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Properties of asymptotically elliptic modular transformations of Teichmüller spaces

By

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Abstract

We survey several properties of the action of a Teichmüller modular transformation that has a fixed point on the asymptotic Teichmüller space. Especially, we consider discreteness of the orbit of such a modular transformation on the fiber over the fixed point.

§ 1. Introduction

A Teichmüller modular transformation is called elliptic if it has a fixed point on the Teichmüller space, or equivalently, the corresponding mapping class can be realized as a conformal automorphism. The action of the Teichmüller modular transformations descends to the asymptotic Teichmüller space, and we can also define asymptotic ellipticity as a property of having a fixed point on the asymptotic Teichmüller space. This is equivalent to saying that the corresponding mapping class is realized as an asymptotically conformal automorphism, which is a quasiconformal automorphism arbitrarily close to conformal near the infinity of the surface.

In this note, we survey the action of asymptotically elliptic modular transformations on Teichmüller spaces. We summarize several results obtained in our previous papers [13], [14] and [15], but try to give more general reasoning for some of those theorems. In a future, our study will go to the investigation of the action of such modular transformations on the asymptotic Teichmüller space. A part of this research has been already done in [10]. Our companion paper [7] also reviews these topics.

An asymptotically elliptic modular transformation can be regarded as a generalization of a Teichmüller modular transformation of an analytically finite Riemann surface.

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Roughly speaking, this is because an asymptotically conformal automorphism gives a deformation of the conformal structure essentially on a compact part of the surface. We also consider a subgroup consisting of asymptotically elliptic modular transformations sharing a fixed point. This group satisfies the same properties of the Teichmüller modular group of an analytically finite Riemann surface such as discreteness of orbits and countability of elements. In this note, we develop our arguments on asymptotically elliptic modular transformations of general Riemann surfaces concerning these properties.

§ 2. Classification of Teichmüller modular transformations

The Teichmüller space $T(R)$ for a given base Riemann surface R is the space of all Teichmüller equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that $f_1 : R \to R_1$ and $f_2 : R \to R_2$ are Teichmüller equivalent if there exists a conformal homeomorphism $h : R_1 \to R_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to h relative to the ideal boundary at infinity of R_1. Namely, the homotopy is assumed to be fixing each boundary point throughout when R has the ideal boundary at infinity. We will use the notation o for the base point [id] of $T(R)$. It is known that $T(R)$ is a complex Banach manifold. Also it has a metric structure such that the distance between $p_1 = [f_1]$ and $p_2 = [f_2]$ is given by $d_T(p_1, p_2) = \log K(f)$, where $K(f)$ is the maximal dilatation of an extremal quasiconformal homeomorphism f in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$, which is called the Teichmüller distance. It is known that d_T is coincident with the Kobayashi distance on $T(R)$.

The *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of a Riemann surface R is the group of all mapping classes $[g]$ that have a quasiconformal automorphism $g : R \to R$ as a representative in each homotopy class. Here, the homotopy is again relative to the ideal boundary at infinity of R. It acts on the Teichmüller space $T(R)$ as the group of biholomorphic automorphisms, which is defined as the *Teichmüller modular group* $\mathrm{Mod}(R)$. It also acts isometrically with respect to the Teichmüller distance d_T.

When R is an analytically finite Riemann surface, Bers [1] classified the Teichmüller modular transformations $\gamma \in \mathrm{Mod}(R)$ analytically according to their translation lengths on $T(R)$.

- elliptic: γ has a fixed point on $T(R)$;
- parabolic: $\inf_{p \in T(R)} d_T(\gamma(p), p) = 0$ but γ has no fixed point on $T(R)$;
- hyperbolic: $\inf_{p \in T(R)} d_T(\gamma(p), p) > 0$.

This has a correspondence to a topological classification of the mapping classes due to Thurston. In the case where $g = 1$, we see the identification $T(R) = \mathbb{H}$ which is the upper half-plane with hyperbolic metric, $\mathrm{MCG}(R) = SL_2(\mathbb{Z})$, and $\mathrm{Mod}(R) = PSL_2(\mathbb{Z})$ which is regarded as a subgroup of fractional linear transformations of $\mathbb{H}$. In this case, the above classification is exactly the same as that of the Möbius transformations.

The orbit $\{\gamma^n(p)\}_{n\in\mathbb{N}}$ of $p \in T(R)$ is bounded if γ is elliptic, whereas $\{\gamma^n(p)\}$ diverges to the infinity, that is, $d_T(\gamma^n(p), o) \to \infty$ as $n \to \infty$ if γ is either parabolic or hyperbolic. This defines the following coarser classification, which we call the *bounded-divergent dichotomy.*

- bounded type: the orbit is bounded;
- divergent type: the orbit diverges to the infinity.

However, there are various kinds of Teichmüller modular transformations once R becomes analytically infinite. In fact, there exists a recurrent modular transformation, which is neither bounded nor divergent. See [13]. Namely, the bounded-divergent dichotomy is not always satisfied when R is analytically infinite.

The following result due to Markovic [11] completely characterizes the boundedness of the orbit. Remark that an elliptic modular transformation can be of infinite order when R is analytically infinite.

Theorem 2.1. *A Teichmüller modular transformation is of bounded type if and only if it is elliptic in all cases.*

§ 3. Asymptotically elliptic modular transformations

The *asymptotic Teichmüller space* $AT(R)$ is a quotient space of the Teichmüller space $T(R)$ obtained by identifying all Teichmüller classes that are equivalent under asymptotically conformal homeomorphisms. Here, an *asymptotically conformal homeomorphism* $f : R \to R'$ is a quasiconformal homeomorphism such that $\inf_V K(f|_{R-V}) = 1$, where the infimum of the maximal dilatation K of f restricted to $R - V$ is taken over all compact subsurfaces V of R. Fundamental results on asymptotic Teichmüller spaces can be found in a series of papers by Earle, Gardiner and Lakic [2], [3], [4].

The asymptotic Teichmüller space $AT(R)$ is endowed with a complex structure such that the quotient map $\alpha : T(R) \to AT(R)$ is holomorphic. It also has the asymptotic Teichmüller metric. The distance d_{AT} induced by this metric is coincident with the quotient distance induced from d_T.

The quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $T(R)$ preserving the fibers of the projection α. This means that $\gamma(T_p) = T_{\gamma(p)}$ for any fiber $T_p \subset T(R)$ over $\alpha(p) \in AT(R)$ containing $p \in T(R)$ and for any $\gamma \in \mathrm{MCG}(R)$. From this, the action of every γ descends on $AT(R)$, which is biholomorphic (see [4]) as well as isometric. Hence we have a representation

$$\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R)).$$

Note that this representation is not faithful if R has a non-abelian fundamental group. We assume that $\pi_1(R)$ is not abelian hereafter.

Definition 3.1. A mapping class $\gamma \in \mathrm{MCG}(R)$ or the corresponding Teichmüller modular transformation $\gamma \in \mathrm{Mod}(R)$ is called *asymptotically elliptic* if it has a fixed point on $AT(R)$.

An elliptic modular transformation is of course asymptotically elliptic because the projection of the fixed point is also fixed. However the converse is not true. A trivial example is a Teichmüller modular transformation caused by a single Dehn twist. This is not elliptic as a Teichmüller modular transformation, but it acts trivially on $AT(R)$ because the deformation can be restricted to a compact subset. In particular, it has a fixed point on $AT(R)$. Petrovic [16] dealt with an asymptotically elliptic modular transformation that acts on $AT(R)$ non-trivially (in fact non-periodically) and that has no fixed point on $T(R)$. Here we give another simpler example of this kind.

Example 3.2. Assume that an analytically infinite Riemann surface R has a conformal automorphism h of order 2 that maps an oriented simple closed geodesic c to another $h(c)$ disjoint from c. Let γ be a mapping class obtained by the composition of the conformal mapping class of order 2 and the double Dehn twists along both c and $h(c)$ for their orientations compatible with h. Then γ^2 is the twice of the double Dehn twists, from which we know that γ is not elliptic as a Teichmüller modular transformation. On the other hand, the action of γ on $AT(R)$ fixes the asymptotic conformal structure and is non-trivial (in fact periodic of order 2). The non-triviality can be easily seen if we assume a certain geometric condition on R. See Remark 2 in the next section.

When R is analytically finite, every Teichmüller modular transformation is asymptotically elliptic since $AT(R)$ consists of a single point. Asymptotically elliptic modular transformations are generalization of Teichmüller modular transformations of analytically finite surfaces in a sense that deformations are essentially given only on compact subsurfaces. As a supporting property for this insight, we see the following similarity to the analytically finite case. The proof has been given in [13] in a very similar method to the arguments for Theorem 4.1 given below.

Theorem 3.3. *Every asymptotically elliptic modular transformation is of either bounded type or divergent type.*

Recall that Teichmüller modular transformations for analytically finite Riemann surfaces have the bounded-divergent dichotomy but this is not valid in general. However, this theorem says that asymptotically elliptic modular transformations keep this dichotomy.

§ 4. Action on the fiber

We consider the action of an asymptotically elliptic modular transformation restricted to the fiber over the fixed point on $AT(R)$. The following theorem has been proved in [15]. A proof will be given in the next section as a consequence of more generalized results.

Theorem 4.1. *For every asymptotically elliptic modular transformation γ, its orbit $\{\gamma^n(p)\}_{n\in\mathbb{Z}}$ of any point p over the fixed point on $AT(R)$ is a discrete set in the fiber T_p.*

As a special case, we first show this theorem for an elliptic modular transformation γ. The proof given below has already appeared in [15], but since the uniqueness of the fixed point of γ in the fiber is newly stated, we repeat it here. Remark that even if an elliptic modular transformation keeps a fiber T_p invariant, it does not necessarily mean that there is a fixed point in the fiber. See also [15] for this fact.

Theorem 4.2. *For every elliptic modular transformation γ, its orbit of any point p over the fixed point on $AT(R)$ is a discrete set in the fiber T_p. When γ is of infinite order, the fixed point of γ in T_p is unique if there is.*

Proof. Let R^* be the complex conjugate of R and $B(R^*)$ the Banach space of all bounded holomorphic quadratic differentials φ on R^*. The Bers embedding $\beta : T(R) \to B(R^*)$ identifies the Teichmüller space $T(R)$ with a bounded domain in $B(R^*)$. We denote the quadratic differential $\beta(p)$ corresponding to $p \in T(R)$ by $\varphi_p \in B(R^*)$. It has been proved by Earle, Markovic and Šarić [5] that each fiber T_p over $AT(R)$ is identified with the intersection

$$\beta(T(R)) \cap \{\varphi_p + B_0(R^*)\}$$

of the Bers embedding and an affine subspace determined by φ_p. Here $B_0(R^*)$ is a closed separable subspace of $B(R^*)$ consisting of all those φ vanishing at infinity, meaning that, for the hyperbolic density ρ on R^*, the function $\rho^{-2}(z)|\varphi(z)|$ converges to zero as z tends to the infinity of R^*.

We may assume that the base point $o \in T(R)$ is a fixed point of the elliptic modular transformation γ. Then the mapping class has a conformal representative g of R. In general, a conformal automorphism g of R (and of R^* by reflection) acts on $B(R^*)$ by $(g_*\varphi)(z)dz^2 := \varphi(g^{-1}(z))dg^{-1}(z)^2$, which is a linear isometry of $B(R^*)$. Then the actions of γ and g are related as $\varphi_{\gamma(p)} = g_*\varphi_p$ for every $p \in T(R)$. Also $B_0(R^*)$ is invariant under g_*.

The fiber T_p is embedded in $\varphi_p + B_0(R^*)$. Since γ preserves T_p, we have $g_*\varphi_p = \varphi_p + \psi$ for some $\psi \in B_0(R^*)$. Then, using this formula inductively, we see that

$$g_*^n\varphi_p = \varphi_p + \sum_{i=0}^{n-1} g_*^i\psi$$

for every integer $n \geq 1$.

We have only to consider the case where γ is of infinite order. Suppose that the orbit $\{\gamma^n(p)\}_{n\in\mathbb{Z}}$ is not discrete. Then there exists an increasing sequence of positive integers $\{n_k\}$ such that $d_T(\gamma^{n_k}(p), p) \to 0$, or equivalently $g_*^{n_k}\varphi_p \to \varphi_p$ as $k \to \infty$. This implies that

$$\sum_{i=0}^{n_k-1} g_*^i\psi \to 0 \quad (k \to \infty)$$

and, by operating g_* once more,

$$\sum_{i=1}^{n_k} g_*^i\psi \to 0 \quad (k \to \infty).$$

Hence, by subtracting the first one from the second, we have $g_*^{n_k}\psi \to \psi$.

Take an arbitrary point $z \in R^*$ and consider

$$\rho^{-2}(z)|(g_*^{n_k}\psi)(z)| = \rho^{-2}(g^{-n_k}(z))|\psi(g^{-n_k}(z))|,$$

which converge to $\rho^{-2}(z)|\psi(z)|$. Since $\langle g\rangle$ acts on R^* discontinuously, $g^{-n_k}(z)$ tend to the infinity as $k \to \infty$. Since ψ vanishes at infinity, we see that the above quantities converge to zero as $k \to \infty$. Hence we have $\psi = 0$ and thus $g_*\varphi_p = \varphi_p$. However, this implies that γ fixes p and hence $\{\gamma^n(p)\} = \{p\}$.

For the second statement, suppose that the conformal representative g of γ of infinite order satisfies $g_*\psi' = \psi'$ for some $\psi' \in B_0(R^*)$. Then by using the same equation

$$\rho^{-2}(z)|(g_*^n\psi')(z)| = \rho^{-2}(g^{-n}(z))|\psi'(g^{-n}(z))|$$

as above and the fact that $g^{-n}(z)$ tend to the infinity as $n \to \infty$, we see that $\psi' = 0$.

If there is another fixed point $q \in T_p$, then $g_*\varphi_q = \varphi_q$ is satisfied. Set $\psi' = \varphi_p - \varphi_q$ which belongs to $B_0(R^*)$. Since $g_*\psi' = \psi'$, we see that $\varphi_p = \varphi_q$. This shows the uniqueness of the fixed point. $\square$

Remark 1. If we assume Theorem 2.1 and Theorem 3.3, then Theorem 4.1 follows from Theorem 4.2. Indeed, since the orbit for a Teichmüller modular transformation of divergent type is of course discrete, we have only to deal with elliptic (=bounded) modular transformations. Actually, we have proved Theorem 4.1 in [15] in this way.

Although Theorem 4.2 says that the orbit of an elliptic modular transformation of infinite order is a discrete set in the fiber over any fixed point on $AT(R)$, it always has an indiscrete orbit in $T(R)$. This has been proved in [12] and [13].

Theorem 4.3. *For every elliptic modular transformation of infinite order, there always exists an orbit in $T(R)$ that is not a discrete set.*

The combination of Theorems 4.2 and 4.3 yields the following consequence as in [15]. Since the proof is very short, we can review it here again.

Corollary 4.4. *No elliptic modular transformation of infinite order acts trivially on $AT(R)$.*

Proof. For an elliptic modular transformation of infinite order, choose a point $p \in T(R)$ whose orbit is not a discrete set by Theorem 4.3. If it acts trivially on $AT(R)$, then the fiber T_p is invariant, but this contradicts the fact in Theorem 4.2 that the orbit in T_p is a discrete set. $\square$

Remark 2. The statement of Corollary 4.4 should be also true for an elliptic modular transformation of finite order, but we can prove it so far under an assumption that R satisfies a *bounded geometry condition*, or more precisely, if the injectivity radii of R are uniformly bounded from above and below. This proof has been done in [10] by a geometric observation completely different from the above argument.

§ 5. Discrete orbits of stabilizer subgroups

We investigate the action of a stabilizer subgroup of $\mathrm{MCG}(R)$ fixing a point on $AT(R)$. When R is analytically finite, the whole $\mathrm{MCG}(R)$ stabilizes *the* point of $AT(R)$. In this case, it is well-known that $\mathrm{MCG}(R)$ acts discontinuously on $T(R)$ and $\mathrm{MCG}(R)$ is a finitely generated group. This situation can be generalized as follows, which has been proved in [14].

Theorem 5.1. *If* $\mathrm{MCG}(R)$ *has a common fixed point* $\alpha(p)$, *then* $\mathrm{MCG}(R)$ *is a countable group and acts discontinuously on* $T(R)$.

Let $\mathrm{MCG}_p(R)$ be the stabilizer of $\alpha(p) \in AT(R)$. Theorem 4.1 implies that any orbit of a cyclic subgroup of $\mathrm{MCG}_p(R)$ is a discrete set in T_p, and Theorem 5.1 implies that this is also the case when $\mathrm{MCG}_p(R) = \mathrm{MCG}(R)$. We propose a problem asking in what extent a subgroup Γ of the stabilizer can satisfy this property. For this problem, there are two different factors on the group $\Gamma \subset \mathrm{MCG}(R)$ to be investigated. One is stationary action of Γ on R which is independent of asymptotic ellipticity and the other is algebraic structure of Γ.

Definition 5.2. We say that a sequence of distinct mapping classes $\{\gamma_i\}_{i\in\mathbb{N}}$ in $\mathrm{MCG}(R)$ is *stationary* if there exists a compact subsurface V of R such that any representative g_i of each mapping class γ_i satisfies $g_i(V) \cap V \neq \emptyset$. On the other hand, a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$ in $\mathrm{MCG}(R)$ is *escaping* if, for every compact subsurface V of R, all but finitely many mapping classes γ_i have representatives g_i satisfying $g_i(V) \cap V = \emptyset$.

Remark that a sequence $\{\gamma_i\}$ itself can be neither stationary nor escaping, but each sequence contains a subsequence that is either stationary or escaping. We can also say a subgroup $\Gamma \subset \mathrm{MCG}(R)$ to be stationary or escaping according to this definition. See [9].

Compactness of a family of normalized quasiconformal homeomorphisms with uniformly bounded maximal dilatation easily yields the following fact. In our situation, the normalization is given by the stationary action, which prevents the images of V from escaping to the infinity.

Proposition 5.3. *Assume that a sequence $\{\gamma_i\} \subset \mathrm{MCG}(R)$ is stationary and satisfies $\gamma_i(p) \to p$ for some point $p \in T(R)$ as $i \to \infty$. Then there are representatives g_i of γ_i such that a subsequence of $\{g_i\}$ converges locally uniformly to a conformal automorphism of finite order on the Riemann surface corresponding to p.*

When Γ is an infinite cyclic group, this proposition makes it possible to exclude the case where both Γ is stationary and some orbit of Γ is indiscrete, as the following lemma asserts. The proof is similar to that of Theorem 6 in [13].

Lemma 5.4. *If a sequence $\{\gamma_i\}$ in an infinite cyclic subgroup $\Gamma \subset \mathrm{MCG}(R)$ is stationary, then, for every $p \in T(R)$, $\{\gamma_i(p)\}$ does not accumulate to p.*

Remark 3. This lemma is also true when Γ is a finitely generated abelian group, which will be proved elsewhere. However, it cannot be applied to an infinitely generated subgroup of $\mathrm{MCG}(R)$. Actually, a counterexample is given by an abelian subgroup Γ generated by infinitely many Dehn twists $\{\gamma_i\}$ along mutually disjoint simple closed geodesics $\{c_i\}$ whose lengths tend to zero.

Another feature of the stationary action is that the orbit is always discrete when we impose a certain geometric condition on a Riemann surface R. In particular, the following result has been proved by Fujikawa [6] and [8].

Lemma 5.5. *If a Riemann surface R satisfies the bounded geometry condition, then, for every stationary sequence $\{\gamma_i\} \subset \mathrm{MCG}(R)$ and for every $p \in T(R)$, $\{\gamma_i(p)\}$ does not accumulate to p.*

Remark 4. When R satisfies the bounded geometry condition, Lemma 5.5 also shows that any stabilizer subgroup $\mathrm{MCG}_p(R)$ is always countable, which has been proved in [14]. Indeed, any subgroup of $\mathrm{MCG}(R)$ contains a stationary subgroup of countable index by σ-compactness of R. Hence, if $\mathrm{MCG}_p(R)$ is uncountable, then it contains an uncountable stationary subgroup Γ. On the other hand, the fiber T_p is a separable subspace. Thus it is impossible that $\Gamma(p)$ is both uncountable and discrete.

As a result of the arguments mentioned above, suppose that we are now in a situation that a sequence $\{\gamma_i\} \subset \Gamma$ in question can be assumed to be escaping. Then, for the discreteness problem of the orbit of Γ in the stabilizer $\mathrm{MCG}_p(R)$, we have to consider the second factor, that is, an algebraic structure on Γ. When Γ is abelian in particular, we have the following theorem, which is crucial for the proof of Theorem 4.1. The arguments are similar to those in Theorem 10 of [13].

Theorem 5.6. *Let Γ be an abelian subgroup of $\mathrm{MCG}_p(R)$. Suppose that there is an escaping sequence $\{\gamma_i\}$ in Γ such that $\gamma_i(p) \to p$ as $i \to \infty$. Then $\gamma(p) = p$ for every $\gamma \in \Gamma$.*

Proof. Without loss of generality, we may assume that p is the base point $o \in T(R)$. Represent R by a Fuchsian group H acting on the unit disk Δ and let $\pi : \Delta \to R = \Delta/H$ be the projection. For the sake of argument, we assume that the limit set of H is $\partial\Delta$. Otherwise, we have to make a little modification but it is not essential. Fix geodesic lines β and β' in Δ such that $\beta \cap \beta' \neq \emptyset$ and $\pi(\beta)$ and $\pi(\beta')$ are closed geodesics on R. For a quasiconformal automorphism g of R in an arbitrary mapping class $\gamma \in \Gamma$, choose its lift $\tilde{g} : \Delta \to \Delta$. Let β_γ and β'_γ be the geodesic lines in Δ determined by the end points of $\tilde{g}(\beta)$ and $\tilde{g}(\beta')$ respectively, and consider the cross-ratio $c(\beta_\gamma, \beta'_\gamma) \in (1, \infty)$ defined by the four end points of β_γ and β'_γ. Then we introduce a real value

$$\xi(\gamma) = \int_2^{c(\beta_\gamma, \beta'_\gamma)} \rho_{\mathbb{C}-\{0,1\}}(x)dx,$$

where $\rho_{\mathbb{C}-\{0,1\}}(z)|dz|$ is the hyperbolic metric on $\mathbb{C} - \{0, 1\}$. This is a signed hyperbolic distance of $c(\beta_\gamma, \beta'_\gamma)$ from 2.

First, we consider the difference of the values

$$\xi(\gamma_i\gamma) - \xi(\gamma) = \int_{c(\beta_\gamma,\beta'_\gamma)}^{c(\tilde{g}_i(\beta_\gamma),\tilde{g}_i(\beta'_\gamma))} \rho_{\mathbb{C}-\{0,1\}}(x)dx$$

for every $\gamma \in \Gamma$. An K-quasiconformal automorphism of Δ changes the cross-ratio by at most $\log K$ with respect to the hyperbolic distance on $\mathbb{C} - \{0,1\}$. Hence

$$|\xi(\gamma_i\gamma) - \xi(\gamma)| \leq d_T(\gamma_i(o), o)$$

for every $\gamma \in \Gamma$. Thus $d_T(\gamma_i(o), o) \to 0$ implies $\xi(\gamma_i\gamma) - \xi(\gamma) \to 0$ as $i \to \infty$.

On the other hand, for every $\gamma \in \Gamma$,

$$\xi(\gamma\gamma_i) - \xi(\gamma_i) = \int_{c(\beta_{\gamma_i},\beta'_{\gamma_i})}^{c(\tilde{g}(\beta_{\gamma_i}),\tilde{g}(\beta'_{\gamma_i}))} \rho_{\mathbb{C}-\{0,1\}}(x)dx$$

tends to 0 as $i \to \infty$. Indeed, this follows from the facts that the mapping class γ has an asymptotically conformal automorphism g of R as a representative and that $\pi(\beta_{\gamma_i} \cup \beta'_{\gamma_i})$ diverge to the infinity of R as $i \to \infty$. Note that the cross-ratios $\{c(\beta_{\gamma_i}, \beta'_{\gamma_i})\}$ are uniformly bounded from above and away from one because β_{γ_i} and β'_{γ_i} are the images of β and β' under quasiconformal automorphisms of bounded dilatations. More detailed arguments can be found in Lemma 8 of [13].

Since $\xi(\gamma_i\gamma) = \xi(\gamma\gamma_i)$, the above two limits conclude that $\lim_{i\to\infty} \xi(\gamma_i) = \xi(\gamma)$ for every $\gamma \in \Gamma$. This in particular implies that ξ is a constant function on Γ, or equivalently, every $\tilde{g}$ does not change the cross-ratio $c(\beta, \beta')$.

Next, take arbitrary four distinct points a_1, a'_1, a_2, a'_2 on $\partial\Delta$. Then there exists a sequence of geodesic lines β in Δ whose projections $\pi(\beta)$ are closed geodesics in R and whose end points converge to a_1 (a'_1) and a_2 (a'_2) respectively. Hence the cross-ratio $c(a_1, a_2, a'_1, a'_2)$ is approximated by a sequence $\{c(\beta, \beta')\}$ for which our estimate can be applied. Since $\tilde{g}$ does not change $c(\beta, \beta')$, continuity of the cross-ratio shows that $\tilde{g}$ does not change $c(a_1, a_2, a'_1, a'_2)$ either. This is true for any four distinct points on $\partial\Delta$. This implies that $\gamma(o) = o$ for every $\gamma \in \Gamma$. See Sorvali [17] for this argument. ☐

Checking two factors we have discussed above, we can obtain the following result for instance, as a combination of Lemma 5.5 and Theorem 5.6. Note that, if the orbit $\Gamma(p)$ is not a discrete set, then we can always find a sequence $\{\gamma_i\} \subset \Gamma$ such that $\gamma_i(p) \neq p$ converge to p as $i \to \infty$.

Corollary 5.7. *If a Riemann surface R satisfies the bounded geometry condition and if a subgroup Γ of the stabilizer $\mathrm{MCG}_p(R)$ is abelian, then the orbit $\Gamma(p)$ is a discrete set in T_p.*

We can further show that the orbit is a discrete set when Γ is solvable. However, if Γ is an infinitely generated free group for instance, then the orbit is not necessarily discrete. Our problem asks for some algebraic conditions upon Γ that guarantee this discreteness. These topics will be discussed elsewhere.

Now Theorem 4.1 immediately follows from Lemma 5.4 and Theorem 5.6.

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