RIMS Kôkyûroku Bessatsu
B15 (2009), 101–111

Global solvability for a chemotaxis system in $\mathbb{R}^2$

By

Toshitaka NAGAI*

Abstract

We consider the Cauchy problem of a parabolic-elliptic system in $\mathbb{R}^2$, which is a mathematical model of chemotaxis. Under a mild restriction on the initial data, we discuss the global existence of nonnegative solutions to the Cauchy problem for the sub-critical case, that is, the total mass is less than 8π.

§ 1. Introduction

We consider the Cauchy problem of the following parabolic-elliptic system, which is a mathematical model of chemotaxis in $\mathbb{R}^2$:

$$(1.1) \qquad \begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^2, \\ -\Delta \psi = u, & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases}$$

where ψ is specified as

$$\psi(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, u(t,y)\, dy.$$

Here $u \geq 0$ denotes the density of microorganisms and ψ the concentration of a chemoattractant secreted by themselves (see [11, 19]). The system is also a model of self-interacting particles in $\mathbb{R}^2$, where u is the density of particles in $\mathbb{R}^2$ interacting with themselves through the potential ψ (see [6, 32]).

Received February 27, 2009. Revised August 20, 2009. Accepted August 23, 2009.
2000 Mathematics Subject Classification(s):
Key Words: chemotaxis system, global solvability, sub-critical case
This paper is based on a work [26] in collaboration with Professor Takayoshi Ogawa of Tohoku University.

*Department of Mathematics, Graduate School of Science, Hiroshima University,
Higashi-Hiroshima, 739-8526, JAPAN
e-mail: `nagai@math.sci.hiroshima-u.ac.jp`

The total mass of u to the system (1.1) is conserved and given by $M := \int_{\mathbb{R}^2} u_0\, dy$, and plays an important role in the global solvability of solutions to (1.1). In fact, there is a critical mass above which all solutions blow up in finite time under additional assumptions (see [6, 8, 21]), and below which they exist globally in time under certain assumptions (see [5, 8]). The critical mass is actually given by $M = 8\pi$. In the critical case $M = 8\pi$, in [5] the global existence of radial solutions to (1.1) was shown for initial data with finite or infinite second moment, and in [7] the global existence of non-radial solutions for initial data with finite second moment. Related results for chemotaxis models, see [12, 14, 15, 17, 25, 27, 28, 29, 30], and for models of self-interacting particles, see [2, 3, 4]. Many of these results for chemotaxis models also can be found in [16, 31].

In this paper we discuss the global existence of solutions in the sub-critical case $M < 8\pi$. The global existence of nonnegative weak solutions to (1.1) has been studied by Blanchet-Dolbeault-Perthame [8] for the nonnegative initial data $u_0 \in L^1$ satisfying

$$u_0 \log u_0,\ \ u_0|x|^2 \in L^1. \tag{1.2}$$

Here and in the sequel $L^p := L^p(\mathbb{R}^2)$ is the Lebesgue space on $\mathbb{R}^2$ for $1 \le p \le \infty$ with the usual norm $\|\cdot\|_p := \|\cdot\|_{L^p}$. One of main tools in [8] is the free energy

$$F[u] = \int_{\mathbb{R}^2} u \log u\, dx - \frac{1}{2}\int_{\mathbb{R}^2} u\psi\, dx,$$

where the first term is the entropy and the second one a potential energy, which gives useful a priori estimates for proving the global existence of weak solutions. Their method to get a priori estimates strongly relies on the free energy inequality

$$F[u(t)] + \int_0^t \int_{\mathbb{R}^2} u|\nabla \log u - \nabla\psi|^2\, dxds \le F[u_0],$$

the logarithmic Hardy-Littlewood-Sobolev inequality (see Lemma 2.4 in [8]) and the second moment identity

$$\int_{\mathbb{R}^2} u(t)|x|^2\, dx = \int_{\mathbb{R}^2} u_0|x|^2\, dx + 4M(1 - \frac{M}{8\pi})t. \tag{1.3}$$

To get a priori estimates, they control $\int_{\mathbb{R}^2} u|\log u|\, dx$ by the free energy and the second moment of u. For this reason assumption (1.2) on the initial data u_0 is essentially needed in their proof of global existence.

We note that the free energy $F[u]$ is well-defined for $u \ge 0$ satisfying

$$u,\ \ u\log u,\ \ u\log(1+|x|) \in L^1, \tag{1.4}$$

because $|\psi|$ is estimated by $\log|x|$ for large $|x|$ and $\int_{\mathbb{R}^2} u\psi\, dx$ is well-defined. We also remark that the condition $u\log(1+|x|) \in L^1$ for $u \in L^1_{loc}$ is necessary and sufficient for $\psi \in L^1_{loc}$. Our aim in this paper is to establish the global solvability of nonnegative solutions satisfying (1.4) for $t > 0$, under the following mild restriction on the initial data $u_0 \in L^1$:

$$u_0 \log(1 + |x|) \in L^1. \tag{1.5}$$

Our method to get a priori estimates of nonnegative solutions is quite different from that in [8], because the second moment identity (1.3) is not useful in our situation. We introduce the modified entropy $\int_{\mathbb{R}^2} (1+u)\log(1+u)\,dx$ being nonnegative and the following free energy

$$\int_{\mathbb{R}^2} (1+u)\log(1+u)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} u\psi\,dx$$

in place of $\int_{\mathbb{R}^2} u\log u\,dx$ and the free energy $F[u]$, respectively, and establish a priori estimate of $\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx$ whose proof is required a careful treatment because $\psi(t) \notin L^p$ for any $1 \le p \le \infty$. We remark that the following two properties

(i) $u,\ u\log u,\ u\log(1+|x|) \in L^1$,

(ii) $(1+u)\log(1+u),\ u\log(1+|x|) \in L^1$

are equivalent, and hence (ii) gives the well-definedness of the free energy $F[u]$.

We give some remarks on the following parabolic-elliptic system in which the sign of the nonlinear term $\nabla \cdot (v\nabla\psi)$ of the first equation is opposite from that of $\nabla \cdot (u\nabla\psi)$ in (1.1):

$$\begin{cases} \partial_t v - \Delta v - \nabla \cdot (v\nabla\psi) = 0, & t > 0, x \in \mathbb{R}^2, \\ -\Delta\psi = v & t > 0, x \in \mathbb{R}^2. \end{cases} \tag{1.6}$$

For the nonnegative solutions of (1.6), we have

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2} v^p\,dx + \frac{4(p-1)}{p^2}\int_{\mathbb{R}^2} |\nabla v^{p/2}|^2\,dx + \frac{1}{p}\int_{\mathbb{R}^2} v^{p+1}\,dx = 0, \tag{1.7}$$

where $p > 1$, which implies that $\|v(t)\|_{L^p} \le \|v(0)\|_{L^p}$ and

$$\frac{d}{dt}\int_{\mathbb{R}^2} v^p\,dx + 4\left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^2} |\nabla v^{p/2}|^2\,dx \le 0. \tag{1.8}$$

Once we get (1.8), techniques in the proof of Proposition 5.2 of [18] give

$$\|v(t)\|_p \le Ct^{-1+1/p}, \quad t > 0,$$

where $1 < p \le \infty$ and C is a constant depending only on p, $\|v(0)\|_1$. On the other hand, for the nonnegative solutions of (1.1), we have (1.7) with the negative sign of the last term that does not give (1.8). Hence it is said that the system (1.6) is stable, but (1.1), on the other hand, is not. A Lipschitz semigroup approach to the system (1.6) has been done in [24]. A drift-diffusion system that has (1.6) as a special case has been studied in [22, 20].

In this paper, Section 2 is devoted to our main results (Theorem 2.1) on the uniqueness, global existence and regularity of nonnegative solutions to (1.1). In Section 3 local existence is discussed. In Section 4 we give the outline of the proof of Theorem 2.1.

§ 2. Main results

In order to mention our main result, we define mild solutions of (1.1).

Definition 2.1. For the initial data $u_0 \in L^1$ satisfying (1.5), a function u on $[0,T) \times \mathbb{R}^2$ is said to be a mild solution of (1.1) on $[0,T)$ if

(i) $u \in C([0,T); L^1) \cap C((0,T); L^{4/3}), \quad \sup_{0<t<T} t^{1/4}\|u(t)\|_{4/3} < \infty,$

(ii) $\sup_{0<t<T} \|u(t)\log(1+|x|)\|_1 < \infty,$

(iii) u satisfies the integral equation

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u(s)\nabla\psi(s))\,ds, \quad 0 < t < T, \tag{2.1}$$

where $e^{t\Delta}$ is the heat semigroup defined by

$$(e^{t\Delta}f)(x) = \int_{\mathbb{R}^2} G(t, x-y)f(y)\,dy, \quad G(t,x) = \frac{1}{4\pi t}\exp(-\frac{|x|^2}{4t}).$$

A function u on $[0,\infty) \times \mathbb{R}^2$ is a global mild solution of (1.1) if u is a mild solution of (1.1) on $[0,T)$ for any $0 < T < \infty$.

Remark. The free energy is well-defined for nonnegative mild solutions to (1.1), because $(1+u(t))\log(1+u(t)) \in L^1$ is satisfied for each $t > 0$ by $u(t) \in L^1 \cap L^{4/3}$, and hence (1.4).

In what follows, we use the following notation.

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2, \quad |\alpha| = \alpha_1 + \alpha_2, \quad \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

Our main theorem is mentioned as follows.

Theorem 2.2. *Assume that $u_0 \in L^1$ is nonnegative and satisfies* (1.5). *If the initial data satisfies*

$$\int_{\mathbb{R}^2} u_0\,dx < 8\pi,$$

then there exists uniquely a nonnegative global mild solution u of (1.1) *with initial data u_0. Moreover, u and ψ satisfy the following:*

(i) $\displaystyle\int_{\mathbb{R}^2} u(t)\,dx = \int_{\mathbb{R}^2} u_0\,dx$ *for $t > 0$.*

(ii) *For all $1 < q \le \infty$,*

$$u \in C((0,\infty); L^q), \qquad \lim_{t\to+0} t^{1-1/q}\|u(t)\|_q = 0.$$

(iii) *For all $1 < q < \infty$, $n \ge 0, \alpha \in \mathbb{Z}_+^2$,*

$$\partial_t^n \partial_x^\alpha u \in C((0,\infty); L^q).$$

(iv) *For all $n \ge 0$, and for all $2 < q < \infty$ in case of $|\alpha| = 0$ and for all $1 < q < \infty$ in case of $|\alpha| \ge 1$,*

$$\partial_t^n \partial_x^\alpha (\nabla\psi) \in C((0,\infty); L^q).$$

(v) *u is a classical solution of* (1.1) *on $(0,\infty) \times \mathbb{R}^2$.*

§ 3. Local existence

In order to get the local existence of solutions to (1.1), we first consider the following Cauchy problem:

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u(\nabla N * u)) = 0, & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{3.1}$$

where

$$N(x) = -\frac{1}{2\pi}\log|x|, \quad (\nabla N * u)(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(t,y)\,dy.$$

Definition 3.1. Given $u_0 \in L^1$, a function u on $[0,T) \times \mathbb{R}^2$ is a mild solution of the Cauchy problem (3.1) if u satisfies

$$u \in C([0,T); L^1) \cap C((0,T); L^{4/3}), \quad \sup_{0<t<T} t^{1/4}\|u(t)\|_{4/3} < \infty,$$

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla\cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds, \quad 0 < t < T.$$

We remark that the equation of (3.1) is rather similar to the vorticity equation in $\mathbb{R}^2$

$$\begin{cases} \partial_t \omega = \Delta\omega - \nabla\cdot(\omega(\nabla^\perp N * \omega)), \quad t>0,\ x\in\mathbb{R}^2, \\ \nabla^\perp N(x) = \dfrac{1}{2\pi}\dfrac{(x_2,-x_1)}{|x|^2}, \quad x=(x_1,x_2), \end{cases}$$

where $\nabla^\perp N * \omega$ is the velocity field of the Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$, which satisfies $\nabla\cdot(\nabla^\perp N * \omega) = 0$. For the global existence, uniqueness and regularity of solutions to the Cauchy problem of the the vorticity equation in $\mathbb{R}^2$, for example, see Giga-Miyakawa-Osada [23], Ben-Artzi [1], Kato [18] and Brezis [9]. To obtain the local existence, uniqueness and regularity of solutions to (3.1), we apply the methods used in Kato [18] and Brezis [9] for the vorticity equation in $\mathbb{R}^2$ to our equation.

Proposition 3.1. Given $u_0 \in L^1$, there exists $T>0$ such that the Cauchy problem (3.1) has a unique mild solution u. Moreover u satisfies the following:

(i) For all $1<q\le\infty$,

$$u\in C((0,T);L^q), \quad \lim_{t\to+0} t^{1-1/q}\|u(t)\|_q = 0.$$

(ii) For all $1<q<\infty, n\ge 0, \alpha\in\mathbb{Z}^2_+$,

$$\partial_t^n\partial_x^\alpha u \in C((0,T);L^q).$$

(iii) For all $n\ge 0$, and for all $2<q<\infty$ in case of $|\alpha|=0$ and for all $1<q<\infty$ in case of $|\alpha|\ge 1$,

$$\partial_t^n\partial_x^\alpha(\nabla N * u)\in C((0,T);L^q).$$

(iv) u is a classical solution of $\partial_t u = \Delta u - \nabla\cdot(u(\nabla N * u))$ on $(0,T)\times\mathbb{R}^2$.

For the nonnegative initial data $u_0\in L^1$ satisfying (1.5), we have the following.

Proposition 3.2. Assume that $u_0\in L^1$ is nonnegative on $\mathbb{R}^2$ and satisfy (1.5). Then the solution u of (3.1) with the initial data u_0 mentioned in Proposition 3.1 satisfies the following:

(i) u is nonengative on $(0,T)\times\mathbb{R}^2$.

(ii) $\displaystyle\int_{\mathbb{R}^2} u(t)\,dx = \int_{\mathbb{R}^2} u_0\,dx$ for $0<t<T$.

(iii) $\displaystyle\sup_{0<t<T}\|u(t)\log(1+|x|)\|_1 < \infty.$

By Propositions 3.1 and 3.2, for the nonnegative solution u of (3.1) with the nonnegative initial data $u_0 \in L^1$ satisfying (1.5), we see that $\psi(t) := (N * u)(t)$ is in $L^1_{loc}(\mathbb{R}^2)$ for any $0 < t < T$ because of the assertion (iii) of Proposition 3.2, and that u is a solution of (1.1) because $\nabla\psi = \nabla N * u$ and $-\Delta\psi = u$. Therefore we have the following.

Proposition 3.3. Assume that $u_0 \in L^1$ is nonnegative on $\mathbb{R}^2$ and satisfy (1.5). Then there exists $T > 0$ such that the Cauchy problem (1.1) has a unique nonnegative mild solution u on $[0, T)$. Moreover, this solution satisfies the conservation of mass ((ii) of Proposition 3.2) and regularity ((i)–(iii) of Proposition 3.1), and is a classical solution of the equations of (1.1) on $(0, T) \times \mathbb{R}^2$.

§ 4. Global existence

We show that the solution u obtained in Proposition 3.3 can be continued to the time interval $[0, \infty)$. For simplicity, we may assume $(1 + u_0)\log(1 + u_0) \in L^1$ by considering $t_0 \in (0, T)$ and $u(t_0)$ as the initial time and the initial data, respectively, because $(1 + u(t_0))\log(1 + u(t_0)) \in L^1$ by virtue of $u(t_0) \in L^q$ for any $1 \le q \le \infty$. To show the global existence of solutions, we need a priori estimates for the solution of (1.1). The core of the proof is to establish the following entropy estimate

$$\int_{\mathbb{R}^2} (1 + u(t))\log(1 + u(t))\,dx \le C(u_0, T) \quad (0 < t < T).$$

Here and in the sequel, $C(u_0, T)$ is a positive constant depending on $\int_{\mathbb{R}^2}(1+u_0)\log(1+u_0)\,dx$, $\int_{\mathbb{R}^2} u_0 \log(1 + |x|)\,dx, T$.

Proposition 4.1. Assume $\int_{\mathbb{R}^2} u_0\,dx < 8\pi$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} (1 + u(t))\log(1 + u(t))\,dx \le C(u_0, T) \quad (0 < t < T),\\
&\int_0^T \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\,dxdt \le C(u_0, T),\\
&\int_0^T \int_{\mathbb{R}^2} u^2\,dxdt \le C(u_0, T).
\end{aligned}$$

Once we get the a priori estimates above, we can obtain a priori estimates on

$$\|u(t)\|_p \ \ (1 < p \le \infty), \quad \|\partial_x^\alpha u(t)\|_2 \ \ (|\alpha| \le 2), \quad \|\partial_t u(t)\|_2.$$

By these a priori estimates, we can establish that the solution u can be continued to $[0, \infty)$.

As we mentioned in the introduction, the proof of the a priori estimates above is required a careful treatment on the behavior of the solution ψ. To our aim we split the derivation of the crucial a priori estimates into two parts, namely, exterior regions and interior regions.

The a priori estimates in exterior regions can be obtained by the smallness of the solution u for sufficiently large $|x|$. For the solution u, let

$$H_{ext}(t;R) = \int_{|x|\geq R} \Big((1+u(t))\log(1+u(t)) - u(t)\Big)\,dx.$$

Proposition 4.2. There exists a sufficiently large constant R_0 depending only on T and $\|u_0\|_1$ such that for all $R \geq R_0$,

$$\sup_{0\leq t<T} H_{ext}(t;R) + \int_0^T \int_{|x|\geq R} \frac{|\nabla u|^2}{1+u}\,dxdt \leq 2H_{ext}(0;R) + C(\|u_0\|_1)T,$$

$$\int_0^T \int_{|x|\geq R} u^2\,dxdt \leq C(\|u_0\|_1)H_{ext}(0;R) + C(\|u_0\|_1)T,$$

where $C(\|u_0\|_1)$ is a positive constant depending only on $\|u_0\|_1$.

This proposition can be obtained by showing the following inequality: There is a sufficiently large natural number N such that for all $n \geq N$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} \Big((1+u(t))\log(1+u(t)) - u(t)\Big)\Phi_n\,dx + \frac{1}{2}\int_{\mathbb{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)}\Phi_n\,dx \\ &\leq C\int_{\mathbb{R}^2} u(t)\Phi_n\,dx + 2^{-2n}\{C(\|u_0\|_1) + C(\|\psi(t)\|_{BMO})\},\end{aligned} \tag{4.1}$$

where $\Phi_n \in C_0^\infty(\mathbb{R}^2)$ is such that

$$\Phi_n(x) = \begin{cases} 1 & (2^n \leq |x| \leq 2^{n+1}), \\ \text{less that 1, positive} & (2^{n-1} < |x| < 2^n,\ 2^{n+1} < |x| < 2^{n+2}), \\ 0 & (|x| \leq 2^{n-1},\ |x| \geq 2^{n+2}), \end{cases}$$

$$1 \leq \sum_{n=1}^{\infty} \Phi_n(x) \leq 2 \quad (|x| \geq 2),$$

and $\|\psi(t)\|_{BMO}$ is the BMO-norm of $\psi(t)$. Using the following inequality

$$\|\psi(t)\|_{BMO} \leq C\|u(t)\|_1 = C\|u_0\|_1$$

in (4.1) and summing up the resulting inequalities with respect to n,we establish the proof of the proposition. We remark that the assumption $\int_{\mathbb{R}^2} u_0 dx < 8\pi$ is not used to prove this proposition.

We next give the entropy estimate in the interior region $\{|x| \leq 4R_0\}$ with R_0 in Proposition 4.2.

Proposition 4.3. There exists a constant $C(u_0,T)$ such that

$$\int_{|x|\le 4R_0}(1+u(t))\log(1+u(t))\,dx \le C(u_0,T) \quad (0<t<T).$$

To prove this proposition, we introduce

$$H_{int}(t;R)=\int_{\mathbb{R}^2}\Big((1+u(t))\log(1+u(t))-u(t)\Big)\Psi_R\,dx-\frac{1}{2}\int_{\mathbb{R}^2}u(t)\psi(t)\Psi_R\,dx,$$

where $\Psi_R\in C_0^\infty(\mathbb{R}^2)$ is such that

$$\Psi_R(x)=\begin{cases}1 & (|x|\le R),\\ \text{less that 1, positive} & (R<|x|<2R),\\ 0 & (|x|\ge 2R).\end{cases}$$

Taking $R=4R_0$ and integrating by parts, we deduce the following differential inequality:

$$\frac{d}{dt}H_{int}(t;4R_0)\le\int_{\mathbb{R}^2}(1+u(t))\log(1+u(t))\Psi_{4R_0}\,dx+F(t),$$

where $F(t)$ consists of the integrals of functions $u(t),\nabla u(t),\psi(t),\nabla\psi(t),\partial_t\psi(t)$ on the annulus region $\{4R_0\le|x|\le 8R_0\}$. $F(t)$ is estimated as

$$\left|\int_0^T F(t)\,dt\right|\le C(u_0,T),$$

using estimates on u and ψ in $\{4R_0\le|x|\le 8R_0\}$ that are obtained by Proposition 4.2. Applying Young's inequality to get

$$(1+u)\log(1+u)\le\frac{1}{a}\{(1+u)\log(1+u)-u-\frac{1}{2}u\psi\}+\frac{1}{a}u+\frac{1-a}{a}\exp\big(\frac{|\psi|}{2(1-a)}\big)$$

for $0<a<1$, we have

$$\begin{aligned}\frac{d}{dt}H_{int}(t;4R_0)\le{}&\frac{1}{a}H_{int}(t;4R_0)+\frac{1}{a}\int_{\mathbb{R}^2}u(t)\Psi_{4R_0}\,dx\\&+\frac{1-a}{a}\int_{\mathbb{R}^2}\exp\big(\frac{|\psi(t)|}{2(1-a)}\big)\Psi_{4R_0}\,dx+F(t).\end{aligned}\tag{4.2}$$

To estimate the last second term in the right-hand side of (4.2), we need the following lemma, which is a consequence of the Brezis-Merle inequality ([10]) under zero-Dirichlet boundary conditions.

Lemma 4.1. Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary. For $g\in L^2(\Omega)$, let $w\in H^2(\Omega)$ satisfy $-\Delta w=g$ in Ω. If $\|g\|_{L^1(\Omega)}<4\pi$, then

$$\int_\Omega\exp(|w|)\,dx\le\frac{4\pi^2}{4\pi-\|g\|_{L^1(\Omega)}}\mathrm{diam}(\Omega)^2\exp(\sup_{\partial\Omega}|w|).$$

We apply this lemma as follows. Since $\psi(t)$ satisfies

$$-\Delta\left(\frac{\psi(t)}{2(1-a)}\right) = \frac{u(t)}{2(1-a)} \quad \text{in } \{|x| < 8R_0\},$$

and $\sup_{|x|=8R_0} |\psi(t)| \le C(u_0, T)$ $(0 < t < T)$, taking $0 < a < 1$ such as $a < (8\pi - \|u_0\|_1)/(8\pi)$, that is,

$$\frac{1}{2(1-a)}\|u(t)\|_1 = \frac{1}{2(1-a)}\|u_0\|_1 < 4\pi,$$

by Lemma 4.1, we have

$$\int_{\mathbb{R}^2} \exp\Big(\frac{|\psi(t)|}{2(1-a)}\Big)\Psi_{4R_0}\, dx \le C(u_0, T).$$

Hence, by (4.2) we establish the proof of Proposition 4.3.

References

[1] M. Ben-Artzi, *Global solutions of two-dimensional Navier-Stokes and Euler equations,* Arch. Rational Mech. Anal., **128**(1994), 329–358.

[2] P. Biler, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, III,* Colloq. Math., **68** (1995), 229–239.

[3] P. Biler, D. Hilhorst, T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, II,* Colloq. Math. **67** (1994), 297–308.

[4] P. Biler, M. Cannone, I. A. Guerra, G. Karch, *Global regular and singular solutions for a model of gravitational particles,* Math. Ann., **330** (2004), 693–708.

[5] P. Biler, G. Karch, Ph. Laurençot, T. Nadzieja, *The 8π-problem for radial symmetric solutions of a chemotaxis model in the plane,* Math. Mech. Appl. Sci., **29** (2006), 1563–1583.

[6] P. Biler, T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interactions of particles I,* Colloq. Math., **66** (1994), 319–334.

[7] A. Blanchet, J. A. Carrillo, N. Masmoudi, *Infinite time aggregation for the critical Paltak-Keller-Segel model in $\mathbb{R}^2$,* Comm. Pure Appl. Math., **61** (2008), 1449–1481.

[8] A. Blanchet, J. Dobeault, B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions,* Electron. J. Differential Equations, **2006** (2006) No. 44, 1–33.

[9] H. Brezis, *Remarks on the preceding paper by M. Ben-Artzi: Global solutions of two-dimensional Navier-Stokes and Euler equations,* Arch. Rational Mech. Anal., **128** (1994), 359–360.

[10] H. Brezis, F. Merle, *Uniform estimate and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions,* Comm. Partial Differential Equations, **16** (1991), 1223–1253.

[11] S. Childress, J. K. Percus, *Nonlinear aspects of chemotaxis,* Math. Biosci., **56** (1981), 217–237.

[12] H. Gajewski, K. Zacharias, *Global behavior of a reaction-diffusion system modelling chemotaxis,* Math. Nachr., **195** (1998), 77–114.

[13] Y. Giga, T. Miyakawa, H. Osada, *Two-dimensional Navier-Stokes flow with measure as initial vorticity,* Arch. Rational Mech. Anal., **104** (1988), 223-250.

[14] M. A. Herrero, J. J. L. Velázquez, *Singularity patterns in a chemotaxis model.* Math. Ann., **306** (1996), 583-623.

[15] M. A. Herrero, J.J.L. Velázquez, *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Norm. Sup. Pisa, **24** (1997), 633–683.

[16] D. Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I,* Jahresber. Deutsch. Math.-Verein., **105** (2003), 103–165.

[17] W. Jäger, S. Luckhaus, *On explosions of solutions to a system of partial differential equations modeling chemotaxis,* Trans. Amer. Math. Soc., **329** (1992), 819-824.

[18] T. Kato, *The Navier-Stokes equation for an incompressible fluid in R^2 with a measure as the initial vorticity,* Differential and Integral Equations, **7** (1994), No 4, 949–966.

[19] E.F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability,* J. Theor. Biol., **26** (1970), 399-415.

[20] R. Kobayashi, S. Kawashima, *Decay estimates and large time behavior of solutions to the drift-diffusion system,* Funkcial. Ekvac., **51** (2008), 371–394.

[21] M. Kurokiba, T. Ogawa, *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type* , Diff. Integral Equations, **16** (2003), 427-452.

[22] M. Kurokiba, T. Ogawa, *Well-posedness for the drift-diffusion type in L^p arising from the semiconductor,* J. Mat. Anal. Appl., **342** (2008), 1052-1067.

[23] Y. Giga, T. Miyakawa, H. Osada, *Two-dimensional Navier-Stokes flow with measures as initial vorticity,* Arch. Rational Mech. Anal., **96** (1988), 223–250.

[24] T. Matsumoto, N. Tanaka, *Lipschitz semigroup approach to drift-diffusion systems,* RIMS Kokyuroku Bessatsu, **B15** (2009), 147–177.

[25] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci.Appl., **5** (1995), 581-601.

[26] T. Nagai, T.Ogawa, *Global existence of solutions to a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$*, preprint.

[27] T. Nagai, T. Senba, T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology*, Hiroshima Math. J., **30** (2000), 463-497

[28] T. Nagai, T.Senba, K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., **40** (1997), 411–433.

[29] T. Senba, T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology.* Adv. Differential Equations, **6** (2001), 21-50.

[30] T. Senba, T. Suzuki, *Blow up behavior of solutions to the rescaled Jäger-Luckhaus system,* Adv. Diff. Equations, **8** (2003), 787-820.

[31] T. Suzuki, *Free Energy and Self-Interacting Particles,* Progress in Nonlinear Differential Equations and their Applications, 62, Birkhäuser Boston Inc., Boston, MA, 2005.

[32] G. Wolansky, *Comparison between two models of self-gravitating clusters: conditions for gravitational collapse,* Nonlinear Anal., **24** (1995), 1119–1129.