On the Lax pairs of the sixth Painlevé equation

Robert Conte

†Service de physique de l'état condensé (URA 2464), CEA–Saclay
F–91191 Gif-sur-Yvette Cedex, France
E-mail: Robert.Conte@cea.fr

Abstract

The dependence of the sixth equation of Painlevé on its four parameters $(2\alpha, -2\beta, 2\gamma, 1 - 2\delta) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2)$ is holomorphic, therefore one expects all its Lax pairs to display such a dependence. This is indeed the case of the second order scalar "Lax" pair of Fuchs, but the second order matrix Lax pair of Jimbo and Miwa presents a meromorphic dependence on θ_∞ (and a holomorphic dependence on the three other θ_j). We analyze the reason for this feature and make suggestions to suppress it.

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Contents

1 Introduction

Consider a second order linear ordinary differential equation for $\psi(t)$ with five Fuchsian singularities, one of them $t = u$ being apparent (i.e. the ratio of two linearly independent solutions remains single valued around it) and the four others having a crossratio x. The condition that the ratio ψ_1/ψ_2 of two linearly independent solutions be singlevalued when t goes around any of these singularities results in one constraint between u and x, which is [2] that the apparent singularity u, considered as a function of the crossratio x, obeys the sixth Painlevé equation P6. In its normalized form (choice $(\infty, 0, 1, x)$ of the four nonapparent Fuchsian singularities), this ODE is [2]

$$\begin{aligned} E(u) \equiv & -u'' + \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right]u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right]u' \\ & + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right] = 0, \end{aligned}$$

its four parameters $\alpha, \beta, \gamma, \delta$ representing the differences θ_j of the two Fuchs indices at the four nonapparent singularities $t = \infty, 0, 1, x$,

$$(2\alpha, -2\beta, 2\gamma, 1-2\delta) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2). \tag{1}$$

The proof by Poincaré [11] of the impossibility to remove the apparent singularity in the second order scalar isomonodromic deformation certainly motivated Jimbo and Miwa to consider, in place of the scalar isomonodromy problem, the matrix isomonodromy problem of the same order (two),

$$\partial_x \psi = L\psi, \quad \partial_t \psi = M\psi, \ [\partial_x - L, \partial_t - M] = 0. \tag{2}$$

There indeed exists a choice [5] of second order matrices (L, M) whose isomonodromy condition also yields P6, in which the singularities of the monodromy matrix M in the t complex plane are four Fuchsian points of crossratio x, without the need for an apparent singularity.

This beautiful result however presents the drawback to have a meromorphic dependence on one of the four monodromy exponents θ_j, while u'' in P6 has a holomorphic such dependence. The purpose of this work is to explore several directions in order to remove this drawback from matrix Lax pairs.

A possibility to achieve that is to consider some simple physical system admitting a Lax pair and a reduction to P6. The corresponding reduction of its Lax pair could then provide a holomorphic Lax pair of P6. One such system if the three-wave resonant interaction, but the resulting Lax pair has third order, and its reduction to second order still encounters some obstacles [1]. The Maxwell-Bloch system [12] could be a better candidate because its Lax pair is second order.

The paper is organized as follows. In section 2, we recall the scalar "Lax" pair of Richard Fuchs, because its expression is required later on.

In section 3, we point out the meromorphic dependence in the second order Lax pair obtained by matrix monodromy.

In section 4, we define in some detail the small amount of required computations in order to obtain a holomorphic Lax pair.

In section 5, we explore the simplest possibility beyond the assumption of Jimbo and Miwa. The resulting Lax pair is linked to a type studied by Kimura [6] and the matrix elements are algebraic functions of u', u, x while in the JM case they are rational functions.

2 Holomorphic Lax pair by scalar isomonodromy

This pair [2, 3], as more nicely written in Ref. [4], is characterized by the two homographic invariants (S, C),

$$\partial_t^2 \psi + (S/2)\psi = 0, \tag{3}$$

$$\partial_x \psi + C\partial_t \psi - (1/2)C_t \psi = 0, \tag{4}$$

with the commutativity condition,

$$X \equiv S_x + C_{ttt} + CS_t + 2C_t S = 0, \tag{5}$$

where

$$-C = \frac{t(t-1)(u-x)}{(t-u)x(x-1)}, \tag{6}$$

$$-\frac{S}{2} = \frac{3/4}{(t-u)^2} + \frac{\beta_1 u' + \beta_0}{(t-u)t(t-1)} + \frac{[(\beta_1 u')^2 - \beta_0^2]\dfrac{u-x}{u(u-1)} + f_{\mathrm{G}}(u)}{t(t-1)(t-x)} + f_{\mathrm{G}}(t), \tag{7}$$

$$\beta_1 = -\frac{x(x-1)}{2(u-x)}, \quad \beta_0 = -u + \frac{1}{2}, \tag{8}$$

$$f_G(z) = \frac{a}{z^2} + \frac{b}{(z-1)^2} + \frac{c}{(z-x)^2} + \frac{d}{z(z-1)}, \tag{9}$$

$$(2\alpha, -2\beta, 2\gamma, 1-2\delta) = (4(a+b+c+d+1), 4a+1, 4b+1, 4c+1). \tag{10}$$

Like u'' in the definition of P6, this scalar Lax pair depends holomorphically on the four θ_j, and also on their squares. Its singularities in the complex plane of t are the five Fuchsian points $t = \infty, 0, 1, x, u$, among which $t = u$ is apparent.

3 Meromorphic Lax pair by matrix isomonodromy

Let us introduce the Pauli matrices σ_k

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_j \sigma_k = \delta_{jk} + i\varepsilon_{jkl}\sigma_l, \tag{11}$$
$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

As proven in [5], the apparent singularity of the scalar Lax pair can be removed by considering a second order matrix Lax pair,

$$\partial_x \Psi = L\Psi,\ \partial_t \Psi = M\Psi, \tag{12}$$

and defining the monodromy matrix M as the sum of four Fuchsian singularities $t = \infty, 0, 1, x$,

$$M = \frac{M_0(x)}{t} + \frac{M_1(x)}{t-1} + \frac{M_x(x)}{t-x},\quad M_\infty + M_0 + M_1 + M_x = 0. \tag{13}$$

However, in order to integrate the differential system of the monodromy conditions,

$$\forall t:\ L_t - M_x + LM - ML = 0. \tag{14}$$

the choice of L is not unique and the type of dependence of $L(x,t)$ on t must be an input. With the very convenient choice [5] of a simple pole at the crossratio $t = x$,

$$L = -\frac{M_x}{t-x},\ M = \frac{M_0(x)}{t} + \frac{M_1(x)}{t-1} + \frac{M_x(x)}{t-x},\quad M_\infty + M_0 + M_1 + M_x = 0, \tag{15}$$

and after minor transformations [10] mainly aimed at making all entries (L_{jk}, M_{jk}) algebraic (not only with algebraic logarithmic derivatives), one obtains the traceless, algebraic Lax pair,

$$\begin{aligned}
L &= -\frac{M_x}{t-x} + L_\infty,\ M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x}, && (16)\\
L_\infty &= -\frac{(\Theta_\infty - 1)(u-x)}{2x(x-1)}\sigma_3, && (17)\\
2M_\infty &= \Theta_\infty \sigma_3, && (18)\\
2M_0 &= z_0\sigma_3 - \frac{u}{x}\sigma^+ + (z_0^2 - \theta_0^2)\frac{x}{u}\sigma^-, && (19)\\
2M_1 &= z_1\sigma_3 + \frac{u-1}{x-1}\sigma^+ - (z_1^2 - \theta_1^2)\frac{x-1}{u-1}\sigma^-, && (20)\\
2M_x &= \left((\theta_0^2 - z_0^2)\frac{x}{u} - (\theta_1^2 - z_1^2)\frac{x-1}{u-1}\right)\sigma^- - \frac{u-x}{x(x-1)}\sigma^+ \\
&\quad -(\Theta_\infty + z_0 + z_1)\sigma_3, && (21)\\
z_0 &= \frac{1}{2\Theta_\infty x(u-1)(u-x)}\Big[(x(x-1)u' - (u-1)(u - \Theta_\infty(u-x)))^2 \\
&\quad -(\Theta_\infty^2 + \theta_0^2)x(u-1)(u-x) + \theta_1^2(x-1)u(u-x) - \theta_x^2 x(x-1)u(u-1)\Big],\\
z_1 &= \frac{-1}{2\Theta_\infty(x-1)u(u-x)}\Big[(x(x-1)u' - u(u-1-\Theta_\infty(u-x)))^2 \\
&\quad +(\Theta_\infty^2 + \theta_1^2)(x-1)u(u-x) - \theta_0^2 x(u-1)(u-x) - \theta_x^2 x(x-1)u(u-1)\Big],\\
&\quad (2\alpha, -2\beta, 2\gamma, 1-2\delta) = ((\Theta_\infty - 1)^2, \theta_0^2, \theta_1^2, \theta_x^2).
\end{aligned}$$

The origin of the meromorphic dependence in (16), as displayed in z_0 and z_1, seems to be the simplifying assumption [5] that the residue M_∞ can be chosen diagonal,

$$M_\infty = \frac{\Theta_\infty}{2}\sigma_3. \tag{22}$$

Indeed, when Θ_∞ vanishes, the residue also vanishes and one singular point is lost, thus preventing to obtain P6 which requires four nonapparent singular points.

As an additional motivation of the present work, this meromorphic feature is also present in many discrete Lax pairs of discrete P6 equations, for instance in the Lax pair found by Jimbo and Sakai [8], as an output to the matrix discrete isomonodromy problem

$$Y(x,qt) = A(x,t)Y(x,t), \tag{23}$$
$$A = A_0(x) + A_1(x)t + A_2(x)t^2, \tag{24}$$

where x is the independent variable, t is the spectral parameter, and the matrix A defines four singular points in the t complex plane. If the residue A_2 at $t=\infty$ is chosen diagonal [8, Eq. (10)],

$$A_2 = \mathrm{diag}(\kappa_1, \kappa_2), \tag{25}$$

then the Lax pair contains the denominator $\kappa_1 - \kappa_2$ and, when $\kappa_1 = \kappa_2$, the isomonodromy problem cannot yield a $\mathrm{q}-\mathrm{P6}$ equation.

4 Towards a holomorphic matrix Lax pair

In order to get rid of this unwanted meromorphic dependence, let us change the assumptions on the matrix Lax pair (L,M) along the lines explored in Ref. [9]. For the assumption (13) on M, which must be kept, we adopt the convention

$$\mathrm{tr}\, M_j = 0,\ \det M_j = -\frac{\theta_j^2}{4} = \text{constant},\ j = \infty, 0, 1, x, \tag{26}$$

and we represent the four residues so as to preserve the invariance under permutation,

$$M_j = \frac{1}{2}\begin{pmatrix} z_j & (\theta_j - z_j)u_j \\ (\theta_j + z_j)u_j^{-1} & -z_j \end{pmatrix},\ j = \infty, 0, 1, x, \tag{27}$$

in which z_j, u_j are functions of x.

After an assumption has been chosen for the dependence of $L(x,t)$ on t, there is no need to integrate the monodromy conditions (14). Indeed, one *a priori* knows that their general solution is expressed in terms of a P6 function. Therefore a “lazy” method to perform the integration is to first convert the matrix Lax pair (12) to scalar form, then to identify the result with the scalar Lax pair (3)–(4).

Let us denote $\Psi = {}^{\mathrm{t}}(\psi_1\ \psi_2)$ the base vectors of the matrix Lax pair after rotation by an arbitrary constant angle φ,

$$P = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},\ \partial_x \Psi = P^{-1}LP\Psi,\ \partial_t \Psi = P^{-1}MP\Psi. \tag{28}$$

After elimination of ψ_2 and removal of the first derivative ψ_1' in the resulting second order linear ODE for ψ_1, the identification of the two sets of coefficients (S,C) will provide L and M in terms of a solution u of P6.

Whatever be the assumption for L, the three scalar conditions of zero sum for the residues,

$$M_\infty + M_0 + M_1 + M_x = 0, \tag{29}$$

under the condition that u_0, u_1, u_x are all different, are first solved for z_0, z_1, z_x,

$$\left\{\begin{array}{l} \frac{J}{u_1-u_x} z_0 = z_\infty \left(u_1^{-1}+u_x^{-1}\right) - (\theta_\infty - z_\infty) u_\infty u_1^{-1} u_x^{-1} - (\theta_\infty + z_\infty) u_\infty^{-1} \\ \qquad + \theta_0 \left(u_0 u_1^{-1} u_x^{-1} - u_0^{-1}\right) + \theta_1 \left(u_x^{-1} - u_1^{-1}\right) + \theta_x \left(u_1^{-1} - u_x^{-1}\right), \\ \frac{J}{u_x-u_0} z_1 = z_\infty \left(u_x^{-1}+u_0^{-1}\right) + (\theta_\infty - z_\infty) u_\infty u_x^{-1} u_0^{-1} - (\theta_\infty + z_\infty) u_\infty^{-1} \\ \qquad + \theta_1 \left(u_1 u_x^{-1} u_0^{-1} - u_1^{-1}\right) + \theta_x \left(u_0^{-1} - u_x^{-1}\right) + \theta_0 \left(u_x^{-1} - u_0^{-1}\right), \\ \frac{J}{u_0-u_1} z_x = z_\infty \left(u_0^{-1}+u_1^{-1}\right) + (\theta_\infty - z_\infty) u_\infty u_0^{-1} u_1^{-1} - (\theta_\infty + z_\infty) u_\infty^{-1} \\ \qquad + \theta_x \left(u_x u_0^{-1} u_1^{-1} - u_x^{-1}\right) + \theta_0 \left(u_1^{-1} - u_0^{-1}\right) + \theta_1 \left(u_0^{-1} - u_1^{-1}\right), \end{array}\right. \tag{30}$$

in which J denotes the Jacobian

$$J \equiv \frac{D(M_{\infty,11}, M_{\infty,12}, M_{\infty,21})}{D(z_0, z_1, z_x)} = -\frac{(u_0-u_1)(u_1-u_x)(u_x-u_0)}{u_0 u_1 u_x}. \tag{31}$$

5 A Kimura-type Lax pair

Following (16) and [9, Eq. (4.18)], let us assume

$$L = -\frac{M_x}{t-x} + L_\infty,\ L_\infty = m(x) M_\infty, \tag{32}$$

which defines the differential system

$$\left\{\begin{array}{l} M_0' = \frac{[M_x, M_0]}{x} - m[M_\infty, M_0], \\ M_1' = \frac{[M_x, M_1]}{x-1} - m[M_\infty, M_1], \\ M_x' = -\frac{[M_x, M_0]}{x} - \frac{[M_x, M_1]}{x-1} - m[M_\infty, M_x]. \end{array}\right. \tag{33}$$

Such a choice ensures that $M_0 + M_1 + M_x$ is a first integral, and therefore M_∞ a constant. The system (33) is equivalent to

$$z_j' = \frac{P_j(u_k, z_k, \theta_k, m)}{x(x-1)u_0 u_1 u_x},\ u_j' = \frac{Q_j(u_k, z_k, \theta_k, m)}{x(x-1)u_0 u_1 u_x},\ j \in \{0,1,x\},\ k \in \{\infty, 0, 1, x\}, \tag{34}$$

in which P_j, Q_j denote polynomials of their arguments, and the closure conditions $z_j' = (z_j)'$ between the systems (34) and (30) are identically satisfied.

The identification of the two C's of the two scalar Lax pairs of the type (3)–(4) is equivalent to the two relations

$$\left\{\begin{array}{l} \left[m + \frac{u-x}{x(x-1)}\right] \left[z_\infty - \theta_\infty \frac{(\cos 2\varphi + 1) u_\infty + (\cos 2\varphi - 1) u_\infty^{-1}}{(\cos 2\varphi + 1) u_\infty - (\cos 2\varphi - 1) u_\infty^{-1} - 2 \sin 2\varphi}\right] = 0, \\ \text{when } \varphi = 0:\ (z_\infty - \theta_\infty) u_\infty u(u-x) + (z_0 - \theta_0) u_0 (u-x) - (z_x - \theta_x) u_x (x-1) u = 0, \end{array}\right. \tag{35}$$

in which, for brevity, the rotation angle φ has been set to 0 in the second relation.

Solving the first equation in (35) for the second factor would result in the vanishing of M_∞ with θ_∞, hence in the same singularity of the Lax pair at $\theta_\infty = 0$ than in (16). Therefore this first equation is solved for m, and in the second one can eliminate z_0, z_1, z_x with (30),

$$\left\{\begin{array}{l} m = -\frac{u-x}{x(x-1)}, \\ F(z_\infty, u_\infty, \theta_\infty, u_0, \theta_0, u_x, \theta_x, u, x, e^{i\varphi}) = 0, \end{array}\right. \tag{36}$$

in which F is a polynomial of its arguments, of degree two in u and each u_j.

Before transformation to the normalized form (3), the second order ODE for ψ_1 is then

$$(t-u)p_1(t)\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t^2} + \frac{p_4(t)}{t(t-1)(t-x)}\frac{\mathrm{d}\psi_1}{\mathrm{d}t} + \frac{p_6(t)}{[t(t-1)(t-x)]^2}\psi_1 = 0, \quad (37)$$

in which p_j denotes polynomials of degree j whose dependence on x has been omitted. The condition that $(t-u)p_1, p_4, p_6$ have a common zero t (otherwise there would be two apparent singularities) results in (when $\varphi = 0$),

$$(z_\infty - \theta_\infty)u_\infty = (z_0 - \theta_0)u_0\frac{x}{u^2} = (z_1 - \theta_1)u_1\frac{x-1}{(u-1)^2} = (z_x - \theta_x)u_x\frac{x(1-x)}{(u-x)^2}, \quad (38)$$

and these relations imply $p_1(t) = t - u$ and a multiplicity two for the zero $t = u$ of the element M_{12} of the monodromy matrix M. As proven in [7], this results in a difference of 3 between the two Fuchs indices at the apparent singularity $t = u$, not 2 like in (7). The Schwarzian associated to (37), which cannot be identified to (7), must then be identified to the Schwarzian of the equation labelled $\mathrm{L}^n_{\mathrm{VI}}$ in [6]. The resulting matrix Lax pair will probably be holomorphic in the four θ_j but surely not rational in u, u', x, since the transformation between the apparent singularities of $\mathrm{L}^n_{\mathrm{VI}}$ and (3) is not birational [7]. Therefore its explicit expression will not be given.

6 Conclusion

In order to build a second order matrix Lax pair of P6(u, x) at the same time holomorphic in θ_j and rational in $u(x), u'(x), x$, it is necessary to make an assumption for L which is different from (32), probably by adding to L a term linear in t like in [6, §6] and [9, Eq. (4.18)]. This will be the subject of future research.

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