{2i}a_{4i}+a_{1i}a_{4i}a_{2j}a_{3j}+a_{1j}a_{4j}a_{2i}a_{3i},\\
&\qquad\qquad \text{for } 5\le i<j\le 2n,
\end{aligned}
\tag{3.86}
$$

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,i,k)+(x_1,x_2,i,k)(x_3,x_4,i,j)\\
&\quad = a_{1i}a_{3i}(x_2,x_4,j,k)+a_{2i}a_{4i}(x_1,x_3,j,k)\\
&\qquad +a_{1i}a_{4i}(x_2,x_3,j,k)+a_{2i}a_{3i}(x_1,x_4,j,k),
\end{aligned}
\tag{3.87}
$$

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,k,j)+(x_1,x_2,k,j)(x_3,x_4,i,j)\\
&\quad = a_{1j}a_{3j}(x_2,x_4,i,k)+a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k)+a_{2j}a_{3j}(x_1,x_4,i,k),
\end{aligned}
\tag{3.88}
$$

$$
\begin{aligned}
&(x_1,x_2,i,j)(x_3,x_4,j,k)+(x_1,x_2,j,k)(x_3,x_4,i,j)\\
&\quad = a_{1j}a_{3j}(x_2,x_4,i,k)+a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k)+a_{2j}a_{3j}(x_1,x_4,i,k),\\
&\qquad\qquad \text{for } 5\le i<j<k\le 2n,
\end{aligned}
\tag{3.89}
$$

which show that all terms in Eq.(3.85) except the last one vanish provided that

$$a_{ik}a_{jk} = 0 \quad \text{for } i,j = 1,2,3,4 \text{ and for } k = 5,6,\cdots,2n. \tag{3.90}$$

We introduce new indices y_i for $i = 1,2,3,4$ in place of x_i.

Let

$$(y_i, y_j) = 0, \tag{3.91}$$

and

$$(y_i, j) = a_{i,j} \quad \text{and} \quad a_{i,k}a_{j,k} = 0, \tag{3.92}$$

for $i,j = 1,2,3,4$ and for $k = 4,5,\cdots,2n$.

Accordingly we write g_1 as

$$g_1 = \overline{g}_1 + h_1, \tag{3.93}$$

where

$$\begin{aligned}
\overline{g}_1 = &\sum_{5\le i<j\le 2n} [a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i},\hat{j})_{n-2}]^2\\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{3i}(x_2,x_4,j,k) + a_{2i}a_{4i}(x_1,x_3,j,k)\\
&\qquad +a_{1i}a_{4i}(x_2,x_3,j,k) + a_{2i}a_{3i}(x_1,x_4,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2}\\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2}\\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k)\\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2}
\end{aligned} \tag{3.94}$$

and

$$h_1 = (y_1,y_2,5,6,\cdots,2n)(y_3,y_4,5,6,\cdots,2n), \tag{3.95}$$

using the condition(3.92).

Interchanging the index 2 with the index 3 in Eq.(3.94) we obtain

$$g_2 = (x_1,x_3,5,6,\cdots,2n)(x_2,x_4,5,6,\cdots,2n) = \overline{g}_2 + h_2, \tag{3.96}$$

where

$$
\begin{aligned}
\overline{g}_2 &= \overline{g}_1|_{2\rightleftharpoons 3}, \\
&= \sum_{5\le i<j\le 2n} [a_{1i}a_{2i}a_{3j}a_{4j} + a_{1j}a_{2j}a_{3i}a_{4i} + a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i},\hat{j})_{n-2}]^2 \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{2i}(x_3,x_4,j,k) + a_{3i}a_{4i}(x_1,x_3,j,k) \\
&\qquad +a_{1i}a_{4i}(x_2,x_3,j,k) + a_{2i}a_{3i}(x_1,x_4,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2} \\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k) \\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2} \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k) \\
&\qquad +a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{,n-2}
\end{aligned} \tag{3.97}
$$

and

$$h_2 = (y_1,y_3,5,6,\cdots,2n)(y_2,y_4,5,6,\cdots,2n). \tag{3.98}$$

Similarly interchanging the index 2 with the index 4 in Eq.(3.94) we obtain

$$g_3 = (x_1,x_4,5,6,\cdots,2n)(x_2,x_3,5,6,\cdots,2n) = \overline{g}_3 + h_3, \tag{3.99}$$

where

$$
\begin{aligned}
\overline{g}_3 &= \overline{g}_1|_{2\rightleftharpoons 4}, \\
&= \sum_{5\le i<j\le 2n} [a_{1i}a_{3i}a_{2j}a_{4j} + a_{1j}a_{3j}a_{2i}a_{4i} + a_{1i}a_{2i}a_{3j}a_{4j} + a_{1j}a_{2j}a_{3i}a_{4i}][(\hat{i},\hat{j})_{n-2}]^2 \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{3i}(x_2,x_4,j,k) + a_{2i}a_{4i}(x_1,x_3,j,k) \\
&\qquad +a_{1i}a_{2i}(x_3,x_4,j,k) + a_{3i}a_{4i}(x_1,x_2,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2} \\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k) \\
&\qquad +a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2} \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{3j}(x_2,x_4,i,k) + a_{2j}a_{4j}(x_1,x_3,i,k) \\
&\qquad +a_{1j}a_{2j}(x_3,x_4,i,k) + a_{3j}a_{4j}(x_1,x_2,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2}
\end{aligned} \tag{3.100}
$$

and

$$h_3 = (y_1,y_4,5,6,\cdots,2n)(y_2,y_3,5,6,\cdots,2n). \tag{3.101}$$

From the expressions of $\overline{g}_1, \overline{g}_2$ and $\overline{g}_3$ we find the following decomposition,

$$\overline{g}_1 = \overline{g}_{12} + \overline{g}_{13}, \tag{3.102}$$

$$\overline{g}_2 = \overline{g}_{12} + \overline{g}_{23}, \tag{3.103}$$

$$\overline{g}_3 = \overline{g}_{13} + \overline{g}_{23}, \tag{3.104}$$

where

$$\begin{aligned}
\overline{g}_{12} = &\sum_{5\le i<j\le 2n} [a_{1i}a_{4i}a_{2j}a_{3j} + a_{1j}a_{4j}a_{2i}a_{3i}][(\hat{i},\hat{j})_{n-2}]^2 \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1i}a_{4i}(x_2,x_3,j,k) + a_{2i}a_{3i}(x_1,x_4,j,k)](\hat{i},\hat{j})_{n-2}(\hat{i},\hat{k})_{n-2} \\
&+ \sum_{5\le i<k<j\le 2n} [a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{k},\hat{j})_{n-2} \\
&+ \sum_{5\le i<j<k\le 2n} [a_{1j}a_{4j}(x_2,x_3,i,k) + a_{2j}a_{3j}(x_1,x_4,i,k)](\hat{i},\hat{j})_{n-2}(\hat{j},\hat{k})_{n-2},
\end{aligned} \tag{3.105}$$

$$g_{13} = \overline{g}_{12}|_{3\rightleftharpoons 4}, \tag{3.106}$$

$$g_{23} = \overline{g}_{12}|_{2\rightleftharpoons 4}. \tag{3.107}$$

Accordingly g_1, g_2, g_3 are decomposed into

$$g_1 = \overline{g}_{12} + \overline{g}_{13} + h_1, \tag{3.108}$$

$$g_2 = \overline{g}_{12} + \overline{g}_{23} + h_2, \tag{3.109}$$

$$g_3 = \overline{g}_{23} + \overline{g}_{23} + h_3. \tag{3.110}$$

§ 3.4. Expressions for h_0, h_1, h_2 and h_3

We have h_0 defined by Eq.(3.66),

$$h_0 = (y_1, y_2, y_3, y_4, 5, 6, \cdots, 2n)(5, 6, \cdots, 2n), \tag{3.111}$$

which is expanded, using Eq.(2.39), in

$$= \sum_{5\le i<j<k<l\le 2n} (y_1, y_2, y_3, y_4, i, j, k, l)(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}. \tag{3.112}$$

Here we assume that the product $(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}$ is decomposed into

$$(\hat{i},\hat{j},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2} = f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) + f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \tag{3.113}$$

The hafnian $(y_1, y_2, y_3, y_4, i, j, k, l)$ in Eq.(3.112) is expanded in three different ways,

$$\begin{aligned}
&(y_1, y_2, y_3, y_4, i, j, k, l)\\
&= (y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)\\
&+(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)\\
&+(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, i) \qquad (3.114)\\
&= (y_1, y_3, i, j)(y_2, y_4, k, l) + (y_1, y_3, k, l)(y_2, y_4, i, j)\\
&+(y_1, y_3, i, k)(y_2, y_4, j, l) + (y_1, y_3, j, l)(y_2, y_4, i, k)\\
&+(y_1, y_3, i, l)(y_2, y_4, j, k) + (y_1, y_3, j, k)(y_2, y_4, i, i) \qquad (3.115)\\
&= (y_1, y_4, i, j)(y_2, y_3, k, l) + (y_1, y_4, k, l)(y_2, y_3, i, j)\\
&+(y_1, y_4, i, k)(y_2, y_3, j, l) + (y_1, y_4, j, l)(y_2, y_3, i, k)\\
&+(y_1, y_4, i, l)(y_2, y_3, j, k) + (y_1, y_4, j, k)(y_2, y_3, i, i). \qquad (3.116)
\end{aligned}$$

Substituting Eqs.(3.114) and (3.113) into Eq.(3.112) we obtain an expression for h_0,

$$\begin{aligned}
h_0 = \sum_{5\le i<j<k<l\le 2n} &[(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)\\
&+(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)\\
&+(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, i, l)(y_3, y_4, j, k)]\\
&\times[f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l})]. \qquad (3.117)
\end{aligned}$$

On the other hand,replacing x_i in Eq.(3.85) by y_i for $i = 1, 2, 3, 4$ and using Eqs.(3.92) we obtain

$$\begin{aligned}
h_1 &= (y_1, y_2, 5, 6, \cdots, 2n)(y_3, y_4, 5, 6, \cdots, 2n),\\
&= \sum_{5\le i<j<k<l\le 2n} [(y_1, y_2, i, j)(y_3, y_4, k, l) + (y_1, y_2, k, l)(y_3, y_4, i, j)](\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2}\\
&\quad +[(y_1, y_2, i, k)(y_3, y_4, j, l) + (y_1, y_2, j, l)(y_3, y_4, i, k)](\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2}\\
&\quad +[(y_1, y_2, i, l)(y_3, y_4, j, k) + (y_1, y_2, j, k)(y_3, y_4, i, l)](\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2}. \qquad (3.118)
\end{aligned}$$

We assume that the products of the $(n-2)$th-order hafnians are decomposed into

$$(\hat{i}, \hat{j})_{n-2}(\hat{k}, \hat{l})_{n-2} = f_{01}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \qquad (3.119)$$

$$(\hat{i}, \hat{k})_{n-2}(\hat{j}, \hat{l})_{n-2} = f_{02}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{12}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}), \qquad (3.120)$$

$$(\hat{i}, \hat{l})_{n-2}(\hat{j}, \hat{k})_{n-2} = f_{03}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{13}(\hat{i}, \hat{j}, \hat{k}, \hat{l}) + f_{23}(\hat{i}, \hat{j}, \hat{k}, \hat{l}). \qquad (3.121)$$

Substituting Eqs.(3.119),(3.120) and (3.121) into Eq.(3.118) we obtain an expression for

h_1,

$$
\begin{aligned}
h_1 = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)] \\
& \quad \times[f_{01}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})] \\
& +[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)] \\
& \quad \times[f_{02}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})] \\
& +[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)] \\
& \quad \times[f_{03}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})].
\end{aligned}
\tag{3.122}
$$

Comparing the expression (3.117) for h_0 with the expression (3.122) for h_1, we obtain common terms h_{01} of h_0 and h_1,

$$
\begin{aligned}
h_{01} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)] \\
& \quad \times f_{01}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\
& +[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)] \\
& \quad \times f_{02}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}) \\
& +[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)] \\
& \quad \times f_{03}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}).
\end{aligned}
\tag{3.123}
$$

Let the decomposition of h_1 be

$$h_1 = h_{01} + \tilde{h}_1. \tag{3.124}$$

Then we find that

$$
\begin{aligned}
\tilde{h}_1 = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)] \\
& \quad \times[f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})] \\
& +[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)] \\
& \quad \times[f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})] \\
& +[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)] \\
& \quad \times[f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})].
\end{aligned}
\tag{3.125}
$$

Interchanging y_2 with y_3 in Eqs.(3.117) and (3.122), we obtain common terms h_{02} of h_0

and h_2,

$$\begin{aligned} h_{02} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)] \\ & \times f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)] \\ & \times f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)] \\ & \times f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \end{aligned} \tag{3.126}$$

Interchanging y_2 with y_3 in Eqs.(3.124) and (3.125), we obtain a decomposition of h_2,

$$h_2 = h_{02} + \tilde{h}_2, \tag{3.127}$$

where

$$\begin{aligned} \tilde{h}_2 = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)] \\ & \times [f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)] \\ & \times [f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})] \\ & +[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)] \\ & \times [f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})+f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})]. \end{aligned} \tag{3.128}$$

Similarly we obtain common terms h_{03} of h_0 and h_3 interchanging y_2 with y_4 in Eqs.(3.117) and (3.122)

$$\begin{aligned} h_{03} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)] \\ & \times f_{01}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)] \\ & \times f_{02}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)] \\ & \times f_{03}(\hat{i},\hat{j},\hat{k},\hat{l}). \end{aligned} \tag{3.129}$$

Interchanging y_2 with y_4 in Eqs.(3.124) and (3.125), we obtain a decomposition of h_3,

$$h_3 = h_{03} + \tilde{h}_3, \tag{3.130}$$

where

$$\tilde{h}_3 = \sum_{5\le i<j<k<l\le 2n} [(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)]$$
$$\times[f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})]$$
$$+[(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]$$
$$\times[f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})]$$
$$+[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]$$
$$\times[f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})]. \tag{3.131}$$

We find, using the expansion formulae (3.114), (3.115), (3.116) and the assumption (3.113), that a simple sum of h_{01}, h_{02} and h_{03} is reduced to h_0,

$$h_{01}+h_{02}+h_{03}$$
$$= \sum_{5\le i<j<k<l\le 2n} (y_1,y_2,y_3,y_4,i,j,k.l)$$
$$\times[f_{01}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{02}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})+f_{03}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})],$$
$$= \sum_{5\le i<j<k<l\le 2n} (y_1,y_2,y_3,y_4,i,j,k.l)(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})_{n-2}(\cdots)_{n-2}$$
$$= h_0. \tag{3.132}$$

Accordingly h_0, h_1, h_2 and h_3 are decomposed into

$$h_0 = h_{01}+h_{02}+h_{03}, \tag{3.133}$$
$$h_1 = h_{01}+\tilde{h}_1, \tag{3.134}$$
$$h_2 = h_{02}+\tilde{h}_2, \tag{3.135}$$
$$h_3 = h_{03}+\tilde{h}_3, \tag{3.136}$$

§ 3.5. Expressions for $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$

We rearrange $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ as

$$\tilde{h}_1 = \sum_{5\le i<j<k<l\le 2n} [(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)$$
$$+(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)]\ f_{12}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})$$
$$+[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)$$
$$+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{13}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l})$$
$$+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)$$
$$+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{23}(\hat{\imath},\hat{\jmath},\hat{k},\hat{l}),$$

(3.137)

$$\begin{aligned}\tilde{h}_2 = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ & +(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ & +[(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ & +(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\ & +(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l})\end{aligned} \tag{3.138}$$

and

$$\begin{aligned}\tilde{h}_3 = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)\\ & +(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ & +[(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)\\ & +(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ & +[(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)\\ & +(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned} \tag{3.139}$$

We have identities of hafnians, Eqs.(2.23),(2.24) and (2.25), which are written as

$$\begin{aligned}&(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\ &+(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\ &=(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)\\ &+(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l),\end{aligned} \tag{3.140}$$

$$\begin{aligned}&(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\ &+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)\\ &=(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\ &+(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)\end{aligned} \tag{3.141}$$

and

$$\begin{aligned}&(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\ &+(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)\\ &=(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ &+(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j).\end{aligned} \tag{3.142}$$

Substituting these identities (3.140),(3.141) and (3.142) into Eq.(3.137) we obtain an expression for $\tilde{h}_1$,

$$\begin{aligned}\tilde{h}_1 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)\\ &\quad +[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\ &\quad +(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ &\quad +(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned}$$

(3.143)

Interchanging y_2 with y_3 in Eq.(3.143) we obtain an expression for $\tilde{h}_2$,

$$\begin{aligned}\tilde{h}_2 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)\\ &\quad +[(y_1,y_4,i,l)(y_2,y_3,j,k)+(y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)\\ &\quad +(y_1,y_4,i,k)(y_2,y_3,j,l)+(y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)\\ &\quad +(y_1,y_4,i,j)(y_2,y_3,k,l)+(y_1,y_4,k,l)(y_2,y_3,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned}$$

(3.144)

Similarly interchanging y_2 with y_4 in Eq.(3.143) we obtain an expression for $\tilde{h}_3$,

$$\begin{aligned}\tilde{h}_3 = \sum_{5\le i<j<k<l\le 2n} &[(y_1,y_3,i,l)(y_2,y_4,j,k)+(y_1,y_3,j,k)(y_2,y_4,i,l)\\ &\quad +[(y_1,y_2,i,l)(y_3,y_4,j,k)+(y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,k)(y_2,y_4,j,l)+(y_1,y_3,j,l)(y_2,y_4,i,k)\\ &\quad +(y_1,y_2,i,k)(y_3,y_4,j,l)+(y_1,y_2,j,l)(y_3,y_4,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l})\\ &+[(y_1,y_3,i,j)(y_2,y_4,k,l)+(y_1,y_3,k,l)(y_2,y_4,i,j)\\ &\quad +(y_1,y_2,i,j)(y_3,y_4,k,l)+(y_1,y_2,k,l)(y_3,y_4,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}).\end{aligned}$$

(3.145)

From the expressions (3.143),(3.144) and (3.145) for $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$, respectively we obtain the following decompositions,

$$\tilde{h}_1 = h_{12} + h_{13}, \tag{3.146}$$

$$\tilde{h}_2 = h_{12} + h_{23}, \tag{3.147}$$

$$\tilde{h}_3 = h_{13} + h_{23}, \tag{3.148}$$

where

$$\begin{aligned} h_{12} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_4,i,l)(y_2,y_3,j,k) + (y_1,y_4,j,k)(y_2,y_3,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,k)(y_2,y_3,j,l) + (y_1,y_4,j,l)(y_2,y_3,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_4,i,j)(y_2,y_3,k,l) + (y_1,y_4,k,l)(y_2,y_3,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}), \end{aligned} \tag{3.149}$$

$$\begin{aligned} h_{13} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_3,i,l)(y_2,y_4,j,k) + (y_1,y_3,j,k)(y_2,y_4,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,k)(y_2,y_4,j,l) + (y_1,y_3,j,l)(y_2,y_4,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_3,i,j)(y_2,y_4,k,l) + (y_1,y_3,k,l)(y_2,y_4,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}) \end{aligned} \tag{3.150}$$

and

$$\begin{aligned} h_{23} = \sum_{5\le i<j<k<l\le 2n} & [(y_1,y_2,i,l)(y_3,y_4,j,k) + (y_1,y_2,j,k)(y_3,y_4,i,l)]\ f_{12}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_2,i,k)(y_3,y_4,j,l) + (y_1,y_2,j,l)(y_3,y_4,i,k)]\ f_{13}(\hat{i},\hat{j},\hat{k},\hat{l}) \\ & +[(y_1,y_2,i,j)(y_3,y_4,k,l) + (y_1,y_2,k,l)(y_3,y_4,i,j)]\ f_{23}(\hat{i},\hat{j},\hat{k},\hat{l}). \end{aligned} \tag{3.151}$$

§ 3.6. Decompositions of f_0, f_1, f_2 and f_3

We have found in the previous subsections the decompositions,

$$h_1 = h_{01} + \tilde{h}_1, \quad h_2 = h_{02} + \tilde{h}_2, \quad h_3 = h_{03} + \tilde{h}_3, \tag{3.152}$$

$$\tilde{h}_1 = h_{12} + h_{13}, \quad \tilde{h}_2 = h_{12} + h_{23}, \quad \tilde{h}_3 = h_{13} + h_{33}, \tag{3.153}$$

which together with Eqs.(3.133), give the decompositions of h_0, h_1, h_2 and h_3,

$$h_0 = h_{01} + h_{02} + h_{03}, \quad h_1 = h_{01} + h_{12} + h_{13}, \tag{3.154}$$

$$h_2 = h_{02} + h_{12} + h_{23}, \quad h_3 = h_{03} + h_{13} + h_{23}. \tag{3.155}$$

On the other hand, we have found the decompositions,

$$f_0 = \overline{f}_1 + \overline{f}_2 + \overline{f}_3 + h_0, \tag{3.156}$$

$$f_1 = \overline{f}_1 + g_1, \quad g_1 = g_{12} + g_{13} + h_1, \tag{3.157}$$

$$f_2 = \overline{f}_2 + g_2, \quad g_2 = g_{12} + g_{23} + h_2, \tag{3.158}$$

$$f_3 = \overline{f}_3 + g_3, \quad g_3 = g_{13} + g_{23} + h_3. \tag{3.159}$$

Combining these decompositions we finally obtain the decompositions,

$$f_0 = f_{01} + f_{02} + f_{03}, \quad f_1 = f_{01} + f_{12} + f_{13}, \tag{3.160}$$

$$f_2 = f_{02} + f_{12} + f_{23}, \quad f_3 = f_{03} + f_{13} + f_{23}, \tag{3.161}$$

where

$$f_{01} = \overline{f}_1 + h_{01}, \quad f_{02} = \overline{f}_2 + h_{02}, \quad f_{03} = \overline{f}_3 + h_{03}, \tag{3.162}$$

$$f_{12} = g_{12} + h_{12}, \quad f_{13} = g_{13} + h_{13}, \quad f_{23} = g_{23} + h_{23}. \tag{3.163}$$

Thus we have proved that the decompositions of the products of nth-order hafnians,

$$f_0 = (1,2,3,4,5,6,\cdots,2n)(5,6,\cdots,2n) \tag{3.164}$$

$$f_1 = (1,2,5,6,\cdots,2n)(3,4,5,6,\cdots,2n) \tag{3.165}$$

$$f_2 = (1,3,5,6,\cdots,2n)(2,4,5,6,\cdots,2n) \tag{3.166}$$

$$f_3 = (1,3,5,6,\cdots,2n)(2,4,5,6,\cdots,2n), \tag{3.167}$$

provided that the decompositions of the products of $(n-2)$th-order hafnians hold. It is easily proved that the decompositions of the products of fourth-order hafnians hold. Hence we have proved by induction that the decompositons of the products of any order hafnians.

§ 4. Ultradiscrete analogue of the identity of pfaffians

It is known that a variety of soliton equations exhibiting multi-soliton solutions expressed by pfaffians give rise to the following identity of pfaffians,

$$\begin{aligned} &\mathrm{pf}(1,2,3,4,5,6,\cdots,2n)\mathrm{pf}(5,6,\cdots,2n) \\ &= \mathrm{pf}(1,2,5,6,\cdots,2n)\mathrm{pf}(3,4,5,6,\cdots,2n) \\ &-\mathrm{pf}(1,3,5,6,\cdots,2n)\mathrm{pf}(2,4,5,6,\cdots,2n) \\ &+\mathrm{pf}(1,4,5,6,\cdots,2n)\mathrm{pf}(2,3,5,6,\cdots,2n). \end{aligned} \tag{4.1}$$

It is known that the pfaffians can not be ultradiscretized due to negativity terms. A remedy for the problem was found by Takahashi and the author of the present article.

They have expressed the multi-soliton solutions to an ultradiscretized soliton equation called "Box and Ball System" by ultradiscretized *permanents* instead of determinants. Accordingly we consider an ultradiscrete form of the identity of hafnians.

We have

$$f_0 = (1,2,3,4,5,6,\cdots,2n)(5,6,\cdots,2n), \tag{4.2}$$

$$f_1 = (1,2,5,6,\cdots,2n)(3,4,5,6,\cdots,2n), \tag{4.3}$$

$$f_2 = (1,3,5,6,\cdots,2n)(2,4,5,6,\cdots,2n), \tag{4.4}$$

$$f_3 = (1,4,5,6,\cdots,2n)(2,3,5,6,\cdots,2n), \tag{4.5}$$

which were decomposed in the previous section into the following form,

$$f_0 = f_{01} + f_{02} + f_{03}, \tag{4.6}$$

$$f_1 = f_{01} + f_{12} + f_{13}, \tag{4.7}$$

$$f_2 = f_{02} + f_{12} + f_{23}, \tag{4.8}$$

$$f_3 = f_{03} + f_{13} + f_{23}. \tag{4.9}$$

We consider a relation of hafnians,

$$f_0 + f_2 = f_1 + f_3, \tag{4.10}$$

which does holds for pfaffians but not for hafnians.

Let

$$f_0 = \exp F_0/\epsilon, \quad f_1 = \exp F_1/\epsilon, \quad f_2 = \exp F_2/\epsilon, \quad f_3 = \exp F_3/\epsilon, \tag{4.11}$$

$$f_{01} = \exp F_{01}/\epsilon, \quad f_{02} = \exp F_{02}/\epsilon, \quad f_{03} = \exp F_{03}/\epsilon, \tag{4.12}$$

$$f_{12} = \exp F_{12}/\epsilon \quad f_{13} = \exp F_{13}/\epsilon, \quad f_{23} = \exp F_{23}/\epsilon. \tag{4.13}$$

Taking a limit $\epsilon \to +0$ in Eq.(4.10), we have an ultradiscrete form of the identity of the hafnians,

$$\max(F_0, F_2) = \max(F_1, F_3). \tag{4.14}$$

By virtue of the decomposition of the hafnians we find that

$$F_0 = \max(F_{01}, F_{02}, F_{03}), \tag{4.15}$$

$$F_1 = \max(F_{01}, F_{12}, F_{13}), \tag{4.16}$$

$$F_2 = \max(F_{02}, F_{12}, F_{23}), \tag{4.17}$$

$$F_3 = \max(F_{03}, F_{13}, F_{23}), \tag{4.18}$$

and Eq.(4.14) is expressed by

$$\max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23}) = \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23}). \tag{4.19}$$

We shall investigate whether Eq.(4.19) does hold or not. Equation (4.19) holds in the following six cases,

(i) $F_{01} \geq \max(F_{02}, F_{03}, F_{12}, F_{13}, F_{23})$,

(ii) $F_{02} = \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23})$,

(iii) $F_{03} \geq \max(F_{01}, F_{02}, F_{12}, F_{13}, F_{23})$,

(iv) $F_{12} \geq \max(F_{01}, F_{02}, F_{03}, F_{13}, F_{23})$,

(v) $F_{13} = \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23})$,

(vi) $F_{23} \geq \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{13})$,

except for the following two cases,

(vii) $F_{02} > \max(F_{01}, F_{03}, F_{12}, F_{13}, F_{23})$,

(viii) $F_{13} > \max(F_{01}, F_{02}, F_{03}, F_{12}, F_{23})$.

In the cases (vii) and (viii) we have the relation, $F_0 = F_2$ and $F_1 = F_3$ respectively. Accordingly we find the algebraic identiy of the ultradiscretized hafnians,

$$(\max(F_0, F_2) - \max(F_1, F_3))(F_0 - F_2)(F_1 - F_3) = 0, \tag{4.20}$$

where F_0, F_1, F_2 and F_3 are the ultradiscrete form of f_0, f_1, f_2 and f_3,respectively.

We have thus proved the ultradiscrete analogue of the identity of the pfaffians.

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