RIMS Kôkyûroku Bessatsu
B15 (2009), 87–99

Self-similar blow-up for a chemotaxis system in higher dimensional domains

By

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§ 1. Introduction and statement of main results

We consider solutions of a parabolic-elliptic system

$$
\begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) & \text{in } \Omega \times (0,T) \\ 0 = \Delta v + u & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}
$$

where either $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$ or $\Omega = \mathbf{R}^N$ with $N \geq 3$. In the former case, we assume $\partial u/\partial \nu - u\partial v/\partial \nu = 0$ and $v = 0$ on $\partial\Omega$, where ν denotes the outer unit normal vector. This system arises in the study of the motion of bacteria by chemotaxis as a simplification of the Keller-Segel model (see [16], [22]). Here, u and v represent the density of the bacteria and the concentration of the chemo-attractant, respectively. This system also has been used as a model for the evolution of self-attracting clusters (see [27], [28], [2]).

In this note we consider the blow-up rate of solutions to the system

$$
\begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) & \text{in } \Omega \times (0,T) \\ 0 = \Delta v + u & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} - u\dfrac{\partial v}{\partial \nu} = 0 \text{ and } v = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(|x|) & \text{in } \Omega, \end{cases} \tag{1.2}
$$

Received Feburary 25, 2009. Revised July 31, 2009. Accepted August 7, 2009.
2000 Mathematics Subject Classification(s): 35K55, 35K57, 92C17.
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where $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$ with $N \geq 3$ and $0 < L < \infty$, and $u_0(r)$ is a nonnegative continuous function on $[0, L]$. We restrict ourselves to the study of radially symmetric solutions. It is known by [2] that the system (1.2) has a unique local classical solution (u, v). It is easy to see that u and v are positive for $0 < t < T$, and that the conservation of the initial mass of u holds, that is,

$$\|u(\cdot, t)\|_1 = \|u_0(\cdot)\|_1 \quad \text{for } 0 < t < T, \tag{1.3}$$

where $\|\cdot\|_p$ denotes the standard $L^p(\Omega)$ norm for $1 \leq p \leq \infty$. A solution (u, v) is said to blow-up at $t = T < \infty$ if (u, v) is classical in $\Omega \times (0, T)$ and satisfies $\limsup_{t \to T} \|u(\cdot, t)\|_\infty = \infty$. A simple argument shows that if u blows up at a finite time $t = T$ then

$$\liminf_{t \to T} (T - t)\|u(\cdot, t)\|_\infty > 0.$$

We say that the blow-up is of type I if u satisfies

$$\limsup_{t \to T} (T - t)\|u(\cdot, t)\|_\infty < \infty.$$

The blow-up is called type II if it is not type I. We note that self-similar solutions, given by (2.1) below, blow up in type I rate.

We briefly review some known results concerning blow-up behavior for (1.1) and related systems. In the case $N = 2$, Herrero and Velázquez [15] considered the system

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\ \tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0, T), \end{cases} \tag{1.4}$$

together with initial conditions

$$u(x, 0) = u_0(|x|) \quad \text{and} \quad v(x, 0) = v_0(|x|) \quad \text{for } x \in \Omega,$$

where $\Omega = \{x \in \mathbf{R}^2 : |x| < L\}$ and $\tau > 0$. It was shown in [15] that (1.4) has radially symmetric solutions such that u develops a Dirac delta-type singularity at the origin in a finite time, and that u blows up in type II rate. See also [13], [14]. Senba and Suzuki [25] considered the system (1.4) in the case where Ω is a bounded smooth domain in $\mathbf{R}^2$ and $\tau = 0$ together with the initial condition $u(x, 0) = u_0(x)$ for $x \in \Omega$. Denote by T the maximal existence time of the solution to (1.4). It was shown in [25] that, if $T < \infty$, then the solution (u, v) of (1.4) satisfies

$$u(x, t) \rightharpoonup \sum_{q \in \mathcal{B}} m(q)\delta_q + f \tag{1.5}$$

in the sense of measures as $t \to T$, where $\mathcal{B}$ is the set of blow-up points, δ_q is the delta function whose support is the point $q \in \overline{\Omega}$, $m(q)$ is the constant satisfying $m(q) \geq 8\pi$ if $q \in \Omega$, and $m(q) \geq 4\pi$ if $q \in \partial\Omega$, and f is a nonnegative function in $L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{B})$. Furthermore, Senba [24] showed that, if $\Omega = \{x \in \mathbf{R}^2 : |x| < L\}$, and if a radial solution (u, v) blows up at $t = T < \infty$, then (1.5) holds with $\mathcal{B} = \{0\}$ and $m(0) = 8\pi$, and u blows up in type II rate. For nonradial case, see [26].

In the case $N = 3$, Herrero et al [11], [12] have investigated the blow-up behavior of solutions by using matched asymptotic expansions. In [12] they showed that (1.2) has a sequence of self-similar blow-up solutions, and they in [11] showed the existence of Burgers like blow-up solutions which are not self-similar. These solutions consist of an imploding smoothed out shock wave that collapses into a Dirac mass when the singularity is formed, and blow up in type II rate. Later, Brenner et al [4] investigated the problem in the case $3 \leq N \leq 9$ by a numerical approach, and showed the existence and stability of both self-similar blow-up solutions and Burgers like blow-up solutions.

For a solution (u, v) to (1.2), putting

$$n(x,t) = \Theta u(x, \Theta t) \quad \text{and} \quad \phi(x,t) = -\Theta v(x, \Theta t)$$

with $\Theta = 1/\|u_0\|_1$, we find that (n, ϕ) solves the problem

$$\text{(1.6)} \qquad \begin{cases} n_t = \nabla \cdot (\Theta \nabla n + n \nabla \phi) & \text{in } \Omega \times (0,T) \\ \Delta \phi = n & \text{in } \Omega \times (0,T) \\ \Theta \dfrac{\partial n}{\partial \nu} + n \dfrac{\partial \phi}{\partial \nu} = 0 \text{ and } v = 0 & \text{on } \partial\Omega \times (0,T) \\ n(x,0) = n_0(x) & \text{in } \Omega, \end{cases}$$

where $n_0(x) = \Theta u_0(|x|)$ for $x \in \Omega$. Note that n_0 satisfies $\|n_0\|_1 = 1$. Guerra and Peletier [10] considered the problem (1.6) in the case $3 \leq N \leq 9$. They in [10] characterize the blow-up behavior of solutions in terms of initial data, and showed that the solution behaves like a self-similar solution near the blow-up point.

In this note, we consider the system (1.2) in the case $3 \leq N \leq 9$, and derive criteria of the blow-up rate of solutions. In particular, we will identify an explicit class of initial data for which the blow-up is of type I rate.

To state our results, define U_0 and V_0, respectively, by

$$\text{(1.7)} \qquad U_0(r) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} u_0(s) ds \quad \text{for } 0 \leq r \leq L$$

and

$$\text{(1.8)} \qquad V_0(r) = \frac{U_0(r)}{r^2} = \frac{1}{r^N} \int_0^r s^{N-1} u_0(s) ds \quad \text{for } 0 \leq r \leq L.$$

For the initial condition, we assume that

$$V_0'(r) \le 0 \quad \text{for } 0 \le r \le L, \tag{1.9}$$

where $' = d/dr$. It is easy to see that (1.9) holds if u_0 satisfies

$$u_0 \in C^1[0, L] \quad \text{and} \quad u_0'(r) \le 0 \quad \text{for } 0 \le r \le L. \tag{1.10}$$

Our first result is the following.

Theorem 1.1. *Let $3 \le N \le 9$, and assume that (1.9) holds.*

(i) *Suppose U_0 satisfies $U_0(r) \le 2$ for $0 \le r \le L$. Then a solution (u, v) of (1.2) does not blow up in finite time.*

(ii) *Suppose that $U_0(r) - 2$ has exactly one zero for $0 \le r < L$ and $U_0(L) > 2$. If a solution (u, v) of (1.2) blows up in finite time, then the blow-up is of type I.*

It should be mentioned that more general criteria will be given in Theorem 3.1 below.

As a consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let $3 \le N \le 9$, and assume that (1.9) holds. Suppose that $U_0(r)$ is increasing for $0 < r < L$. If a solution (u, v) of (1.2) blows up in finite time, then the blow-up is of type I.*

Note that U_0 satisfies

$$(r^{N-1}U_0'(r))' = r^{N-1}(2u_0(r) + ru_0'(r)) \quad \text{for } 0 < r < L.$$

Assume that u_0 satisfies (1.10) and

$$ru_0'(r) + 2u_0(r) \text{ has at most one zero for } 0 \le r \le L. \tag{1.11}$$

Then one easily see that $U_0'(r)$ has at most one zero for $(0, L]$, and that $U_0'(r) > 0$ for $0 < r < r_0$ and $U_0'(r) < 0$ for $r_0 < r \le L$ if $U_0'(r)$ has a zero at $r_0 \in (0, L)$. Assume, in addition, that

$$\int_0^L s^{N-1}u_0(s)ds > 2L^{N-2}. \tag{1.12}$$

Then $U_0(L) > 2$ and $U_0(r) - 2$ has exactly one zero for $0 \le r < L$. By Theorem 1.1 (ii) we obtain the following:

Corollary 1.3. *Let $3 \le N \le 9$, and assume that u_0 satisfies (1.10), (1.11) and (1.12). If a solution of (1.2) blows up in finite time, then the blow-up is of type I.*

We recall here some sufficient conditions for blow-up in finite time by [2], [21].

Proposition 1.4. *Let $N \geq 3$. Assume that one of the following* (i)-(iii) *holds:*

(i) $U_0(L) > 2N$;

(ii) $U_0(L) \geq 4$ *and* U_0 *satisfies, with some* $T_0 > 0$,

$$U_0(r) \geq \frac{4r^2}{2(N-2)T_0 + r^2} \quad \text{for } 0 \leq r \leq L; \tag{1.13}$$

(iii) u_0 *satisfies*

$$\int_{|x|\leq L} |x|^N u_0(|x|)dx < \left(\frac{\|u_0\|_1^{(2N-2)/N}}{4(N-1)\omega_N} \right)^{N/(N-2)}, \tag{1.14}$$

where ω_N *is the surface area of the unit sphere in* $\mathbf{R}^N$.

Then a solution (u, v) *of* (1.2) *blows up in finite time* $t = T < \infty$. *Furthermore, in the case* (ii), *the solution blows up at time* T *with* $T \leq T_0$.

The blow-up of solutions was shown in the case (i) by Biler [3, Theorem 3]. We can show the blow-up of solutions in the case (ii) by the comparison argument, and in the case (iii) by following the argument due to Nagai [21, Theorem 3.1]. For the proof of Proposition 1.4, see [20].

As a consequence of Corollaries 1.2 and 1.3 and Proposition 1.4, we can show the existence of solutions which blow up with type I rate. As a simple example, let $u_0(r) \equiv \ell$ with $\ell > 2N/L$. Then a solution (u, v) of (1.2) blows up in finite time with type I rate by Corollary 1.2 and Proposition 1.4 (i). (See [10, Corollary 1.2].) For another example, let $u_0(r) = \ell G(r, \tau)$ with $\tau > 0$ and $\ell > 0$, where $G(r, t) = (4\pi t)^{-N/2} e^{-r^2/4t}$ is the heat kernel. Then (1.10) and (1.11) hold, and it is easy to see that

$$\|u_0\|_1 = \omega_N \int_0^L s^{N-1} u_0(s)ds \to \ell \quad \text{and} \quad \int_{|x|\leq L} |x|^N u_0(|x|)dx \to 0$$

as $\tau \to 0$. Combining Corollary 1.3 and Proposition 1.4 (iii), we obtain the following:

Corollary 1.5. *Let* $3 \leq N \leq 9$, *and let* $u_0(r) = \ell G(r, \tau)$ *with* $\tau > 0$ *and* $\ell > 2L^{N-2}\omega_N$. *Then there exists* $\tau_0 > 0$ *such that, if* $\tau \in (0, \tau_0]$, *then a solution* (u, v) *of* (1.2) *blows up in finite time with type I rate.*

We note that, in Corollary 1.5, the initial function u_0 converges to a Dirac delta function in the sense of measure as $\tau \to 0$. Thus this corollary suggests that self-similar blow-up may be seen even if initial function is close to a Dirac delta function.

Next, we consider the local blow-up profile of solutions to (1.2). Assume that V_0, defined by (1.8), satisfies

$$(V_0)_{rr} + \frac{N+1}{r}(V_0)_r + N(V_0)^2 + rV_0(V_0)_r \geq 0 \quad \text{for } 0 \leq r \leq L. \tag{1.15}$$

Guerra and Peletier [10] showed that, when $N \geq 3$ and (1.9) and (1.15) hold, any type I blow-up solution behaves like a self-similar solution near the singularity $x = 0$.

Our result is the following.

Theorem 1.6. *Let* $3 \leq N \leq 9$*, and assume that* (1.9) *and* (1.15) *hold. If a solution* (u, v) *of* (1.2) *blows up in finite time, then the blow-up is of type I.*

Remark. Define the average density function V by

$$V(r,t) = \frac{1}{r^N}\int_0^r s^{N-1}u(s,t)ds \quad \text{for } 0 \leq r \leq L,\ 0 \leq t < T. \tag{1.16}$$

It was shown by Guerra and Peletier [10, Theorem 2.3] that when $N \geq 3$ and (1.9) and (1.15) hold, if a solution (u, v) blows up in finite time $t = T$ with type I rate, then V satisfies

$$\lim_{t\to T}(T-t)V(\rho\sqrt{T-t}, t) = \Phi(\rho)/\rho^2 \tag{1.17}$$

uniformly on compact set $|\rho| \leq C$ for every $C > 0$, where Φ is a certain positive function. Combining with Theorem 1.6, we find that when $3 \leq N \leq 9$ and (1.9) and(1.15) hold, if a solution (u, v) blows up in finite time, then (1.17) holds. Note here that the condition (1.15) ensures that $V_t \geq 0$ for all $0 < t < T$. (See (3.6) below.) It is still an open problem whether (1.17) holds for type I blow-up solutions without the condition (1.15).

We note that similar results hold for the well-studied problem for semilinear heat equation

$$\begin{cases} u_t - \Delta u = u^p & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(|x|) & \text{in } \Omega, \end{cases} \tag{1.18}$$

where $p > 1$, $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$, and $u_0(r)$ is nonnegative and nonincreasing for $0 \leq r \leq L$. A simple comparison argument shows that any blow-up solution satisfies

$$\liminf_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_\infty > 0.$$

Assume that $u_0 = u_0(|x|)$ satisfies

$$\Delta u_0 + u_0^p \geq 0 \qquad \text{in } \Omega. \tag{1.19}$$

By the maximum principle, the condition (1.19) implies that $u_t \geq 0$ for all $0 < t < T$. Friedman and McLeod [6] showed that, if (1.19) holds, then any blow-up solution satisfies

$$\limsup_{t \to T} (T-t)^{1/(p-1)} \|u(\cdot,t)\|_\infty < \infty. \tag{1.20}$$

Bebernes and Eberly showed in [1] that, under the condition (1.19), finite time blow-up solutions are asymptotically self-similar. Precisely, any solution u of (1.18) which blows up in finite time $t = T$ satisfies

$$\lim_{t \to T} (T-t)^{1/(p-1)} u((T-t)^{1/2} y, t) = \kappa$$

uniformly on compact set $|y| \leq C$ for every $C > 0$ with $\kappa = (p-1)^{-1/(p-1)}$. It should be mentioned that Matos [19] later showed that any blow-up solution which satisfies (1.20) is asymptotically self-similar in the supercritical case without the condition (1.19). For the precise characterization of the behavior of blow-up solutions to (1.18), we refer to Giga and Kohn [7, 8, 9] in the subcritical case and Matano and Merle [17, 18] in the supercritical case.

This note is organized as follows: In Section 2, we will show the existence of a sequence of self-similar solutions to (1.1) with $\Omega = \mathbf{R}^N$. In Section 3, we derive criteria of the blow-up rate of solutions, and give the proof of Theorems 1.1 and 1.6.

§ 2. Backward self-similar solutions

The proof of Theorems 1.1 and 1.6 are based on the study of the properties of backward self-similar solutions to the system (1.1) with $\Omega = \mathbf{R}^N$. The system (1.1) with $\Omega = \mathbf{R}^N$ is invariant under the scaling

$$(u, v) \mapsto (u_\lambda, v_\lambda) = (\lambda^2 u(\lambda x, \lambda^2 t), v(\lambda x, \lambda^2 t))$$

for $\lambda > 0$. A solution (u, v) is called *self-similar* if $(u, v) = (u_\lambda, v_\lambda)$ for each $\lambda > 0$, and is called *backward* if (u, v) is defined for all $t < 0$. By the transformation in the time, a backward self-similar solution has the form

$$u(x,t) = \frac{1}{T-t}\phi(x/\sqrt{T-t}) \quad \text{and} \quad v(x,t) = \psi(x/\sqrt{T-t}) \tag{2.1}$$

for $x \in \mathbf{R}^N$ and $t < T$, where (ϕ, ψ) satisfies

$$\begin{cases} \Delta\phi - \nabla \cdot \left(\phi\left(\frac{x}{2} + \nabla\psi\right)\right) + \frac{N-2}{2}\phi = 0, & x \in \mathbf{R}^N \\ 0 = \Delta\psi + \phi, & x \in \mathbf{R}^N. \end{cases} \tag{2.2}$$

We will obtain the existence of a sequence of self-similar solutions to (2.2) together with the properties of solutions. For the proof, see [20].

Theorem 2.1. *Let $3 \le N \le 9$. Then the system* (2.2) *has radially symmetric solutions $\{(\phi_j, \psi_j)\}_{j=1}^{\infty}$ such that $\phi_j(r) > 0$ for $r \ge 0$ and $\phi_j(0) \to \infty$ as $j \to \infty$. For each $j = 1, 2, \ldots,$ define*

$$\Phi_j(r) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} \phi_j(s) ds, \quad r > 0. \tag{2.3}$$

Then $\Phi_j(r) - 2$ has exactly $2j$ zeros on $(0, \infty)$ and no zeros on (R_0, ∞) with $R_0 = 2\sqrt{N-1}$. Furthermore, there exists a sequence $\{\alpha_j\}_{j=1}^{\infty}$ satisfying $0 < \cdots < \alpha_{j+1} < \alpha_j < \cdots < \alpha_1$ and $\alpha_j \to 0$ as $j \to \infty$ such that the following (i) *and* (ii) *hold.*

(i) *For any constant $c > 0$,*

$$\inf_{0<r<c\alpha_j} \frac{\Phi_j(r)}{r^2} \to \infty \quad \text{as } j \to \infty.$$

(ii) *For any $\varepsilon > 0$, there exist a constant $c_0 = c_0(\varepsilon) > 0$ and an integer $j_0 = j_0(\varepsilon) \in \mathbf{N}$ such that if $j \ge j_0$ then*

$$\sup_{r \ge c_0\alpha_j} |\Phi_j(r) - 2| < \varepsilon \quad \text{and} \quad \sup_{r \ge c_0\alpha_j} |r\Phi_j'(r)| < \varepsilon.$$

Remark. For each fixed $r > 0$, we have

$$r^2 \phi_j(r) \to 2(N-2) \quad \text{as } j \to \infty. \tag{2.4}$$

In fact, it follows from (2.3) that

$$r^2 \phi_j(r) = (N-2)\Phi_j(r) + r\Phi_j'(r) \quad \text{for } r > 0.$$

Since (ii) holds and $\alpha_j \to 0$ as $j \to \infty$, we obtain (2.4).

The existence of self-similar solutions was already shown by [11, 4, 23]. It seems, however, that the properties on the location of zeros and properties (i) and (ii) are new, and these properties play an important role in the proof of the theorems.

§ 3. Proof of Theorems 1.1 and 1.6 (sketch)

We restrict our attention to radially symmetric solutions to (1.2) of the form $u = u(r,t)$ and $v = v(r,t)$, $r = |x|$, and consider the system

$$\begin{cases} r^{N-1}u_t = (r^{N-1}u_r)_r - (r^{N-1}uv_r)_r, & 0 < r < L,\ 0 < t < T, \\ 0 = (r^{N-1}v_r)_r + r^{N-1}u, & 0 < r < L,\ 0 < t < T, \\ u_r(0,t) = u_r(L,t) - u(L,t)v_r(L,t) = 0, & 0 < t < T, \\ v_r(0,t) = v(L,t) = 0, & 0 < t < T, \\ u(r,0) = u_0(r), & 0 \le r \le L. \end{cases} \tag{3.1}$$

Put

$$M = \int_0^L r^{N-1} u_0(r)dr.$$

Then, by the third formula in (3.1), it follows that

$$\int_0^L r^{N-1} u(r,t)dr = M \quad \text{for } 0 \le t < T. \tag{3.2}$$

Put $\hat{u}$ by

$$\hat{u}(r,t) = \int_0^r s^{N-1} u(s,t)ds = -r^{N-1} v_r(r,t).$$

Here we have used the second formula in (3.1). Then the system (3.1) can be reduced to a single equation

$$\hat{u}_t = r^{N-1}(r^{1-N}\hat{u}_r)_r + r^{1-N}\hat{u}\hat{u}_r.$$

Define

$$U(r,t) = r^{2-N}\hat{u}(r,t) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} u(s,t)ds. \tag{3.3}$$

Then U satisfies

$$U_t = U_{rr} + \frac{N-3}{r} U_r - \frac{2(N-2)}{r^2} U + \frac{(N-2)U^2 + rUU_r}{r^2} \tag{3.4}$$

for $0 < r < L, 0 < t < T$ and

$$U(0,t) = \lim_{r \to 0} U(r,t) = 0 \quad \text{and} \quad U(L,t) = ML^{2-N} \quad \text{for } 0 \le t < T.$$

Note here that, by using l'Hospital's rule, we obtain

$$\lim_{r \to 0} \frac{U(r,t)}{r^2} = \lim_{r \to 0} \frac{\int_0^r s^{N-1} u(s,t)ds}{r^N} = \frac{u(0,t)}{N} \quad \text{for } 0 < t < T. \tag{3.5}$$

Put $V(r,t) = U(r,t)/r^2$. Then V satisfies

$$V_t = V_{rr} + \frac{N+1}{r} V_r + NV^2 + rVV_r \quad \text{for } 0 < r < L, 0 < t < T \tag{3.6}$$

and $V(L,t) = ML^{-N}$ for $0 < t < T$. We will show here that

$$V_r(0,t) = \lim_{r \to 0} V_r(r,t) = 0 \quad \text{for } 0 < t < T. \tag{3.7}$$

In fact,

$$\begin{aligned} V_r(r,t) &= \frac{1}{r}\left(u(r,t) - Nr^{-N} \int_0^r s^{N-1} u(s,t)ds\right) \\ &= \frac{u(r,t) - u(0,t)}{r} - N \frac{\int_0^r s^{N-1}(u(s,t) - u(0,t))ds}{r^{N+1}}. \end{aligned}$$

By using l'Hospital's rule, we obtain

$$\lim_{r\to 0}\frac{\displaystyle\int_0^r s^{N-1}(u(s,t)-u(0,t))ds}{r^{N+1}} = \frac{1}{N+1}\lim_{r\to 0}\frac{u(r,t)-u(0,t)}{r} = \frac{u_r(0,t)}{N+1}.$$

Since $u_r(0,t)=0$ for $0\le t<T$, we obtain (3.7).

Let (ϕ_j,ψ_j) be a radially symmetric solutions of (2.2) obtained in Theorem 2.1, and put Φ_j by (2.3). Take $T>0$, and put

$$U_j(r,t)=\Phi_j(r/\sqrt{T-t})=\frac{1}{r^{N-2}(T-t)}\int_0^r s^{N-1}\phi_j(s/\sqrt{T-t})ds \tag{3.8}$$

for $0\le r\le L$, $0\le t<T$. Then $U=U_j$ solves (3.4) and

$$U_j(0,t)=0 \quad\text{and}\quad U_j(L,t)=\Phi_j(L/\sqrt{T-t}) \quad \text{for } 0<t<T.$$

By the similar argument as in (3.5) we obtain

$$\lim_{r\to 0}\frac{U_j(r,t)}{r^2}=\frac{\phi_j(0)}{N(T-t)} \quad \text{for } 0<t<T. \tag{3.9}$$

Put $V_j(r,t)=U_j(r,t)/r^2$. Then $V=V_j$ solves (3.6) and

$$(V_j)_r(0,t)=0 \quad\text{and}\quad V_j(L,t)=\Phi_j(L/\sqrt{T-t})L^{-2} \quad \text{for } 0<t<T.$$

By using the zero number properties of solutions for linear parabolic equations [5], we will derive criteria of the blow-up rate of solutions to (1.2) in terms of the function U defined by (3.3). We obtain Theorems 1.1 and 1.6 as a consequence of the following result.

Theorem 3.1. *Let $3\le N\le 9$, and assume that* (1.9) *holds. Let (u,v) be a radially symmetric solution of* (1.2) *for $0\le t<T$, and define U by* (3.3).

(i) *Assume that there exist $t_0\in[0,T)$ and $r_0\in(0,L]$ such that*

$$U(r,t_0)\le 2 \quad \textit{for } 0\le r\le r_0 \quad \textit{and} \quad U(r_0,t)\le 2 \quad \textit{for } t_0\le t<T.$$

Then the solution (u,v) does not blow up at $t=T$.

(ii) *Assume that there exist $t_0\in[0,T)$ and $r_0\in(0,L]$ such that $U(r,t_0)-2$ has exactly one zero for $0\le r\le r_0$ and $U(r_0,t)>2$ for $t_0\le t<T$. If the solution (u,v) blows up at $t=T<\infty$ then the blow-up is of type I.*

It is clear that U_0, defined by (1.7), satisfies $U_0(r)=U(r,0)$ for $0\le r\le L$. By the property (3.2) we see that $U(L,t)=U_0(L)$ for $0\le t<T$. Then, by applying Theorem 3.1 with $r_0=L$ and $t_0=0$, we obtain Theorem 1.1 immediately.

Proof of Theorem 1.6. Assume that (u,v) blows up at $t=T<\infty$. By Theorem 3.1 (i) we have $\{r\in(0,L):U(r,t)>2\}\neq\emptyset$ for any $0\le t<T$. It is easy to see that there exist $t_0\in(0,T)$ and $r_0\in(0,L)$ such that $U(r_0,t_0)>2$ and $U(r,t_0)-2$ has exactly one zero for $0<r<r_0$. Note that the condition (1.15) ensures that $V_t\ge 0$ for all $t\in(0,T)$. Then $U(r_0,t)$ is nondecreasing in $t\in(t_0,T)$, and hence $U(r_0,t)>2$ for $t_0\le t<T$. By Theorem 3.1 (ii), the blow-up is of type I. □

We will give a sketch of the proof of Theorem 3.1. For the detail, see [20].

Proof of Theorem 3.1. (Sketch). (i) By using the comparison argument, we may assume that $U(r,t)<2$ for $0\le r<r_0$ and $t_0<t\le T$. Take $\hat{T}>T$, and define $\hat{U}_j$ by

$$\hat{U}_j(r,t)=\Phi_j\left(\frac{r}{\sqrt{\hat{T}-t}}\right)\quad\text{for } 0\le r\le L, t_0\le t<\hat{T},$$

where $\{\Phi_j\}_{j=1}^{\infty}$ is a sequence of function obtained in Theorem 2.1. By using the properties in Theorem 2.1, we will find that there exists $j_0\in\mathbf{N}$ such that, if $j=j_0$, then

$$\hat{U}_j(r,t_0)>U(r,t_0)\quad\text{for } 0<r\le r_0, \tag{3.10}$$

and

$$\hat{U}_j(r_0,t)>U(r_0,t)\quad\text{for } t_0\le t\le T. \tag{3.11}$$

Put V and $\hat{V}_j$ by

$$V(r,t)=\frac{U(r,t)}{r^2}\quad\text{and}\quad\hat{V}_j(r,t)=\frac{\hat{U}_j(r,t)}{r^2},$$

respectively. Then V and $\hat{V}$ solve (3.6) and satisfy $V_r(0,t)=\hat{V}_r(0,t)=0$ for $t_0\le t<T$. It follows from (3.10) and (3.11) that

$$\hat{V}_j(r,t_0)>V(r,t_0)\quad\text{for } 0\le r\le r_0\quad\text{and}\quad\hat{V}_j(r_0,t)>V(r_0,t)\quad\text{for } t_0\le t<T.$$

Then, by the maximum principle, we obtain $V(r,t)<\hat{V}_j(r,t)$ for $0\le r\le r_0,\ t_0\le t<T$. From (3.5) and (3.9) we see that

$$\lim_{r\to 0}V(r,t)=\lim_{r\to 0}\frac{U(r,t)}{r^2}=\frac{u(0,t)}{N}\quad\text{and}\quad\lim_{r\to 0}\hat{V}_j(r,t)=\lim_{r\to 0}\frac{\hat{U}_j(r,t)}{r^2}=\frac{\phi_j(0)}{N(\hat{T}-t)}.$$

This implies that $u(0,t)<\phi_j(0)/(\hat{T}-t)$ for $t_0\le t<T<\hat{T}$. Note that (1.9) implies $u(0,t)=\|u(\cdot,t)\|_\infty$. Then $\sup_{0\le t<T}\|u(\cdot,t)\|_\infty<\infty$, and hence (u,v) does not blow up at $t=T$.

(ii) For a continuous function ψ defined on an interval J, we define the zero number of the function ψ on J by $\mathcal{Z}_J[\psi] = \#\{r \in J : \psi(r) = 0\}$. We will find that $\mathcal{Z}_{[0,r_0]}[U(\cdot,t) - 2] = 1$ for $t_0 \le t < T$, and we may assume that $T - t_0 > 0$ is small enough so that

$$\frac{r_0}{\sqrt{T - t_0}} > R_0, \tag{3.12}$$

where R_0 is the constant which appears in Theorem 2.1.

Define U_j by (3.8) for $j = 1, 2, \ldots$. First we show that, for each $j = 1, 2, \ldots$,

$$U_j(r_0, t) < U(r_0, t) \quad \text{for } t_0 \le t < T. \tag{3.13}$$

Since $\Phi_j(r) - 2$ has exactly $2j$ zeros on $(0, R_0]$ and no zeros on (R_0, ∞) by Theorem 2.1, we see that that $\Phi_j(r) < 2$ for $r > R_0$. From (3.12) we obtain $U_j(r_0, t) < 2$ for $t_0 \le t < T$. Since $U(r_0, t) > 2$ for $t_0 \le t < T$, we obtain (3.13).

By using the properties in Theorem 2.1, we will find that there exists $j_0 \in \mathbf{N}$ such that, if $j \ge j_0$, then $U_j(r, t_0) - U(r, t_0)$ has exactly one zero for $0 \le r < r_0$. Put V and V_j by $V(r,t) = U(r,t)/r^2$ and $V_j(r,t) = U(r,t)/r^2$, respectively. Then $\mathcal{Z}_{[0,r_0]}[V_j(\cdot, t_0) - V(\cdot, t_0)] = 1$. By the zero number property, we obtain $\mathcal{Z}_{[0,r_0]}[V_j(\cdot,t) - V(\cdot,t)] = 1$ for $t_0 < t < T$. We denote by $\tilde{r}(t)$ a unique zero of $V_j(r,t) - V(r,t)$. Then $\tilde{r}(t) \in (0, r_0)$ for $t_0 \le t < T$, and $V(r,t) < V_j(r,t)$ for $0 \le r < \tilde{r}(t)$. From (3.5) and (3.9) we obtain, for each $t \in [t_0, T)$,

$$\lim_{r \to 0} V(r,t) = \lim_{r \to 0} \frac{U(r,t)}{r^2} = \frac{u(0,t)}{N} \quad \text{and} \quad \lim_{r \to 0} V_j(r,t) = \lim_{r \to 0} \frac{U_j(r,t)}{r^2} = \frac{\phi_j(0)}{N(T-t)}.$$

Then it follows that $(T-t)u(0,t) \le \phi_j(0)$ for $t_0 \le t < T$. This implies that the blow-up is of type I. $\square$

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