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Asymptotic Expansion of Solution to the Nernst-Planck Drift-Diffusion Equation

By

Masakazu YAMAMOTO*

Abstract

We discuss the asymptotic profiles of the solution to the initial value problem for the Nernst-Planck type drift-diffusion equation in $\mathbb{R}^3$. It was shown that the time global existence and decay of the solutions to the equation with large initial data. Furthermore the second order asymptotic expansion for the solution was already given. In this paper we show the asymptotic expansion of the solution up to the higher terms as $t \to \infty$.

§ 1. Introduction

We consider the large time behavior of the solution to the following drift-diffusion equation arising in a model of the plasma dynamics:

$$
\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, \ x \in \mathbb{R}^3, \\ -\Delta \psi = -u, & t > 0, \ x \in \mathbb{R}^3, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.1}
$$

Here, the unknown function $u(t, x)$ denotes the density of charges, and $\psi(t, x)$ stands for the electric potential. The drift-diffusion equation describes the model for the dissipative dynamics of carriers in a monopolar semiconductor device.

The drift-diffusion equation was first considered as the initial boundary value problem in a bounded domain (see, for example [2, 5, 8, 20] and references therein). In this case, the global existence of the solution and its asymptotic stability to the corresponding steady state solution was discussed. For the Cauchy problem (1.1), the result for the

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Supported by the Tohoku University Global COE Program — Weaving Science Web beyond Particle-Matter Hierarchy.
*Mathematical Institute, Tohoku University, Sendai 980-8578, Japan.
e-mail: sa5m27@math.tohoku.ac.jp

time local well-posedness was given. Moreover, the time global existence and the decay of the solution have already been proved (for example, we refer to [15, 16, 17, 28]). The first order asymptotic expansion of the solution was considered by Biler-Dolbeault [1] and Kawashima-Kobayashi [12]. In [27], Ogawa and the author found out the second order asymptotic expansion of the solution. In this result, the asymptotic expansion of the solution contains the correction term. There are similar equations appearing in the other context. For example, the incompressible fluid dynamics governed by the Navier-Stokes equations (see [10, 11, 18]), and the Keller-Segel equation in a model of the chemotaxis (see also [7, 13, 14, 21, 22, 23, 29, 30]). The asymptotic profiles of the solution as $t \to \infty$ was observed by Escobedo-Zuazua [4] for the convection diffusion equation. For the Navier-Stokes equation, the asymptotic expansion of the solution was considered by Carpio [3] and Fujigaki-Miyakawa [6]. The asymptotic profiles of the time global solution to the Keller-Segel equation was given by Nagai-Syukuinn-Umesako [24], Nagai-Yamada [25] and Nishihara [26]. Moreover, M.Kato [9] and Yamada [31] showed the higher order asymptotic expansion of the solution to the following parabolic Keller-Segel system:

$$(1.2)\qquad \begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \partial_t \psi - \Delta \psi + \psi = u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 1$. In these results, there exists a logarithmic term appeared in the asymptotic expansion if n is even. On the other hand, when n is odd, the asymptotic expansions for (1.2) contains only algebraic decay rates.

Our aim here is to obtain the third order asymptotic expansion for the solution to (1.1). Especially, by considering an asymptotic expansion of the solution, we shall bring out the contrast between the Keller-Segel equation and our equation. Before stating our result, we introduce the following integral equation:

$$(1.3)\qquad u(t) = e^{t\Delta} u_0 + \int_0^t \nabla e^{(t-s)\Delta} \cdot (u \nabla (-\Delta)^{-1} u)(s) ds,$$

where $\{e^{t\Delta}\}_{t \geq 0}$ is the heat semi-group and the operator $(-\Delta)^{-1}$ is represented as

$$(-\Delta)^{-1} f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \quad \text{for} \quad f \in L^p(\mathbb{R}^3),\ 1 < p < 3/2.$$

The solution of (1.3) is called a mild solution to (1.1). It is known that the mild solution u solves the original Cauchy problem (1.1) if u satisfies $u \in C([0,T); L^p(\mathbb{R}^n)) \cap C\left((0,T); W^{2,p}(\mathbb{R}^n)\right) \cap C^1((0,T); L^p(\mathbb{R}^n))$ for a proper exponent p. Hereafter, we analyze (1.3) to give the asymptotic expansion of the solution. Throughout this paper, we always assume that $u_0 \in L^1_2(\mathbb{R}^3) \cap L^\infty_1(\mathbb{R}^3)$, where $L^p_\mu(\mathbb{R}^3) := \{f \in L^p(\mathbb{R}^3) | |x|^\mu f \in$

$L^p(\mathbb{R}^3)\}$ for $\mu \in \mathbb{R}$. Let $G = G(t,x)$ be the heat kernel, that is, $G(t,x) := (4\pi t)^{-3/2}e^{-|x|^2/(4t)}$ and set the following notations:

$$m_0 := \int_{\mathbb{R}^3} u_0(y)dy, \quad \vec{m} := \int_{\mathbb{R}^3} y u_0(y)dy.$$

Then we define the functions $J_0 = J_0(t,x)$, $V_1 = V_1(t,x)$ and $V_2 = V_2(t,x)$ by

$$\begin{aligned} &J_0(t,x) := \int_0^t \nabla e^{(t-s)\Delta} \cdot (G\nabla(-\Delta)^{-1}G)(s)ds, \\ &V_1(t,x) := m_0 G(t,x), \quad V_2(t,x) := -\vec{m}\cdot\nabla G(t,x) + m_0^2 J_0(t,x). \end{aligned} \tag{1.4}$$

By the definition, it is easy to observe that the following scaling relation holds for V_1, J_0 and V_2 :

$$V_1(t,x) = \lambda^3 V_1\left(\lambda^2 t, \lambda x\right), \;\; J_0(t,x) = \lambda^4 J_0\left(\lambda^2 t, \lambda x\right), \;\; V_2(t,x) = \lambda^4 V_2\left(\lambda^2 t, \lambda x\right) \tag{1.5}$$

for any $\lambda > 0$. In [27], it is shown the second order asymptotic expansion of the solution: Namely, for the mild solution u of (1.1),

$$\|u(t) - V_1(1+t) - V_2(1+t)\|_p = o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}\right) \quad \text{as} \quad t\to\infty \tag{1.6}$$

for $1 \le p \le \infty$. The estimate (1.6) states that the second order asymptotic expansion of the solution has only the algebraic decay rate. In the proof of this argument, we chose the approximation of $u\nabla(-\Delta)^{-1}u$ by $V_1\nabla(-\Delta)^{-1}V_1$ in order to obtain the asymptotic expansion for the nonlinear term of (1.3). In this paper, we find out more detailed approximation of $u\nabla(-\Delta)^{-1}u$ to obtain a higher order asymptotic expansion up to third order for the solution. For this purpose, we introduce the following functions:

$$\tilde{V}_3(t,x) := -\frac{m_0^3}{3}\Delta G(t,x)\int_{\mathbb{R}^3} y\cdot\left(G\nabla(-\Delta)^{-1}J_0 + J_0\nabla(-\Delta)^{-1}G\right)(1,y)dy \tag{1.7}$$

and

$$\begin{aligned} V_3(t,x) := &\sum_{2l+|\beta|=2} \frac{\partial_t^l \nabla^\beta G(t,x)}{\beta!}\int_{\mathbb{R}^3}(-1)^l(-y)^\beta u_0(y)dy \\ &- \int_0^\infty \int_{\mathbb{R}^3} (y\cdot\nabla)\nabla G(t,x)\cdot\big(u\nabla(-\Delta)^{-1}u(s,y) \\ &\qquad - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big)dyds \\ &+ \frac{2m_0^2}{3}\Delta G(t,x)\int_{\mathbb{R}^3} y\cdot(G\nabla(-\Delta)^{-1}G)(1,y)dy \\ &- m_0\int_0^t \nabla e^{(t-s)\Delta}\cdot(\vec{m}\cdot\nabla)\left(G\nabla(-\Delta)^{-1}G\right)(s)ds \\ &- 2\int_{\mathbb{R}^3}(y\cdot\nabla)\nabla G(t,x)\cdot\left((\vec{m}\cdot\nabla)G\,\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\right)(1,y)dy \\ &- \frac{2m_0^4}{3}\Delta G(1+t,x)\int_{\mathbb{R}^3} y\cdot\left(J_0\nabla(-\Delta)^{-1}J_0\right)(1,y)dy \end{aligned} \tag{1.8}$$

$$+ m_0^3 \int_0^t \int_{\mathbb{R}^3} (\nabla G(t-s, x-y) + (y \cdot \nabla) \nabla G(t,x)) \cdot \left(J_0 \nabla (-\Delta)^{-1} G + G \nabla (-\Delta)^{-1} J_0 \right) (s,y) dy ds.$$

We should remark that V_3 contains some extra terms. For instance, in the second term on the right hand side of (1.8), the integrands $y_j V_1 \partial_k (-\Delta)^{-1} V_1$ $(j \neq k)$ are vanishing. For simplicity, we leave those terms. We denote that $\tilde{V}_3$ and V_3 satisfy the following scaling relation:

$$\tilde{V}_3(t,x) = \lambda^5 \tilde{V}_3 \left(\lambda^2 t, \lambda x\right), \quad V_3(t,x) = \lambda^5 V_3 \left(\lambda^2 t, \lambda x\right) \quad \text{for any } \lambda > 0. \tag{1.9}$$

For the solution to (1.1) and the functions which are given by (1.4), (1.7) and (1.8), we obtain the following estimate.

Theorem 1.1. *Assume that* $u_0 \in L^1_2(\mathbb{R}^3) \cap L^\infty_1(\mathbb{R}^3)$ *and* $\|u_0\|_{L^1_1 \cap L^\infty_1}$ *is sufficiently small. Let* u *be the solution to* (1.1), *and* $V_1, V_2, \tilde{V}_3, V_3$ *be defined by* (1.4), (1.7) *and* (1.8). *Then the following estimate holds:*

$$\begin{aligned} &\|u(t) - V_1(1+t) - V_2(1+t) - \log(2+t)\tilde{V}_3(1+t) - V_3(1+t)\|_p \\ =&o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\right) \quad \text{as} \quad t \to \infty \end{aligned} \tag{1.10}$$

for $1 \leq p \leq \infty$. *Moreover,* $\tilde{V}_3(t) \not\equiv 0$ *if* $\int_{\mathbb{R}^3} u_0 dx \neq 0$.

We should notice that when the initial data satisfies the mass zero condition $m_0 = 0$, the auxiliary term $\tilde{V}_3(t)$ vanishes. Theorem 1.1 states that the third order asymptotic expansion of the solution to (1.1) contains the logarithmic terms if $m_0 \neq 0$ since the integrand $y \cdot (G \nabla (-\Delta)^{-1} J_0 + J_0 \nabla (-\Delta)^{-1} G)(1,y)$ does not vanish. In contrast, for the solution to the three-dimensional Keller-Segel equation (1.2) with space-time decay conditions, it has been already proved in [31] that the third order asymptotic expansions never contain the logarithmic terms. This difference of asymptotic behavior between the two systems appears from the structural differences of the nonlinear terms. Namely, in those systems, the equations solved by ψ are different. As a result, we can obtain the estimate $\|\nabla \psi(t)\|_p = O\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}}\right)$ for our equation (for the details, see Proposition 2.3 and Lemma 2.6 in Section 2). On the other hand, it is known that the estimate for the other case, we have the faster decay rate $\|\nabla \psi(t)\|_p = O\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}\right)$.

In this paper, we use the following notation. The convolution of functions f, g over $\mathbb{R}^3$ is denoted by $f * g$. The Fourier transform and the Fourier inverse transform are defined by $\mathcal{F}[f](\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx$, $\mathcal{F}^{-1}[f](x) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix\cdot\xi} f(\xi) d\xi$. For simplicity, we describe the Fourier transform by $\mathcal{F}[f](\xi) = [f]^\wedge(\xi) = \hat{f}(\xi)$. We denote by L^p the Lebesgue space for $1 \leq p \leq \infty$. The norm of $L^p(\mathbb{R}^3)$ is represented as $\|\cdot\|_p$. Let $L^p_\mu(\mathbb{R}^3)$ be the weighted L^p space with $\|f\|_{L^p_\mu} := \|(1+|x|)^\mu f\|_p$. The set of nonnegative integers represented as $\mathbb{Z}_+$. Various constants are simply denoted by C.

§ 2. Preliminaries

Before proving Theorem 1.1, we prepare several lemmas and propositions. In order to obtain an estimate for $\nabla\psi$ in (1.1) and some fractional integrals, we use the following lemma.

Lemma 2.1 (Hardy-Littlewood-Sobolev's inequality). *Let* $1 < p < 3,\ 3/2 < p_* < \infty$ *with* $\frac{1}{p_*} = \frac{1}{p} - \frac{1}{3}$. *Then, there exists a constant* $C > 0$ *such that*

$$\|\nabla(-\Delta)^{-1} f\|_{p_*} \le C\|f\|_p \quad \text{for any} \quad f \in L^p(\mathbb{R}^3).$$

Proof. For the proof of Lemma 2.1, see [32, p. 86]. Hence, we omit the proof. □

Lemma 2.1 suggests that the estimate of $\nabla\psi$ can be obtained by using the estimate of u. Indeed, we see that $\|\nabla\psi(t)\|_{p_*} \le C\|u(t)\|_p$ for $1 < p < 3,\ 3/2 < p* < \infty$ with $\frac{1}{p_*} = \frac{1}{p} - \frac{1}{3}$, since $\nabla\psi = -\nabla(-\Delta)^{-1}u$.

The following lemma is well-known for the estimates of the heat kernel.

Lemma 2.2. *Let* $\alpha, \beta \in \mathbb{Z}_+^3,\ l \in \mathbb{Z}_+$ *and* $1 \le p \le \infty$. *Then the heat kernel* G *satisfies the following estimate:*

$$\left\|x^\alpha \partial_t^l \nabla^\beta G(t)\right\|_p \le C t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{2l+|\beta|-|\alpha|}{2}} \quad \text{for any} \quad t > 0.$$

Moreover, the functions V_1 and V_2 which are defined by (1.4) satisfy the following equalities.

Proposition 2.3. *Let* V_1 *and* V_2 *be defined by* (1.4). *Let* $\alpha \in \mathbb{Z}_+^3$ *with* $|\alpha| \le 1$ *and* $1 \le p \le \infty$. *Then,* $\|x^\alpha V_k(1)\|_p$ *is bounded for* $k = 1, 2$, *and the following equality holds:*

$$\|x^\alpha V_k(t)\|_p = t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k-1}{2}+\frac{|\alpha|}{2}} \|x^\alpha V_k(1)\|_p.$$

Moreover, $\|x^\alpha \nabla(-\Delta)^{-1} V_k(1)\|_p$ *is bounded for* $k = 1, 2$ *when* $\frac{3}{2-|\alpha|} < p < \infty$, *and the following equality is satisfied:*

$$\left\|x^\alpha \nabla(-\Delta)^{-1} V_k(t)\right\|_p = t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{k-2}{2}+\frac{|\alpha|}{2}} \left\|x^\alpha \nabla(-\Delta)^{-1} V_k(1)\right\|_p.$$

Proof. The scaling argument (1.5) immediately gives the desired equalities. □

Next, we give the estimates for the moment of the solution to (1.1).

Proposition 2.4. *Let* $u_0 \in L^1_m(\mathbb{R}^3) \cap L^\infty_m(\mathbb{R}^3)$ *for* $m \in \mathbb{Z}_+$ *and* $\|u_0\|_{L^1_m \cap L^\infty_m}$ *be sufficiently small. Then, the solution* u *to* (1.1) *satisfies*

$$\left\|x^\beta u(t)\right\|_p \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)+\frac{|\beta|}{2}}$$

for any $\beta \in \mathbb{Z}_+^3$ *with* $|\beta| \le m$ *and* $1 \le p \le \infty$.

Proof. The idea of the proof is the same as in Miyakawa [19]. Hence, we omit it. □

Moreover, we also need the following second order asymptotic expansion.

Proposition 2.5. *Under the same assumption as in Theorem* 1.1*, the solution* u *to* (1.1) *satisfies the following estimate for any* $1 \le p \le \infty$*:*

$$\|u(t) - V_1(1+t) - V_2(1+t)\|_p \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-1} \log(2+t). \tag{2.1}$$

In order to prove Proposition 2.5, we prepare the following auxiliary lemma.

Lemma 2.6. *Assume that* $u_0 \ge 0$ *or* $\|u_0\|_{L^1 \cap L^\infty}$ *is sufficiently small. Let* u *be the solution to* (1.1) *and the function* V_1 *be defined by* (1.4)*. Then, for any* $1 \le p \le \infty$*, there is a constant* $C > 0$ *such that*

$$\|u(t) - V_1(1+t)\|_p \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

For $3/2 < p < \infty$*, the following estimate holds:*

$$\left\|\nabla(-\Delta)^{-1}u(t) - \nabla(-\Delta)^{-1}V_1(1+t)\right\|_p \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}.$$

Moreover, $u\nabla(-\Delta)^{-1}u$ *satisfies*

$$\left\|u\nabla(-\Delta)^{-1}u(t) - V_1\nabla(-\Delta)^{-1}V_1(1+t)\right\|_p \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{3}{2}}$$

for $1 \le p < \infty$*.*

Proof. For the proof of Lemma 2.6, see also [12, 27]. We omit the details. □

Proof of Proposition 2.5. First, Lemma 2.6 and Proposition 2.3 immediately give the uniformly boundedness of $\|u(t) - V_1(1+t) - V_2(1+t)\|_p$. Indeed,

$$\begin{aligned}\|u(t) - V_1(1+t) - V_2(1+t)\|_p &\le \|u(t) - V_1(1+t)\|_p + \|V_2(1+t)\|_p \\ &\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}} \le C.\end{aligned}$$

Now we claim that u can be represented as

$$u(t) = V_1(1+t) + V_2(1+t) + I_1(t) + I_2(t) + I_3(t) - I_4(t), \tag{2.2}$$

where

$$\begin{aligned}
I_1(t,x) :=& e^{t\Delta}u_0 - \sum_{|\beta|\le 1} \nabla^\beta G(1+t,x) \int_{\mathbb{R}^3} (-y)^\beta u_0(y)dy, \\
I_2(t,x) :=& \int_0^{t/2} \int_{\mathbb{R}^3} (\nabla G(t-s,x-y) - \nabla G(1+t,x)) \\
&\qquad \cdot (u\nabla(-\Delta)^{-1}u(s,y) - V_1\nabla(-\Delta)^{-1}V_1(1+s,y))dyds, \\
I_3(t,x) :=& \int_{t/2}^t \nabla e^{(t-s)\Delta} \cdot (u\nabla(-\Delta)^{-1}u(s) - V_1\nabla(-\Delta)^{-1}V_1(1+s))ds,
\end{aligned}$$

$$
\begin{aligned}
I_4(t,x) :=& \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot (V_1 \nabla(-\Delta)^{-1} V_1)(s) ds \\
&- \int_0^t \nabla e^{(t-s)\Delta} \cdot (V_1 \nabla(-\Delta)^{-1} V_1)(1+s) ds.
\end{aligned}
$$

Indeed, the nonlinear term on the right hand side of (1.3) is split into

$$
\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta} \cdot (u \nabla(-\Delta)^{-1} u)(s) ds \\
=& \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u \nabla(-\Delta)^{-1} u(s) - V_1 \nabla(-\Delta)^{-1} V_1(1+s)\big)\, ds \\
&+ \int_0^t \nabla e^{(t-s)} \cdot (V_1 \nabla(-\Delta)^{-1} V_1)(1+s) ds.
\end{aligned} \tag{2.3}
$$

Applying the integration by parts, the integrand $u\nabla(-\Delta)^{-1}u - V_1\nabla(-\Delta)^{-1}V_1$ is vanishing. Hence, the right hand side of (2.3) is represented as

$$
\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u \nabla(-\Delta)^{-1} u(s) - V_1 \nabla(-\Delta)^{-1} V_1(1+s)\big)\, ds \\
=& \int_0^{t/2} \int_{\mathbb{R}^3} (\nabla G(t-s, x-y) - \nabla G(1+t, x)) \\
&\qquad \cdot \big(u \nabla(-\Delta)^{-1} u(s,y) - V_1 \nabla(-\Delta)^{-1} V_1(1+s,y)\big)\, dyds \\
&+ \int_{t/2}^t \nabla e^{(t-s)\Delta} \cdot \big(u \nabla(-\Delta)^{-1} u(s,y) - V_1 \nabla(-\Delta)^{-1} V_1(1+s,y)\big)\, ds \\
=& I_2(t) + I_3(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \nabla e^{(t-s)} \cdot (V_1 \nabla(-\Delta)^{-1} V_1)(1+s) ds \\
=& \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot (V_1 \nabla(-\Delta)^{-1} V_1)(s) ds - I_4(t) \\
=& m_0^2 J_0(1+t) - I_4(t).
\end{aligned}
$$

Substituting those equalities into (2.3), we obtain (2.2). In order to conclude the proof, we confirm that $\|I_j(t)\|_p \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}} \log(2+t)$ for $j = 1,2,3,4$. It is well known that

$$
\|I_1(t)\|_p \le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-1}. \tag{2.4}
$$

We show the estimate for I_2. By the mean-valued theorem, we see that

$$\begin{aligned} I_2(t,x) = &- \int_0^{t/2} \int_{\mathbb{R}^3} \int_0^1 (1+s)\partial_t \nabla G(1+t-\sigma(1+s), x-y) \\ &\qquad \cdot (u\nabla(-\Delta)^{-1}u(s,y) - V_1\nabla(-\Delta)^{-1}V_1(1+s,y)) d\sigma dy ds \\ &- \int_0^{t/2} \int_{\mathbb{R}^3} \int_0^1 (y\cdot\nabla)\nabla G(1+t, x-\sigma y) \\ &\qquad \cdot (u\nabla(-\Delta)^{-1}u(s,y) - V_1\nabla(-\Delta)^{-1}V_1(1+s,y)) d\sigma dy ds. \end{aligned}$$

Thus, by the Hausdorff-Young inequality, Lemma 2.2, Propositions 2.3, 2.4 and Lemma 2.6, we obtain that

$$\|I_2(t)\|_p \le Ct^{-\frac{3}{2}(1-\frac{1}{p})-1} \log(2+t). \tag{2.5}$$

For any $1 \le p \le \infty$, let $1 \le q < \infty$ and $1 \le r < 3/2$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$. Then, by the similar argument, we have the estimate for I_3 that

$$\begin{aligned} \|I_3(t)\|_p \le &\int_{t/2}^t \|\nabla G(t-s)\|_r \|u\nabla(-\Delta)^{-1}u(s) - V_1\nabla(-\Delta)^{-1}V_1(1+s)\|_q ds \\ \le &C\int_{t/2}^t (t-s)^{-\frac{3}{2}(1-\frac{1}{r})-\frac{1}{2}} s^{-\frac{3}{2}(1-\frac{1}{q})-\frac{3}{2}} ds \le Ct^{-\frac{3}{2}(1-\frac{1}{p})-1}. \end{aligned} \tag{2.6}$$

Next, we show the estimate for I_4. Since the odd integrand $G\nabla(-\Delta)^{-1}G$ is vanishing, we can split I_4 into the following three parts,

$$\begin{aligned} I_4(t,x) =&m_0^2 \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot (G\nabla(-\Delta)^{-1}G)(s) ds \\ &- m_0^2 \int_0^t \nabla e^{(t-s)\Delta} \cdot (G\nabla(-\Delta)^{-1}G)(1+s) ds \\ =&m_0^2 \Big\{ \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot (G\nabla(-\Delta)^{-1}G)(1+s) ds \\ &\quad - \int_0^t \nabla e^{(t-s)\Delta} \cdot (G\nabla(-\Delta)^{-1}G)(1+s) ds \Big\} \\ &- m_0^2 \int_0^{\frac{1+t}{2}} \int_{\mathbb{R}^3} (\nabla G(1+t-s, x-y) - \nabla G(1+t,x)) \\ &\qquad \cdot \big(G\nabla(-\Delta)^{-1}G)(1+s) - G\nabla(-\Delta)^{-1}G)(s) \big) dy ds \\ &- m_0^2 \int_{\frac{1+t}{2}}^{1+t} \nabla e^{(1+t-s)\Delta} \cdot \big(G\nabla(-\Delta)^{-1}G)(1+s) - G\nabla(-\Delta)^{-1}G)(s) \big) ds. \end{aligned}$$

Moreover, applying the mean-valued theorem, I_4 can be represented as

$$\begin{aligned}
&I_4(t,x)\\
=&m_0^2\int_0^1 \partial_\tau\left(\int_0^\tau \nabla e^{(\tau-s)\Delta}\cdot(G\nabla(-\Delta)^{-1}G)(1+s)ds\right)\Big|_{\tau=t+\sigma}d\sigma\\
&+m_0^2\int_0^{\frac{1+t}{2}}\int_{\mathbb{R}^3}\int_0^1\int_0^1 (s\partial_t\nabla G(1+t-\sigma s,x-y)+(y\cdot\nabla)\nabla G(1+t,x-\sigma y))\\
&\qquad\cdot\partial_s(G\nabla(-\Delta)^{-1}G)(s+\mu,y)d\mu d\sigma dyds\\
&-m_0^2\int_{\frac{1+t}{2}}^{1+t}\int_0^1\nabla e^{(1+t-s)\Delta}\cdot\partial_s(G\nabla(-\Delta)^{-1}G)(s+\mu)d\mu ds.
\end{aligned}\tag{2.7}$$

The first term on the right hand side of (2.7) is split into

$$\begin{aligned}
&\int_0^1 \partial_\tau\left(\int_0^\tau \nabla e^{(\tau-s)\Delta}\cdot(G\nabla(-\Delta)^{-1}G)(1+s)ds\right)\Big|_{\tau=t+\sigma}d\sigma\\
=&\int_0^1\nabla\cdot(G\nabla(-\Delta)^{-1}G)(1+t+\sigma)d\sigma\\
&+\int_0^1\int_0^{\frac{t+\sigma}{2}}\nabla\Delta e^{(t+\sigma-s)\Delta}\cdot(G\nabla(-\Delta)^{-1}G)(1+s)dsd\sigma\\
&+\int_0^1\int_{\frac{t+\sigma}{2}}^{t+\sigma}\nabla e^{(t+\sigma-s)\Delta}\cdot\Delta(G\nabla(-\Delta)^{-1}G)(1+s)dsd\sigma.
\end{aligned}$$

Hence, by Lemma 2.2 and the Hausdorff-Young inequality, we obtain that

$$\begin{aligned}
&\|I_4(t)\|_p\\
\le&m_0^2\int_0^1\left\|\nabla\cdot\left(G\nabla(-\Delta)^{-1}G\right)(1+t+\sigma)\right\|_p d\sigma\\
&+C\int_0^1\int_0^{\frac{t+\sigma}{2}}(t+\sigma-s)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{3}{2}}\left\|G\nabla(-\Delta)^{-1}G(1+s)\right\|_1 dsd\sigma\\
&+C\int_0^1\int_{\frac{t+\sigma}{2}}^{t+\sigma}(t+\sigma-s)^{-1/2}\left\|\Delta(G\nabla(-\Delta)^{-1}G)(1+s)\right\|_p dsd\sigma\\
&+C\int_0^{\frac{1+t}{2}}\int_0^1(1+t-s)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{3}{2}}s\left\|\partial_s(G\nabla(-\Delta)^{-1}G)(s+\mu)\right\|_1 d\mu ds\\
&+C\int_0^{\frac{1+t}{2}}\int_0^1(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\left\|y\partial_s(G\nabla(-\Delta)^{-1}G)(s+\mu)\right\|_1 d\mu ds\\
&+C\int_{\frac{1+t}{2}}^{1+t}\int_0^1(1+t-s)^{-1/2}\left\|\partial_s(G\nabla(-\Delta)^{-1}G)(s+\mu)\right\|_p d\mu ds.
\end{aligned}\tag{2.8}$$

When $p=\infty$, by employing the L^∞-L^1 estimate for the Fourier transform, we obtain the estimates of $\left\|\nabla\cdot\left(G\nabla(-\Delta)^{-1}G\right)(1+t+\sigma)\right\|_\infty$, $\left\|\Delta(G\nabla(-\Delta)^{-1}G)(1+s)\right\|_\infty$ and

$\left\|\partial_s(G\nabla(-\Delta)^{-1}G)(s+\mu)\right\|_\infty$. Namely,

$$\begin{aligned}\left\|\Delta(G\nabla(-\Delta)^{-1}G)(1+s)\right\|_\infty =&\left\|\mathcal{F}\left[|\xi|^2\int_{\mathbb{R}^3}e^{-(1+s)|\xi-\eta|^2}\frac{\eta}{|\eta|^2}e^{-(1+s)|\eta|^2}d\eta\right]\right\|_\infty\\ \leq& C\left\||\xi|^2\int_{\mathbb{R}^3}e^{-(1+s)|\xi-\eta|^2}\frac{\eta}{|\eta|^2}e^{-(1+s)|\eta|^2}d\eta\right\|_1\\ \leq& C\int_{\mathbb{R}^3}\frac{e^{-(1+s)|\eta|^2}}{|\eta|}\int_{\mathbb{R}^3}|\xi|^2e^{-(1+s)|\xi-\eta|^2}d\xi d\eta\\ =&C(1+s)^{-7/2}.\end{aligned}$$

The other two estimates are given by the same calculation. Combining those arguments and Lemma 2.1 with (2.8), we have that

$$\begin{aligned}\|I_4(t)\|_p\leq& Ct^{-\frac{3}{2}(1-\frac{1}{p})-1}+C\int_0^{t/2}\int_0^1(t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{3}{2}}s(s+\mu)^{-2}d\mu ds\\ &+Ct^{-\frac{3}{2}(1-\frac{1}{p})-1}\int_0^{\frac{1+t}{2}}\int_0^1(s+\mu)^{-3/2}d\mu ds\\ &+C\int_{\frac{1+t}{2}}^{1+t}\int_0^1(1+t-s)^{-1/2}(s+\mu)^{-\frac{3}{2}(1-\frac{1}{p})-2}d\mu ds\\ \leq& Ct^{-\frac{3}{2}(1-\frac{1}{p})-1}.\end{aligned}\tag{2.9}$$

Summing up (2.4), (2.5), (2.6) and (2.9), we have the desired estimate. □

Proposition 2.5 gives the following estimates.

Corollary 2.7. *Under the same assumption as in Proposition* 2.5, $\nabla(-\Delta)^{-1}u$ *satisfies the following estimate for* $3/2<p<\infty$*:*

$$\left\|\nabla(-\Delta)^{-1}\left(u(t)-V_1(1+t)-V_2(1+t)\right)\right\|_p\leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}\log(2+t).\tag{2.10}$$

Moreover, the following estimate holds for $1\leq p<\infty$*:*

$$\begin{aligned}&\left\|u\nabla(-\Delta)^{-1}u(t)-(V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+t)\right\|_p\\ \leq& C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-2}\log(2+t).\end{aligned}\tag{2.11}$$

Proof. The estimate (2.10) is given by Lemma 2.1 and Proposition 2.5. Also, Propositions 2.3, 2.4, 2.5 and the estimate (2.10) give the estimate (2.11). □

§ 3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The following proposition is essential for the proof.

Proposition 3.1. *Under the same assumption as in Theorem* 1.1*, there exists the function* $\rho = \rho(t,x)$ *such that*

$$u(t,x) = V_1(1+t,x) + V_2(1+t,x) + \log(2+t)\tilde{V}_3(1+t,x) + V_3(1+t,x) + \rho(t,x)$$

and $\|\rho(t)\|_p = o\left(t^{-\frac{3}{2}(1-\frac{1}{p})-1}\right)$ *as* $t \to \infty$ *for* $1 \le p \le \infty$.

Proof. We split u into the following parts:

$$\begin{aligned}
&u(t,x)\\
=&V_1(1+t,x) + V_2(1+t,x)\\
&+ \sum_{2l+|\beta|=2} \frac{\partial_t^l \nabla^\beta G(1+t,x)}{\beta!}(-1)^l \int_{\mathbb{R}^3} (-y)^\beta u_0(y)dy\\
&- \int_0^\infty \int_{\mathbb{R}^3} (y\cdot\nabla)\nabla G(1+t,x)\\
&\qquad \cdot \big(u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big)\, dyds\\
&+ \int_0^t \nabla e^{(t-s)\Delta} \cdot \big((V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)\big)(1+s)ds - m_0^2 J_0(1+t,x)\\
&+ \rho_1(t,x) + \rho_2(t,x) + \rho_3(t,x) - \rho_4(t,x),
\end{aligned} \tag{3.1}$$

where V_1 and V_2 are defined by (1.4), and

$$\begin{aligned}
\rho_1(t,x) :=&e^{t\Delta}u_0 - \sum_{2l+|\beta|\le 2} \frac{\partial_t^l \nabla^\beta G(1+t,x)}{\beta!}(-1)^l \int_{\mathbb{R}^3} (-y)^\beta u_0(y)dy,\\
\rho_2(t,x) :=& \int_0^{t/2} \int_{\mathbb{R}^3} \Big(\nabla G(t-s,x-y) - \sum_{|\beta|\le 1} \nabla^\beta \nabla G(1+t,x)(-y)^\beta\Big)\\
&\qquad \cdot \big(u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big)\, dyds,\\
\rho_3(t,x) :=& \int_{t/2}^t \nabla e^{(t-s)\Delta} \cdot (u\nabla(-\Delta)^{-1}u(s) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s))ds,\\
\rho_4(t,x) :=& \sum_{|\beta|=1} \nabla^\beta \nabla G(1+t,x) \cdot \int_{t/2}^\infty \int_{\mathbb{R}^3} (-y)^\beta \big(u\nabla(-\Delta)^{-1}u(s,y)\\
&\qquad - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big)dyds.
\end{aligned}$$

Indeed, the nonlinear term on the right hand side of (1.3) is represented as

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta} \cdot (u\nabla(-\Delta)^{-1}u)(s)ds\\
=& \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u\nabla(-\Delta)^{-1}u(s) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s)\big)\, ds\\
&+ \int_0^t \nabla e^{(t-s)\Delta} \cdot (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s)ds.
\end{aligned} \tag{3.2}$$

Applying the integration by parts, the integrand $u\nabla(-\Delta)^{-1}u-(V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)$ is vanishing. Hence, the first term on the right hand side of (3.2) is split into

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta}\cdot\big(u\nabla(-\Delta)^{-1}u(s)-(V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s)\big)\,ds\\
=&\sum_{|\beta|=1}\nabla^\beta\nabla G(1+t)\cdot\int_0^\infty\int_{\mathbb{R}^3}(-y)^\beta\big(u\nabla(-\Delta)^{-1}u(s,y)\\
&\qquad\qquad -(V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big)dyds\\
&+\rho_2(t)+\rho_3(t)-\rho_4(t).
\end{aligned}$$

Combining (1.3) and (3.2) with this equality, we obtain (3.1). Next we consider the following term on the right hand side of (3.1):

$$\begin{aligned}
&\int_0^t \nabla e^{(t-s)\Delta}\cdot\big((V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)\big)(1+s)ds-m_0^2J_0(1+t)\\
=&\left\{m_0^2\int_0^t\nabla e^{(t-s)\Delta}\cdot\big(G\nabla(-\Delta)^{-1}G\big)(1+s)ds-m_0^2J_0(1+t)\right\}\\
&-m_0\int_0^t\nabla e^{(t-s)\Delta}\cdot(\vec{m}\cdot\nabla)\big(G\,\nabla(-\Delta)^{-1}G\big)(1+s)ds\\
&+\int_0^t\nabla e^{(t-s)\Delta}\cdot\big((\vec{m}\cdot\nabla)G\,\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\big)(1+s)ds\\
&+m_0^4\int_0^t\nabla e^{(t-s)\Delta}\cdot(J_0\nabla(-\Delta)^{-1}J_0)(1+s)ds\\
&+m_0^3\int_0^t\nabla e^{(t-s)\Delta}\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1+s)ds-\rho_5(t),
\end{aligned}\tag{3.3}$$

where

$$\rho_5(t,x):=m_0^2\int_0^t\nabla e^{(t-s)\Delta}\cdot\big((\vec{m}\cdot\nabla)G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\big)(1+s)ds.$$

Now, we give the expansion for the right hand side of (3.3). First, we expand the fifth term on the right hand side of (3.3). Since the odd integrands $G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G$ and $y_j(G\partial_k(-\Delta)^{-1}J_0+J_0\partial_k(-\Delta)^{-1}G)$ $(j\neq k)$ are vanishing, the fifth term on the right hand side of (3.3) is written as follows:

$$\begin{aligned}
&m_0^3\int_0^t\nabla e^{(t-s)\Delta}\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1+s)ds\\
=&m_0^3\int_0^t\int_{\mathbb{R}^3}\Big(\nabla G(t-s,x-y)-\sum_{|\beta|\le1}\nabla^\beta\nabla G(t,x)(-y)^\beta\Big)\\
&\qquad\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1+s,y)dyds\\
&-m_0^3\int_0^t\int_{\mathbb{R}^3}(y\cdot\nabla)\nabla G(t,x)\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1+s,y)dyds
\end{aligned}\tag{3.4}$$

$$
\begin{aligned}
=& m_0^3 \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s, x-y) - \sum_{|\beta| \le 1} \nabla^\beta \nabla G(t,x)(-y)^\beta \Big) \\
&\qquad \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(s,y) dyds \\
&- \frac{m_0^3}{3} \Delta G(t,x) \int_0^t \int_{\mathbb{R}^3} y \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(1+s,y) dyds \\
&+ m_0^3 \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s, x-y) - \sum_{|\beta| \le 1} \nabla^\beta \nabla G(t,x)(-y)^\beta \Big) \\
&\qquad \cdot \big((G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G)(1+s,y) \\
&\qquad\qquad - (G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G)(s,y) \big) dyds \\
=& m_0^3 \int_0^{1+t} \int_{\mathbb{R}^3} \left(\nabla G(1+t-s, x-y) + (y \cdot \nabla) \nabla G(1+t,x) \right) \\
&\qquad \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(s,y) dyds \\
&- \frac{m_0^3}{3} \Delta G(1+t,x) \int_0^{1+t} \int_{\mathbb{R}^3} y \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(1+s,y) dyds \\
&+ \rho_6(t),
\end{aligned}
$$

where

$$
\begin{aligned}
&\rho_6(t,x) \\
:=& m_0^3 \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s, x-y) - \sum_{|\beta| \le 1} \nabla^\beta \nabla G(t,x)(-y)^\beta \Big) \\
&\qquad \cdot \big((G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G)(1+s,y) \\
&\qquad\qquad - (G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G)(s,y) \big) dyds \\
&+ m_0^3 \int_0^t \int_{\mathbb{R}^3} \left(\nabla G(t-s, x-y) + (y \cdot \nabla) \nabla G(t,x) \right) \\
&\qquad \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(s,y) dyds \\
&- m_0^3 \int_0^{1+t} \int_{\mathbb{R}^3} \left(\nabla G(1+t-s, x-y) + (y \cdot \nabla) \nabla G(1+t,x) \right) \\
&\qquad \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(s,y) dyds \\
&- \frac{m_0^3}{3} \Delta G(t,x) \int_0^t \int_{\mathbb{R}^3} y \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(1+s,y) dyds \\
&+ \frac{m_0^3}{3} \Delta G(1+t,x) \int_0^{1+t} \int_{\mathbb{R}^3} y \cdot \left(G\nabla(-\Delta)^{-1} J_0 + J_0 \nabla(-\Delta)^{-1} G \right)(1+s,y) dyds.
\end{aligned}
$$

By the scaling argument for G and (1.5), the second term on the right hand side of (3.4) satisfies

$$\begin{aligned}
&\int_0^{1+t}\int_{\mathbb{R}^3} y\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1+s,y)dyds\\
&=\int_0^{1+t}(1+s)^{-5/2}\int_{\mathbb{R}^3}(1+s)^{-1/2}y\\
&\qquad\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1,(1+s)^{-1/2}y)dyds\\
&=\int_0^{1+t}(1+s)^{-1}ds\int_{\mathbb{R}^3}\eta\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1,\eta)d\eta,
\end{aligned}$$

where we put $\eta := (1+s)^{-1/2}y$ in the second equality. Substituting this relation into (3.4), the fifth term on the right hand side of (3.3) is represented as follows:

$$\begin{aligned}
&m_0^3\int_0^t \nabla e^{(t-s)\Delta}\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1+s)ds\\
&=m_0^3\int_0^{1+t}\int_{\mathbb{R}^3}(\nabla G(1+t-s,x-y)+(y\cdot\nabla)\nabla G(1+t,x))\\
&\qquad\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(s,y)dyds\\
&-\frac{m_0^3}{3}\Delta G(1+t,x)\log(2+t)\int_{\mathbb{R}^3}y\cdot\big(G\nabla(-\Delta)^{-1}J_0+J_0\nabla(-\Delta)^{-1}G\big)(1,y)dy\\
&+\rho_6(t,x).
\end{aligned}\tag{3.5}$$

Similarly, the other terms on the right hand side of (3.3) are expanded as

$$\begin{aligned}
&m_0^2\int_0^t\nabla e^{(t-s)\Delta}\cdot\big(G\nabla(-\Delta)^{-1}G\big)(1+s)ds-m_0^2J_0(1+t)\\
&=\frac{2m_0^2}{3}\Delta G(1+t,x)\int_{\mathbb{R}^3}y\cdot(G\nabla(-\Delta)^{-1}G)(1,y)dy+\rho_7(t,x),\\
&m_0\int_0^t\nabla e^{(t-s)\Delta}\cdot(\vec{m}\cdot\nabla)\big(G\,\nabla(-\Delta)^{-1}G\big)(1+s)ds\\
&=m_0\int_0^{1+t}\nabla e^{(1+t-s)\Delta}\cdot(\vec{m}\cdot\nabla)\big(G\,\nabla(-\Delta)^{-1}G\big)(s)ds+\rho_8(t,x),\\
&\int_0^t\nabla e^{(t-s)\Delta}\cdot\big((\vec{m}\cdot\nabla)G\,\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\big)(1+s)ds\\
&=-2\int_{\mathbb{R}^3}(y\cdot\nabla)\nabla G(1+t,x)\cdot\big((\vec{m}\cdot\nabla)G\,\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\big)(1,y)dy+\rho_9(t,x),\\
&m_0^4\int_0^t\nabla e^{(t-s)\Delta}\cdot(J_0\nabla(-\Delta)^{-1}J_0)(1+s)ds\\
&=-\frac{2m_0^4}{3}\Delta G(1+t,x)\int_{\mathbb{R}^3}y\cdot(J_0\nabla(-\Delta)^{-1}J_0)(1,y)dy+\rho_{10}(t,x),
\end{aligned}\tag{3.6}$$

where

$$\rho_7(t,x) := m_0^2 \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s, x-y) - \sum_{|\beta| \le 1} \nabla^\beta \nabla G(1+t, x)(-y)^\beta \Big) \cdot \big((G\nabla(-\Delta)^{-1}G)(1+s, y) - (G\nabla(-\Delta)^{-1}G)(s, y) \big)\, dyds$$
$$- \frac{2m_0^2}{3} \Big((1+t)^{1/2} - t^{1/2} \Big) \Delta G(1+t, x) \int_{\mathbb{R}^3} y \cdot (G\nabla(-\Delta)^{-1}G)(1, y) dy$$
$$- m_0^2 (J_0(1+t) - J_0(t)),$$

$$\rho_8(t,x) := m_0 \int_0^t \nabla e^{(t-s)\Delta} \cdot (\vec{m} \cdot \nabla)(G\nabla(-\Delta)^{-1}G)(1+s) ds$$
$$- m_0 \int_0^{1+t} \nabla e^{(1+t-s)\Delta} \cdot (\vec{m} \cdot \nabla)(G\nabla(-\Delta)^{-1}G)(s) ds,$$

$$\rho_9(t,x) := \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s, x-y) - \sum_{|\beta| \le 1} \nabla^\beta \nabla G(1+t, x)(-y)^\beta \Big) \cdot \big((\vec{m} \cdot \nabla) G\, \nabla(-\Delta)^{-1} (\vec{m} \cdot \nabla) G \big) (1+s, y) dyds$$
$$+ 2(1+t)^{-1/2} \int_{\mathbb{R}^3} (y \cdot \nabla) \nabla G(1+t, x) \cdot \big((\vec{m} \cdot \nabla) G\, \nabla(-\Delta)^{-1} (\vec{m} \cdot \nabla) G \big) (1, y) dy,$$

$$\rho_{10}(t,x) := m_0^4 \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s, x-y) - \sum_{|\beta| \le 1} \nabla^\beta \nabla G(1+t, x)(-y)^\beta \Big) \cdot \big(J_0 \nabla(-\Delta)^{-1} J_0 \big) (1+s, y) dyds$$
$$+ \frac{2m_0^4}{3} (1+t)^{-1/2} \Delta G(1+t, x) \int_{\mathbb{R}^3} y \cdot \big(J_0 \nabla(-\Delta)^{-1} J_0 \big) (1, y) dy.$$

Combining (3.3), (3.5) and (3.6) with (3.1), we obtain

$$u(t) = V_1(1+t) + V_2(1+t) + \log(2+t)\tilde{V}_3(1+t) + V_3(1+t)$$
$$+ \rho_1(t) + \rho_2(t) + \rho_3(t) - \rho_4(t) - \rho_5(t) + \rho_6(t) + \rho_7(t) - \rho_8(t) + \rho_9(t) + \rho_{10}(t).$$

Now, we put $\rho(t) := (\rho_1 + \rho_2 + \rho_3 - \rho_4 - \rho_5 + \rho_6 + \rho_7 - \rho_8 + \rho_9 + \rho_{10})(t)$. In order to conclude the proof, we should confirm that $\{\rho_k(t)\}_{k=1}^{10}$ satisfy the following estimate:

$$\|\rho_k(t)\|_p = o\Big(t^{-\frac{3}{2}(1-\frac{1}{p})-1} \Big) \quad \text{as} \quad t \to \infty \quad \text{for} \quad 1 \le p \le \infty. \tag{3.7}$$

The estimate for $\rho_1(t)$ is well known that

$$\|\rho_1(t)\|_p = o\Big(t^{-\frac{3}{2}(1-\frac{1}{p})-1} \Big) \quad \text{as} \quad t \to \infty. \tag{3.8}$$

In order to show the estimate (3.7) for $\rho_2(t)$, we introduce the auxiliary function $R(t) > 0$ such that $\lim_{t\to\infty} R(t) = \infty$ and $R(t) = o\left(t^{1/2}\right)$ as $t \to \infty$. By the mean-valued theorem, we can represent $\rho_2(t)$ as

$$\rho_2(t,x) = -\rho_{21}(t,x) + \rho_{22}(t,x) + \rho_{23}(t,x), \tag{3.9}$$

where

$$\rho_{21}(t,x) := \int_0^{t/2}\int_{\mathbb{R}^3}\int_0^1 (1+s)\partial_t \nabla G(1+t-\sigma(1+s), x-y) \cdot (u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y))d\sigma dyds,$$

$$\rho_{22}(t,x) := \frac{1}{2}\int_0^{t/2}\int_{|y|\le R(t)}\int_0^1 (y\cdot\nabla)^2\nabla G(1+t, x-\sigma y)(1-\sigma) \cdot (u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y))d\sigma dyds,$$

$$\rho_{23}(t,x) := \int_0^{t/2}\int_{|y|\ge R(t)} \Big(\nabla G(1+t,x-y) - \sum_{|\beta|\le 1}\nabla^\beta\nabla G(1+t,x)(-y)^\beta\Big) \cdot \big(u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big)\, dyds.$$

We employ the Hausdorff-Young inequality, Propositions 2.3, 2.4 and Corollary 2.7 to have that

$$\|\rho_{21}(t)\|_p + \|\rho_{22}(t)\|_p \le Ct^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{3}{2}}\left(\log(2+t) + R(t)\right).$$

Since $R(t) = o\left(t^{1/2}\right)$ as $t\to\infty$, we obtain that

$$\|\rho_{21}(t)\|_p + \|\rho_{22}(t)\|_p = o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\right) \quad \text{as} \quad t\to\infty. \tag{3.10}$$

We check that ρ_{23} satisfies $\|\rho_{23}(t)\|_p = o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\right)$ as $t\to\infty$. By the mean-valued theorem, $\rho_{23}(t)$ can be represented as follows:

$$\rho_{23}(t,x) = -\int_0^{t/2}\int_{|y|\ge R(t)}\left(\int_0^1 (y\cdot\nabla)\nabla G(1+t,x-\sigma y)d\sigma - (y\cdot\nabla)\nabla G(1+t,x)\right) \cdot (u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y))dyds.$$

By Propositions 2.3, 2.4 and Corollary 2.7, we see that

$$\int_0^\infty\int_{\mathbb{R}^3}|y|\left|u\nabla(-\Delta)^{-1}u(s,y) - (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\right| dyds < \infty.$$

Thus, we obtain that

$$\begin{aligned}\|\rho_{23}(t)\|_p \le& Ct^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\int_0^\infty\int_{|y|\ge R(t)}|y|\big|u\nabla(-\Delta)^{-1}u(s,y) \\ &- (V_1+V_2)\nabla(-\Delta)^{-1}(V_1+V_2)(1+s,y)\big|dyds \\ =& Ct^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}o(1) = o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\right) \quad \text{as} \quad t\to\infty,\end{aligned} \tag{3.11}$$

since $R(t)\to\infty$ as $t\to\infty$. We confirm the estimate (3.7) for $\rho_3(t)$ and $\rho_4(t)$. By Propositions 2.3, 2.4 and Corollary 2.7, we obtain the desired estimate

$$\|\rho_3(t)\|_p + \|\rho_4(t)\|_p = o\left(t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-1}\right) \quad \text{as} \quad t\to\infty. \tag{3.12}$$

Applying the mean-valued theorem yields the estimates (3.7) for $\rho_6(t), \dots, \rho_{10}(t)$:

$$\|\rho_6(t)\|_p + \cdots + \|\rho_{10}(t)\|_p = o\left(t^{-\frac{3}{2}(1-\frac{1}{p})-1}\right). \tag{3.13}$$

Finally, we check the estimate (3.7) for $\rho_5(t)$. Now, we remember the definition of $\rho_5(t)$:

$$\rho_5(t) := m_0^2 \int_0^t \nabla e^{(t-s)\Delta} \cdot \left((\vec{m}\cdot\nabla)G\nabla(-\Delta)^{-1}J_0 + J_0\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\right)(1+s)ds.$$

By integrating by parts we see that

$$\int_{\mathbb{R}^3} \left((\vec{m}\cdot\nabla)G\nabla(-\Delta)^{-1}J_0 + J_0\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\right)(1+s,y)dy = 0.$$

In addition, since J_0 is even in y, the integrands

$$y_j\left((\vec{m}\cdot\nabla)G\partial_k(-\Delta)^{-1}J_0 + J_0\partial_k(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\right)(1+s,y), \quad j,k=1,2,3$$

are odd in y. Hence $\rho_5(t)$ is represented as

$$\begin{aligned}\rho_5(t,x) =& m_0^2 \int_0^t \int_{\mathbb{R}^3} \Big(\nabla G(t-s,x-y) - \sum_{|\beta|\le 1} \nabla^\beta \nabla G(t,x)(-y)^\beta\Big) \\ &\cdot \left((\vec{m}\cdot\nabla)G\nabla(-\Delta)^{-1}J_0 + J_0\nabla(-\Delta)^{-1}(\vec{m}\cdot\nabla)G\right)(1+s,y)dyds.\end{aligned}$$

Thus by the mean-valued theorem, we have the estimate (3.7) for $\rho_5(t)$:

$$\|\rho_5(t)\|_p = o\left(t^{-\frac{3}{2}(1-\frac{1}{p})-1}\right). \tag{3.14}$$

As a result, the desired estimate (3.7) follow from (3.8), (3.10)-(3.14). □

To complete the proof of Theorem 1.1, we confirm that the logarithmic term $\tilde{V}_3(t)$ in (1.10) does not vanish if $m_0 := \int_{\mathbb{R}^3} u_0 dy \neq 0$. We remember the definition of $\tilde{V}_3(t)$:

$$\tilde{V}_3(t,x) = -\frac{m_0^3 C_0}{3}\Delta G(t,x), \quad C_0 := \int_{\mathbb{R}^3} y\cdot\left(G\nabla(-\Delta)^{-1}J_0 + J_0\nabla(-\Delta)^{-1}G\right)(1,y)dy,$$

where J_0 is defined by (1.4). We should check that $C_0 \neq 0$. By Perseval's equality, the constant C_0 is represented as

$$\begin{aligned}C_0 =& \int_{\mathbb{R}^3} [yG(1,y)]^\wedge(\xi)\cdot\left[\nabla(-\Delta)^{-1}J_0(1,y)\right]^\wedge(\xi)d\xi \\ &+ \int_{\mathbb{R}^3} [J_0(1,y)]^\wedge(\xi)\left[y\cdot\nabla(-\Delta)^{-1}G(1,y)\right]^\wedge(\xi)d\xi \\ =& \int_{\mathbb{R}^3}\left(4-\frac{1}{|\xi|^2}\right)e^{-|\xi|^2}[J_0(1,y)]^\wedge(\xi)d\xi.\end{aligned} \tag{3.15}$$

Since the odd integrand $\xi e^{-|\xi|^2}$ is vanishing, we have that

$$
\begin{aligned}
&[J_0(1,x)]^\wedge(\xi) \\
&= \int_0^1 [\nabla G(1-s)]^\wedge \cdot \int_{\mathbb{R}^3} [G(s)]^\wedge(\eta-\xi)\left[\nabla(-\Delta)^{-1}G(s)\right]^\wedge(\eta)d\eta ds \\
&= \int_0^1 (i\xi)e^{-(1-s)|\xi|^2} \cdot \int_{\mathbb{R}^3} e^{-s|\eta-\xi|^2}\frac{i\eta}{|\eta|^2}e^{-s|\eta|^2}d\eta ds \\
&= -\int_0^1 \int_{\mathbb{R}^3} \frac{\xi\cdot\eta}{|\eta|^2}e^{-|\xi|^2-2s|\eta|^2+2s\xi\cdot\eta}d\eta ds \\
&= -\int_0^1 \int_{\mathbb{R}^3} \frac{\xi\cdot\eta}{|\eta|^2}e^{-|\xi|^2-2s|\eta|^2}\left(e^{2s\xi\cdot\eta}-1\right)d\eta ds.
\end{aligned}
\tag{3.16}
$$

Substituting (3.16) into (3.15), and applying the mean-valued theorem, we obtain that

$$
\begin{aligned}
C_0 =& -4\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\xi\cdot\eta}{|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2+2s\xi\cdot\eta}d\eta d\xi ds \\
&+\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\xi\cdot\eta}{|\xi|^2|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2}\left(e^{2s\xi\cdot\eta}-1\right)d\eta d\xi ds \\
=& -4\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\xi\cdot\eta}{|\eta|^2}e^{-2\left|\xi-\frac{s}{2}\eta\right|^2-s\left(2-\frac{s}{2}\right)|\eta|^2}d\eta d\xi ds \\
&+2\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s \frac{(\xi\cdot\eta)^2}{|\xi|^2|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2+2\sigma\xi\cdot\eta}d\sigma d\eta d\xi ds.
\end{aligned}
\tag{3.17}
$$

We now calculate the first term on the right hand side of (3.17). We put $\zeta := \xi - \frac{s}{2}\eta$, since the odd integrand is vanishing, we have that

$$
\begin{aligned}
&\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\xi\cdot\eta}{|\eta|^2}e^{-2\left|\xi-\frac{s}{2}\eta\right|^2-s\left(2-\frac{s}{2}\right)|\eta|^2}d\eta d\xi ds \\
&= \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\left(\zeta+\frac{s}{2}\eta\right)\cdot\eta}{|\eta|^2}e^{-2|\zeta|^2-s\left(2-\frac{s}{2}\right)|\eta|^2}d\eta d\zeta ds \\
&= \frac{1}{2}\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} s e^{-2|\zeta|^2-s\left(2-\frac{s}{2}\right)|\eta|^2}d\eta d\zeta ds.
\end{aligned}
\tag{3.18}
$$

On the other hand, we show the upper bound for the second term on the right hand side of (3.17):

$$
\begin{aligned}
&\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s \frac{(\xi\cdot\eta)^2}{|\xi|^2|\eta|^2}e^{-2|\xi|^2-2s|\eta|^2+2\sigma\xi\cdot\eta}d\sigma d\eta d\xi ds \\
&< \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s e^{-2\left|\xi-\frac{\sigma}{2}\eta\right|^2-\left(2s-\frac{\sigma^2}{2}\right)|\eta|^2}d\sigma d\eta d\xi ds \\
&= \int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_0^s e^{-2|\zeta|^2-\left(2s-\frac{\sigma^2}{2}\right)|\eta|^2}d\sigma d\eta d\zeta ds.
\end{aligned}
\tag{3.19}
$$

Substituting (3.18) and (3.19) into (3.17), we see that $C_0 < 0$. Thus, we conclude the proof of Theorem 1.1. □

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