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Structure theorem for holomorphic self-covers of Riemann surfaces and its applications

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Abstract

Non-injective holomorphic self-covers of a hyperbolic Riemann surface are rather mysterious objects. Non-trivial cases appear only when Riemann surfaces are of topologically infinite type, and the theory of such Riemann surfaces is still in its infancy. The structure theorem proved in our previous work gives a powerful tool to clarify the nature of topologically infinite Riemann surfaces and we can investigate non-injective holomorphic self-covers as an agent.

In this paper, we recall the structure theorem with several typical examples, and then explain two applications. First, we give a brief survey of the natural interpretation of the situation from the viewpoint of the Teichmüller theory. Second, we give concentrated discussions about the Denjoy-Wolff phenomena.

§ 0. Preface

This paper is a supplemental version of our previous work [3]. In Sections 1 and 2, we deal with holomorphic self-covers f of Riemann surfaces R and holomorphic self-embeddings f^* of the Teichmüller spaces $T(R)$ induced by f. We focus on the structure theorem of non-injective self-covers and its application to the Teichmüller spaces. Since the detailed arguments and proofs have been given in the original paper [3], we try to supply more examples and commentaries in these sections rather than making the arguments self-contained.

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On the other hand, the contents of Section 3 are new, which provide arguments around the Denjoy-Wolff theorem on Riemann surfaces. Proofs are given for all statements. Although the structure theorem is not applied until we consider the dynamics of holomorphic non-injective self-covers (Theorem 3.8), the arguments on Denjoy-Wolff points and absorbing domains have their own interests and provide the background of our work.

§ 1. Structure theorem

Throughout this paper, we always assume that a Riemann surface R admits a hyperbolic metric and has a non-cyclic fundamental group. Then R is represented as the quotient space Δ/Γ of the unit disk $\Delta \subset \mathbb{C}$ by a non-elementary torsion-free Fuchsian group Γ. Covering is always meant to be unlimited and unbranched unless we specifically mention otherwise.

We start with reviewing the structure theorem for non-injective holomorphic self-covers of a Riemann surface, which has been given in [3]. Similar results have appeared in Jørgensen, Marden and Pommerenke [6], Beardon [1] and McMullen and Sullivan [9].

Theorem 1.1 (Structure theorem). *Let R be a Riemann surface of topologically infinite type, $\pi : \Delta \to R$ a holomorphic universal cover, and $\Gamma \subset \mathrm{Aut}(\Delta)$ the covering transformation group for π, which is a non-elementary torsion-free Fuchsian group. Suppose that there exists a non-injective holomorphic self-cover $f : R \to R$. Then the following claims are satisfied.*

1. *There exists a conformal automorphism $g \in \mathrm{Aut}(\Delta)$ such that*
$$f \circ \pi = \pi \circ g.$$
The conjugate $\Gamma_1 = g^{-1}\Gamma g$ properly contains Γ, which is the covering transformation group for $f \circ \pi$.

2. *Set $\Gamma_n = g^{-n}\Gamma g^n$ for each $n \in \mathbb{N}$. They are the covering transformation groups for $f^n \circ \pi$ and the following proper inclusion relations hold.*
$$\Gamma = \Gamma_0 \subsetneqq \Gamma_1 \subsetneqq \cdots \subsetneqq \Gamma_{n-1} \subsetneqq \Gamma_n \subsetneqq \cdots.$$

3. *Set*
$$\Gamma_\infty = \bigcup_{n=0}^{\infty} \Gamma_n.$$
Then Γ_∞ is discrete and torsion-free. Actually it is the geometric limit of the sequence $\{\Gamma_n\}$.

4. *The conformal automorphism g belongs to the normalizer of the Fuchsian group Γ_∞, that is,*
$$g^{-1}\Gamma_\infty g = \Gamma_\infty.$$
Let $R_\infty = \Delta/\Gamma_\infty$ and g_∞ the conformal automorphism of R_∞ induced by g. Then g_∞ is of infinite order and R_∞ is of topologically infinite type.

5. *Let $f_\infty : R \to R_\infty$ be the holomorphic cover corresponding to the inclusion relation $\Gamma \subset \Gamma_\infty$. Then it satisfies*
$$g_\infty \circ f_\infty = f_\infty \circ f.$$
Summing up, the following diagram commutes:
$$\begin{array}{ccc} \Delta & \xrightarrow{g} & \Delta \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ R & \xrightarrow{f} & R \\ \downarrow{\scriptstyle f_\infty} & & \downarrow{\scriptstyle f_\infty} \\ R_\infty & \xrightarrow{g_\infty} & R_\infty \end{array}$$

6. *Let $\hat{\Gamma} = \langle \Gamma, g \rangle$ be the Fuchsian group generated by Γ and g. Then it is represented as a semi-direct product $\hat{\Gamma} = \Gamma_\infty \rtimes \langle g \rangle$. The quotient $R_\infty/\langle g_\infty \rangle$ of R_∞ by the cyclic group of the conformal automorphism g_∞ is the Riemann surface $\hat{R} = \Delta/\hat{\Gamma}$.*

7. *Suppose that there are a holomorphic cover $\underline{f} : R \to \underline{R}$ and a biholomorphic automorphism $\underline{g} : \underline{R} \to \underline{R}$ of a Riemann surface $\underline{R}$ satisfying $\underline{g} \circ \underline{f} = \underline{f} \circ f$. Then there is a holomorphic cover $\hat{f} : R_\infty \to \underline{R}$ such that $\underline{g} \circ \hat{f} = \hat{f} \circ g_\infty$.*
$$\begin{array}{ccc} R & \xrightarrow{f} & R \\ \downarrow{\scriptstyle f_\infty} & & \downarrow{\scriptstyle f_\infty} \\ R_\infty & \xrightarrow{g_\infty} & R_\infty \\ \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \hat{f}} \\ \underline{R} & \xrightarrow{\underline{g}} & \underline{R} \end{array}$$
In other words, $f_\infty : R \to R_\infty$ is the nearest holomorphic cover from R among such $\underline{f} : R \to \underline{R}$ as above.

We should show that there do exist non-injective holomorphic self-covers of Riemann surfaces. The following theorem answers this, since there are Riemann surfaces of topologically infinite type admitting conformal automorphisms of infinite order.

Theorem 1.2 (Existence of holomorphic self-cover). *For every Riemann surface $\underline{R}$ of topologically infinite type with a conformal automorphism $\underline{g} : \underline{R} \to \underline{R}$ of infinite order, there exist a holomorphic cover $\underline{f} : R \to \underline{R}$ and a non-injective holomorphic self-cover $f : R \to R$ such that $\underline{g} \circ \underline{f} = \underline{f} \circ f$.*

We can characterize R_∞ and $\hat{R}$ in the structure theorem from the dynamical viewpoint.

Definition 1.3. The *grand orbit* of $x \in R$ under f is the set of all points $x' \in R$ such that $f^n(x) = f^m(x')$ for some $n \geq 0$ and $m \geq 0$. The *small orbit* of $x \in R$ under f is the set of all points $x' \in R$ such that $f^n(x) = f^n(x')$ for some $n \geq 0$.

Proposition 1.4. *The quotient space $R/\sim_f$ by the small orbit equivalence relation for f is coincident with the Riemann surface $R_\infty = \Delta/\Gamma_\infty$. The quotient space $R/\approx_f$ by the grand orbit equivalence relation for f is coincident with the Riemann surface $\hat{R} = \Delta/\hat{\Gamma}$.*

We give several examples of non-injective self-covers. First one gives a finite-sheeted self-cover.

Example 1.5. Let f be a rational map of the Riemann sphere having an immediate attracting or a parabolic basin D. Suppose that the grand orbit $\hat{O}$ of the critical points of f is discrete in D or f has a non-critical attracting fixed point in D. We consider a Riemann surface $R = D - \mathrm{cl}(\hat{O})$. The restriction of f to R gives a finite-sheeted non-injective holomorphic self-cover, and $\hat{R} = R/\approx_f$ is an analytically finite Riemann surface.

The following two examples give infinite-sheeted self-covers.

Example 1.6. Suppose that $0 < \lambda < 1/e$ and set $f(z) = \lambda e^z$. Then f has an attracting fixed point z_0. The complement D of the Julia set is the immediate attracting basin of z_0, and the grand orbit $\hat{O}$ of the critical point 0 is discrete in $D - \{z_0\}$. We consider a Riemann surface $R = D - \mathrm{cl}(\hat{O})$. The restriction of f to R gives an infinite-sheeted holomorphic self-cover, and $\hat{R} = R/\approx_f$ is a once-punctured torus.

Example 1.7. A pair of pants is a hyperbolic surface homeomorphic to a three-punctured sphere having three geodesic boundary components. Choose a pair of pants P whose boundary components c_0, c_1 and c_2 have the same length. First, glue two copies of

P along the 2 boundary components c_1 and c_2 of P, which results in a hyperbolic surface P_1 with 5 boundary components. Next, glue four copies of P along the 4 boundary components of P_1 coming from c_1 and c_2, which results in a hyperbolic surface P_2 with 9 boundary components. Continuing this process infinitely many times, we have a hyperbolic surface P_∞ with the boundary component c_0. Let Γ be a Fuchsian group such that $R = \Delta/\Gamma$ is the Nielsen extension of P_∞ beyond c_0. On the other hand, for a connected component R' of $P_\infty - P$, the subgroup Γ' of Γ corresponding to the fundamental group of R' is properly contained in Γ but it is conformally conjugate to Γ. This implies that the Riemann surface R admits a non-injective holomorphic self-cover.

The structure theorem can be generalized to the case of holomorphic branched self-covers.

Theorem 1.8. *Let $f : R \to R$ be a holomorphic branched self-cover of a Riemann surface R of topologically infinite type. Suppose that the grand orbit of the critical points of f is discrete in R. Then there exist a holomorphic branched cover $f_\infty : R \to R_\infty$ with R_∞ of topologically infinite type, and a conformal automorphism $g_\infty : R_\infty \to R_\infty$ of infinite order such that $g_\infty \circ f_\infty = f_\infty \circ f$.*

In the case where the grand orbit of the critical points of f is *not* discrete in R, Theorem 1.8 does not hold. For example, we choose a number c outside of the Mandelbrot set and consider the quadratic polynomial $f(z) = z^2 + c$. Then the complement R of the Julia set in $\mathbb{C}$ is of topologically infinite type, but the grand orbit of the critical points of f is not discrete in R. Recall that $f|_R : R \to R$ is a holomorphic branched self-cover, which is usually reduced to a Böttcher map $z^2 : \Delta^* \to \Delta^*$, where $\Delta^* = \Delta - \{0\}$.

A typical example of holomorphic branched self-covers satisfying the assumptions in Theorem 1.8 can be constructed as follows.

Example 1.9. Consider the cubic polynomial $f(z) = z^3 - 3\epsilon^2 z$ with a sufficiently small $\epsilon > 0$ such that f belongs to the class of Milnor's type A_1. Let D be the immediate attracting basin of the attracting fixed point 0, and $\hat{O}_\pm$ the grand orbit of the critical points $\pm\epsilon$. Then $\hat{O}_\pm$ are discrete in $D - \{0\}$. For $R := D - \mathrm{cl}(\hat{O}_-)$, the branched self-cover $f|_R : R \to R$ satisfies the assumptions of Theorem 1.8, and we have the same R_∞ as in the first case. Recall that $f|_D : D \to D$ is usually reduced to a Schröder map $-3\epsilon^2 z : \mathbb{C} \to \mathbb{C}$.

§ 2. Holomorphic self-embeddings of a Teichmüller space

In this section, we explain an application of the structure theorem to holomorphic self-embeddings of Teichmüller spaces. The *Teichmüller space* $T(R)$ of a Riemann surface $R = \Delta/\Gamma$ is the set of equivalence classes $[f]$ of quasiconformal homeomorphisms

f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are *Teichmüller equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity of R. Here the homotopy is considered to be relative to the ideal boundary at infinity of R.

A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by

$$d_{T(R)}([f_1],[f_2]) = \frac{1}{2}\log K(f),$$

where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then $d_{T(R)}$ is a complete distance on $T(R)$ which is called the Teichmüller distance.

Let Δ^c be the complement of $\overline{\Delta}$ in the Riemann sphere and $B(\Gamma)$ the complex Banach space of all bounded holomorphic quadratic differentials for Γ on Δ^c endowed with the hyperbolic supremum norm. Then the Teichmüller space $T(R)$ is a complex Banach manifold modeled on $B(\Gamma)$. In fact, $T(R)$ is embedded in $B(\Gamma)$ as a bounded contractible domain $T_B(\Gamma)$. More precisely, for a holomorphic universal cover $\pi : \Delta \to R$, we have an injection $\beta_\pi : T(R) \to B(\Gamma)$ whose image is $T_B(\Gamma)$. This is called the *Bers embedding* of $T(R)$. If R is analytically infinite, then $T(R)$ is infinite dimensional, and vice versa.

The Teichmüller distance $d_{T(R)}$ is coincident with the Kobayashi distance on the complex manifold $T(R)$ for every Riemann surface (see [4]). Every biholomorphic automorphism is an isometry with respect to the Kobayashi distance. Also it has the non-expanding property for holomorphic maps. Concerning the Kobayashi distance, one can refer to [7].

Every holomorphic cover $f : R \to R'$ of a Riemann surface R onto another Riemann surface R' induces a holomorphic injection $f^* : T(R') \to T(R)$ between their Teichmüller spaces. Such an f^* is said to be *geometric*. Moreover, a holomorphic cover is non-injective if and only if the induced holomorphic injection between Teichmüller spaces is non-surjective. In particular, a holomorphic self-cover f induces a holomorphic self-embedding $f^* : T(R) \to T(R)$. Hence f^* is non-expanding and if f^* is biholomorphic then it is isometric.

The diagram in the structure theorem yields the following diagram.

$$\begin{array}{ccc}
T(\Delta) & \xleftarrow{\;g^*\;} & T(\Delta) \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\
T(R) & \xleftarrow{\;f^*\;} & T(R) \\
\uparrow{\scriptstyle f_\infty^*} & & \uparrow{\scriptstyle f_\infty^*} \\
T(R_\infty) & \xleftarrow{\;g_\infty^*\;} & T(R_\infty)
\end{array}$$

Recall that in the diagram above, the holomorphic self-embedding f^* preserves the base point and the infinity. Here we say that a holomorphic self-embedding f^* preserves the infinity if bounded sets are preserved.

Remark. In general, we say that a continuous map $f: X \to Y$ between metric spaces *preserves the infinity* if the preimage of every bounded set of Y is bounded. Recall that a continuous map $f: X \to Y$ between metric spaces is *proper* if the preimage of every compact set of Y is compact. Since infinite dimensional Teichmüller spaces are not locally compact, we adopt the condition preserving the infinity instead of properness.

Thus we exclude trivial cases such as a self-map of a bounded domain D that compresses D in a relatively compact open ball contained in D. Nevertheless there are still so many non-surjective holomorphic self-embeddings of infinite-dimensional complex manifolds preserving the infinity. A typical example is the forward shift

$$(z_1, \cdots) \mapsto (0, z_1, \cdots)$$

of $\mathbb{C}^\infty$ equipped with either L^p-norm for $p \geq 1$ or L^∞-norm. Note that this embedding is even isometric. Another example comes from a holomorphic amenable non-injective self-cover $f: R \to R$. In fact, we know that the corresponding geometric self-embedding $f^*: T(R) \to T(R)$ is non-surjective but isometric.

On the other hand, a geometric self-embedding $f^*: T(R) \to T(R)$ induced by a holomorphic self-cover $f: R \to R$ is not necessarily isometric, but is at least a strongly bounded contraction. Here, we say that a geometric self-embedding f^* is a strongly bounded contraction if there exists a uniform constant $c > 0$ such that

$$c\, d_{T(R)}(p, q) \leq d_{T(R)}((f^*)^n(p), (f^*)^n(q)) \leq d_{T(R)}(p, q)$$

for every p and q in $T(R)$ and for every $n \in \mathbb{N}$.

In [3], we have the following theorem.

Theorem 2.1. *For a geometric self-embedding f^*, the full cluster set*

$$C(f^*) \left(= \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} (f^*)^n(T(R))} \right) = \bigcap_{n=1}^{\infty} (f^*)^n(T(R))$$

is identified with $T(R_\infty)$, or more precisely, it is coincident with $f_\infty^*(T(R_\infty))$.

Remark. The recurrent set and the limit set of f^* are coincident, and contained in $C(f^*)$ as a nowhere dense subset. See [2]. In our original paper [3], we have observed that f^* is not uniformly contracting. We gave a quick reasoning for this fact based on a property that $C(f^*)$ is not a singleton, but this was not sufficient. We have to look at the recurrent set of f^*, as we have actually done in the arguments on the distribution of isometric tangent vectors.

An important example of non-isometric holomorphic self-maps is a holomorphic retract, which also admits invariant proper submanifolds. However, in general, a holomorphic retract does not preserve the infinity. Moreover the above fact implies that, in the case where a self-cover f is non-amenable, the induced self-embedding f^* of $T(R)$ has completely different nature from that of holomorphic retracts even on the full cluster sets, which are special invariant proper submanifolds.

§ 3. Denjoy-Wolff phenomena

In this section, we give an application of the structure theorem, which is related to the Denjoy-Wolff theorem. The so-called Denjoy-Wolff theorem is one of fundamental and important results in the geometric function theory of one complex variable. Originally, it was a result on the unit disk, but it has been generalized as the following theorem. See [5] and [9].

Theorem 3.1 (Denjoy-Wolff on Riemann surfaces)**.** *For every holomorphic endomorphism f of a Riemann surface R, the dynamics of f is described by one of the following* (*mutually exclusive*) *possibility.*

1. (Escaping) *For every $p \in R$, the orbit $\{f^n(p)\}$ escapes from compact sets of R as n tends to ∞. Namely, for every compact set K, there is an integer N such that $f^n(p) \notin K$ for every $n \geq N$.*
2. (Attracting) *There exists a fixed point $p_0 \in R$ of f such that*
$$\lim_{n\to\infty} f^n(p) = p_0$$
for every $p \in R$.
3. (Periodic) *The endomorphism f is a periodic automorphism of R.*
4. (Irrational rotation) *The Riemann surface R is biholomorphically equivalent either Δ or $\{z \mid 0 \leq r < |z| < 1\}$, and f corresponds to the restriction of an irrational rotation around* 0.

From the structure theorem and hyperbolic geometry, we can easily see that every non-injective holomorphic self-cover is escaping. This fact is a key to the proof of the Denjoy-Wolff theorem in [9]. Indeed, suppose to the contrary that there were a non-escaping and non-injective self-cover $f : R \to R$. Then there should exist an accumulation point p_∞ of $\{f^n(p_0)\}$ for some p_0, which implies that $\{g_\infty^n(f_\infty(p_0))\}$ accumulate to $f_\infty(p_\infty)$. However, since g_∞ is an isometry of infinite order and $\langle g_\infty \rangle$ acts discontinuously on R, this is a contradiction.

In the attracting case, there are various kinds of results which clarify phenomena relating the Denjoy-Wolff theorem more precisely. However there seems to be not so many in the escaping case. We discuss such results in this section. First, we consider the Denjoy-Wolff points. The point p_0 in the attracting case is called the *Denjoy-Wolff point* for f. Moreover, even in the escaping case, we can associate f with such a "point". For this purpose, we introduce an ideal boundary of R.

Definition 3.2. Let ∂R denote the free boundary $(\partial\Delta - \Lambda(\Gamma))/\Gamma$ of a Riemann surface $R = \Delta/\Gamma$, where Γ is a torsion-free Fuchsian group acting on Δ and $\Lambda(\Gamma)$ denotes the limit set of Γ. If ∂R is empty, then we set $S = R$, and if not, let S be the double $(\widehat{\mathbb{C}} - \Lambda(\Gamma))/\Gamma$ of R. Let S^* be the Kerékjártó-Stoïlow end compactification of S, and $\overline{R}^{S^*}$ the closure of R in S^*. Consequently, if ∂R is empty, then $\overline{R}^{S^*} = R^*$. Set $dR = \overline{R}^{S^*} - R$, and we call dR the *ideal boundary* of R. A point of dR is called an *ideal boundary point* of R.

We have the following theorem. Note that, in [5], a similar result has been proved for the Kerékjártó-Stoïlow boundary (topological ends) $R^* - R$ of an arbitrary Riemann surface R. Also, in [5] and [11], our theorem have been proved in a special case that R is a compact bordered Riemann surface for which $dR = \partial R$. Our ideal boundary dR is something like a hybrid between the Kerékjártó-Stoïlow boundary and the free boundary .

Theorem 3.3 (Ideal Denjoy-Wolff point). *Let $f : R \to R$ be an escaping holomorphic endomorphism of a Riemann surface R. Then there exists a unique point $\xi \in dR$ such that $f^n(x)$ converge to ξ as $n \to \infty$ for every point $x \in R$.*

Proof. Consider the orbit $\{f^n(x)\}$ of a given point $x \in R$. Since $\overline{R}^{S^*}$ is compact and satisfies the second countability axiom, it has an accumulation point. Since f is escaping, all accumulation points must be in the ideal boundary dR. Assume that the orbit $\{f^n(x)\}$ has two distinct accumulation points ξ_1 and ξ_2 in dR.

We first consider the case where there is a simple closed geodesic α in R such that α *separates* ξ_1 and ξ_2 in R, namely, the connected components U_1 and U_2 of $\overline{R}^{S^*} - \alpha$ containing ξ_1 and ξ_2, respectively, are disjoint. Let A be the compact $2L$-neighborhood of α in R with $L = d_R(x, f(x)) > 0$. Since $d_R(f^n(x), f^{n+1}(x)) \leq L$ by the non-expanding property with respect to the hyperbolic distance, the orbit $\{f^n(x)\}$ contains infinitely many points in both of U_1 and U_2 only if so does it in A, which is impossible.

Hence we have only to consider the case where there are no such simple closed geodesic α. Then the two ideal boundary points ξ_1 and ξ_2 should correspond to the same boundary point of the end compactification R^* of R, and this in particular implies that $\partial R \neq \emptyset$. Hence, there exists an end domain Ω of R such that the closure $\overline{\Omega}^{S^*}$ in S^*

contains both ξ_1 and ξ_2. Here, we see that $\overline{\Omega}^{S^*}$ contains an accumulation point of the orbit $\{f^n(x)\}$ belonging to $\partial R \subset S^*$. Indeed, if one of ξ_1 and ξ_2 is in ∂R, then there is nothing to prove. If not, then ξ_1 and ξ_2 belong to $S^* - S$ and there is a simple closed geodesic α' in S such that α' separates ξ_1 and ξ_2 in S. As before, we take the compact $2L$-neighborhood A' of α' in S. Since the identical inclusion $\iota : R \to S$ is non-expanding with respect to the hyperbolic metric, we have an accumulating point $\eta \in A'$ of the orbit $\{f^n(x)\}$. Furthermore, since f is escaping, we have $\eta \in A' \cup dR \subset \partial R$, and hence it is a desired point.

Take an open disk D with center $\eta \in \partial R$ in S. Then $V := D \cap R$ is simply connected. Set $l = D \cap \partial R$. By taking sufficiently small D if necessary, we may assume that at least one of ξ_1 and ξ_2 lies outside of D. Consider the universal cover $\pi : \Delta \to R = \Delta/\Gamma$. We may also assume that $\gamma(V) \cap V = \emptyset$ for every non-trivial $\gamma \in \Gamma$. Then every connected component of $\pi^{-1}(V)$ is biholomorphic to V. We choose a point $x_0 \in V \cap \{f^n(x)\}$ so close to η that $x_1 = f(x_0)$ also lies in V. Fix a connected component $\tilde{V}$ of $\pi^{-1}(V)$, and let z_0 and z_1, respectively, be the unique preimages of x_0 and x_1 in $\tilde{V}$. Then, f can be lifted to such a holomorphic endomorphism $g \in \mathrm{End}(\Delta)$ that $\pi \circ g = f \circ \pi$ and $g(z_0) = z_1$. Also, let $\tilde{l} \subset \partial\Delta$ and $\tilde{\eta} \in \partial\Delta$ be the boundary arc and the boundary point of $\tilde{V}$ corresponding to l and η, respectively.

We take a smooth compact arc c in V connecting x_0 and $x_1 = f(x_0)$, and set

$$C = \bigcup_{n=1}^{\infty} f^n(c).$$

Then $f(C) \subset C$. Though C is not entirely contained in V, the assumptions that f is escaping and that there is another accumulation point outside of D imply that there is a sequence $\{C_m\}_{m=0}^{\infty}$ $(c \subset C_0)$ of connected components of $C \cap V$ converging to a non-degenerate subarc of l in the sense of Hausdorff. Correspondingly, letting $\tilde{C}_m$ be the unique lift of C_m on $\tilde{V}$ for every m, we see that $\tilde{C}_m$ converge to a non-degenerate subarc of $\tilde{l}$ in the sense of Hausdorff. Let $\tilde{C}^{(m)}$ be the connected component of $\pi^{-1}(C)$ that contains $\tilde{C}_m$. Since z_0 and z_1 are in $\tilde{C}_0$ and the lift g is chosen so that $g(z_0) = z_1$, we see that $g(\tilde{C}^{(0)}) \subset \tilde{C}^{(0)}$. Moreover, by continuity of g, we also see that $g(\tilde{C}^{(m)}) \subset \tilde{C}^{(m)}$ for every m.

Set

$$\delta = \max_{y \in c} d_R(y, f(y)).$$

Then by non-expanding property with respect to the hyperbolic metric, we see that

$$\sup_{w \in \tilde{C}_m} d_\Delta(w, g(w)) \leq \delta$$

for every m. Since the Euclidean distance $|w - g(w)|$ tend to 0 uniformly if $d_\Delta(w, g(w))$ are bounded and w tend to $\partial\Delta$, we conclude that $\sup_{w \in \tilde{C}_m} |w - g(w)|$ tend to 0 as m

tend to ∞. Hence, a classical theorem due to Koebe on bounded holomorphic functions (see Theorem X.16 in [12]) implies that g, and hence f, should be the identical map, which contradicts the assumption.

Thus, we have proved that the accumulation point of the orbit $\{f^n(x)\}$ is unique, which we denote by $\xi \in dR$. For the reminder part of the assertions, we have to show that the orbit $\{f^n(x')\}$ also converges to the same ξ even if we change the base point x to a different point $x' \in R$. However, since $d_R(x, x') < \infty$ and f is non-expanding, we can prove this fact by a standard argument possibly repeating a part of the above proof. ☐

Remark. In [11], the existence of the Denjoy-Wolff point for an escaping map to ∂R on a Riemann surface R of parabolic end type has been proved. Theorem 3.3 extends this result without any assumption on R. Denjoy-Wolff phenomena are also discussed in [8], [10] and [11] by using the Martin boundary of a Riemann surface.

The next issue is to find a canonical domain associated to f "near" the ideal Denjoy-Wolff point. In the attracting case, there is a neighborhood U of p_0 (with compact smooth boundary) such that

- the orbit $\{f^n(x)\}$ of every point $x \in R$ has intersection with U,
- $f(U) \subset U$, and
- $\bigcap_{n=1}^{\infty} f^n(U) = \{p_0\}$,

which we call an *absorbing domain* for an attracting f. We classify the attracting case into two sub-cases according as there is an absorbing domain U such that U is *simple*, namely $f|_U : U \to U$ is injective, or not. If no simple absorbing domains exist, then we call f *super-attracting.*

We can consider absorbing domains also in the escaping case. A typical example is an attracting petal in the immediate parabolic basin.

Definition 3.4. For an escaping holomorphic endomorphism f of R, we say that a domain $U \subset R$ is an *absorbing domain* for f if

- the orbit $\{f^n(x)\}$ of every point $x \in R$ has intersection with U,
- $f(U) \subset U$, and
- $\bigcap_{n=1}^{\infty} f^n(U) = \emptyset$.

Furthermore, if f is injective on U, then we call U *simple.*

Remark. Let U be a simple absorbing domain for $f : R \to R$. Then $U - f(U)$ gives a fundamental set for the action of f on R in the sense that the identification of the relative boundaries ∂U and $\partial f(U)$ under f yields the quotient $R/\approx_f$ by the grand orbit relation.

For an escaping f, there always exists an absorbing domain.

Theorem 3.5 (Absorbing domain). *For every escaping holomorphic endomorphism f of a Riemann surface R, there exists an absorbing domain U for f.*

Proof. Recall that the assumption implies that for every pair of compact sets E and F in R, the orbit of E *escapes from* F, namely, there is an N such that $f^n(E) \cap F = \emptyset$ for every $n \geq N$. Fix a point $p_0 \in R$ arbitrarily, and let c be an arc connecting p_0 with $f(p_0)$. Consider the open 1-neighborhood of c, which is denoted by G_0. Take an exhaustion of R by the compact bordered subsurfaces

$$R_k = \{p \in R \mid d(p, p_0) \leq k + \operatorname{diam}(G_0)\}$$

for every $k \geq 1$, and let G_k $(\supset G_0 \supset c)$ be the interior of R_k. Then for every k, we can find an N_k such that

$$\left(\bigcup_{n=N_k}^{\infty} f^n(G_k) \right) \bigcap R_k = \emptyset.$$

Here we may also assume that N_k are non-decreasing with respect to k.

Since $f^n(G_k) \cap f^{n+1}(G_k) \neq \emptyset$ $(\ni f^{n+1}(p_0))$, we see that $U_k := \bigcup_{n=N_k}^{\infty} f^n(G_k)$ is connected. Also $f^n(G_k) \subset f^n(G_{k+1})$ gives that $U := \bigcup_{k=1}^{\infty} U_k$, is a connected open set. Then U is an absorbing domain for f. Indeed, for every $x \in R$, there is some k such that $x \in G_k$ and hence $f^{N_k}(x) \in U_k \subset U$. The condition $f(U) \subset U$ is clearly satisfied, for $f(U_k) \subset U_k$. Finally, for any m, we see

$$f^{N_m}\left(\bigcup_{k=m}^{\infty} U_k \right) \cap R_m \subset \left(\bigcup_{k=m}^{\infty} U_k \right) \cap R_m = \emptyset$$

and

$$f^{N_m}\left(\bigcup_{k=1}^{m-1} U_k \right) \cap R_m \subset \left(\bigcup_{n=N_m}^{\infty} f^n(G_m) \right) \cap R_m = \emptyset.$$

Hence $f^{N_m}(U) \cap R_m = \emptyset$ for every m, from which $\bigcap_{n=1}^{\infty} f^n(U) = \emptyset$ follows. □

Corollary 3.6. *For every escaping holomorphic automorphism f of a Riemann surface R, there exists a simple absorbing domain U for f.*

On the other hand, simple absorbing domains do not necessarily exist if we allow us to consider branched holomorphic self-covers.

Example 3.7. There is a branched holomorphic self-cover with a given finite positive number of branch points which admits no simple absorbing domains.

Indeed, let f be a rational map having a super-attracting fixed point z_0. Let D be the immediate super-attracting basin of z_0, and $\hat{O}(z_0)$ and C the grand orbits of z_0 and all critical points of f other than z_0, respectively. Assume that $\hat{O}(z_0) \cap D \neq \{z_0\}$ and $C \cap \hat{O}(z_0) = \emptyset$. Set $R = D - \hat{O}(z_0)$. Since $\hat{O}(z_0)$ is discrete in D, the map $f : R \to R$ is a branched holomorphic self-cover, which clearly can admit no simple absorbing domains.

A rational map f satisfying all of above assumptions actually exists. An example with a single branch point comes from the cubic polynomial $f(z) = z^3 + z^2$.

Nevertheless, we can show, as an application of the structure theorem, that a simple absorbing domain does exist for every holomorphic non-injective self-cover other than Böttcher self-covers $z^n : \Delta^* \to \Delta^*$ $(n \geq 2)$. The rest of this section is devoted to the proof of the following theorem.

Theorem 3.8 (Simple absorbing). *For a holomorphic non-injective self-cover $f : R \to R$ of a Riemann surface R other than Böttcher self-covers, there exists a simple absorbing domain U.*

Recall that holomorphic non-injective self-covers other than Böttcher self-covers appear only when R is of topologically infinite type. Hence we may assume that R is of topologically infinite type.

To prove Theorem 3.8, we first recall the following lemma which is used in [3] to prove the existence theorem (Theorem 1.2).

Lemma 3.9. *Let $\hat{R} = \Delta/\hat{\Gamma}$, $R_\infty = \Delta/\Gamma_\infty$, and g_∞ the same as in Theorem 1.2. Taking a base point $x_0 \in R_\infty$ arbitrarily, we have a Dirichlet fundamental domain*

$$W = \{x \in R_\infty \mid d_{R_\infty}(x, x_0) < d_{R_\infty}(x, g_\infty^{2k}(x_0)) \text{ for all } k \in \mathbb{Z} - \{0\}\}$$

of $\langle g_\infty^2 \rangle$ in R_∞ centered at x_0. Let $V_0 = W \cap g_\infty^{-1}(W)$ and $V_1 = g_\infty(V_0)$. Then the subgroups H, J_0 and J_1 of Γ_∞ corresponding to the fundamental groups of W, V_0 and V_1, respectively, give the representation of $\hat{\Gamma}$ by the HNN-extension.

We take an exhaustion of $\hat{R}$ by a sequence $\{\hat{R}_k\}_{k \in \mathbb{N}}$ of the interiors of suitable compact bordered subsurfaces. Let $R_k \subset R_\infty$ be the preimage of $\hat{R}_k$ under the projection $R_\infty \to \hat{R}$ and set $W_k = W \cap R_k$. Then $\{W_k\}_{k \in \mathbb{N}}$ is an exhaustion of W. Assume that $x_\infty \in W_1$ and let H_k be a subgroup of $H \subset \Gamma_\infty$ corresponding to the fundamental group of W_k. Obviously, $\{H_k\}$ gives an exhaustion of H.

For every $k \in \mathbb{N}$ and $m \in \mathbb{Z}$, we define

$$\Gamma^{(k,m)} = \langle g^n h g^{-n} \mid h \in H_k,\ n \geq m \rangle,$$

which is a subgroup of Γ_∞ generated by the elements $g^n h g^{-n}$ for all $h \in H_k$ and for all $n \geq m$. Let

$$U^{(k,m)} = \bigcup_{n \geq m} g_\infty^n(W_k),$$

and V a neighborhood of the appropriate path from x_∞. Then $\Gamma^{(k,m)}$ corresponds to the fundamental group of $V \cup U^{(k,m)}$.

Lemma 3.10. *There exists a sequence $\{m_k\}_{k=1}^\infty$ such that $m_k > m_{k-1}$ and $\Gamma^{(k,m_k)} \subset \Gamma$ for every k.*

Proof. For $k = 1$, we can choose $m_1 \in \mathbb{Z}$ such that $\Gamma^{(1,m_1)}$ is contained in Γ. Indeed, since the sequence $\Gamma_n = g^{-n}\Gamma g^n$ gives the exhaustion of Γ_∞ and since H_1 is a finitely generated subgroup of Γ_∞, there exists m_1 such that, for every $n \geq m_1$, a finite system of the generators of H_1 is contained in Γ_n, namely, $H_1 \subset \Gamma_n$. Then $g^n H_1 g^{-n} \subset \Gamma$ for every $n \geq m_1$, which implies that $\Gamma^{(1,m_1)} \subset \Gamma$.

For $k = 2$, similarly we can choose $m_2 > m_1$ so that $\Gamma^{(2,m_2)} \subset \Gamma$. Inductively, for each k, we can choose desired m_k. □

Let Γ^+ be the subgroup of $\Gamma \subset \Gamma_\infty$ generated by the elements in $\Gamma^{(k,m_k)}$ for all k. Set

$$U_\infty = \bigcup_{k=1}^\infty U^{(k,m_k)}.$$

Then U_∞ is a domain and Γ^+ corresponds to the fundamental group of $V \cup U_\infty$.

Finally, let $x \in R$ be a point determined by the conditions that $f_\infty(x) = x_\infty$ and that the inclusion $\Gamma \subset \Gamma_\infty$ corresponds to the natural injection from the fundamental group of R with respect to x into that of R_∞ with respect to x_0. Let U be the connected component of the inverse image $f_\infty^{-1}(U_\infty)$ which contains x.

From the construction, we see that f_∞ is injective on U. Here we modify U by replacing each boundary component of U with a geodesic in the same homotopy class and, if necessary, by pasting an annulus or half-disk to each boundary component facing to the ideal boundary. This modification does not affect the property that f_∞ is injective on U. Thus, the following lemma implies Theorem 3.8.

Lemma 3.11. *The domain U is a simple absorbing for f.*

Proof. We take an arbitrary compact set K in R and consider the projection $K_\infty = f_\infty(K)$ on R_∞, which is also compact. It is clear from the definition of U_∞ that there exists $N \in \mathbb{N}$ such that $g_\infty^n(U_\infty) \cup K_\infty = \emptyset$ for every $n \geq N$. Since $g_\infty \circ f_\infty = f_\infty \circ f$, we see that $\bigcap_{n=1}^\infty f^n(U) = \emptyset$.

From the definition of U_∞, $g_\infty(U_\infty) \subset U_\infty$ and, for every point $y_\infty \in R_\infty$, there exists $n_0 \in \mathbb{N}$ such that $g_\infty^{n_0}(y_\infty) \in U_\infty$. The facts that $U_\infty = f_\infty(U)$ and $g_\infty \circ f_\infty = f_\infty \circ f$ imply that $f_\infty(f(U)) \subset f_\infty(U)$. Since f_∞ maps U onto U_∞ bijectively, we conclude that $f(U) \subset U$. Also $g_\infty \circ f_\infty = f_\infty \circ f$ implies that, for every $y \in R$, there is $n_0 \in \mathbb{N}$ such that $f_\infty(f^{n_0}(y)) \in f_\infty(U)$. This yields $f^{n_0}(y) \in U$ and thus U is an absorbing domain.

Finally, since $f_\infty : R \to R_\infty$ has a factorization including $f : R \to R$, we see that f is injective on U if so is f_∞. This means that U is simple. □

From the above proof, we also have the following fact in a special case where the Riemann surface $\hat{R}$ obtained by the grand orbit relation is topologically finite. Note that, if R comes from an invariant Fatou component for a rational map f, it satisfies this condition. See [9].

Corollary 3.12. *Assume that, for a non-injective holomorphic self-cover $f : R \to R$, the Riemann surface $\hat{R} = R/\approx_f$ is topologically finite. Then a simple absorbing domain U can be taken so that the number of the connected components of the relative boundary ∂U in R is finite.*

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