Linear monodromy of the sixth Painlevé transcendents which are meromorphic around a fixed singularity

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1 Introduction

The Painlevé equation can be obtained by an isomonodromic deformation of a linear equation. We call the monodromy data of the linear equation *a linear monodromy* of the Painlevé function. In general, the linear monodromy cannot be calculated explicitly, and we will study Painlevé functions whose linear monodromy can be explicitly determined. In this paper we call such Painlevé functions *monodromy solvable*. It is A.V. Kitaev who first found the monodromy solvable solution with any value of parameter which is included in the Painlevé equation [9]. He constructed the so-called symmetric solution with any value of parameter α for the second Painlevé equation. By using his method, we have found the monodromy solvable solutions with any values of parameters for the fourth, fifth and sixth Painlevé equations [4],[5],[6]. In [6], the author has found twelve sets of meromorphic solutions around a fixed singularity for the sixth Painlevé equation

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}\\ &\quad+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right]\end{aligned}\tag{1.1}$$

and calulated the linear morodromy explicitly using Garnier-Okamoto's second order single equation as isomonodromic deformation equation [10].

In this paper we will give the same results for the sixth Painlevé equation using Jimbo-Miwa's 2×2 matrix type equation as isomonodromic deformation equation, which has the same polynomial Hamiltonian system as Garnier-Okamoto's single equation. For the calculation of the monodromy and asymptotic expansion of τ-function, it is convenient to normalize the linear equation so that the linear monodromy belongs to SL(2,$\mathbf{C}$). In [8], the normalized Jimbo-Miwa's 2×2 matrix type equation is introduced but its explicit form is not presented. In section 2.2 we explicitly write down the normalized Jimbo-Miwa's equation, which is a rational Hamiltonian system. Between these two kinds of Jimbo-Miwa's systems, there is a canonical transformation.

For the sixth Painlevé equation, A.D. Bryuno and I.V. Goryuchkina construct asymptotic solutions around the fixed singularity [2] and D. Guzzetti presents the leading term of the critical behavior at the fixed singularity [3]. We construct four meromorphic solutions of (1.1) at $t = 0$ for generic values of parameters $\alpha, \beta, \gamma, \delta$. By using some Bäcklund transformations we obtain four meromorphic solutions at the other singular points $t = 1$ and $t = \infty$ as well. These twelve meromorphic solutions are invariant under the action of the Bäcklund transformation group and further, monodromy solvable. It is pointed out by K. Iwasaki that these twelve solutions pass through the intersection points of 24 lines among 27 lines included in Fricke's cubic surface.

One of our solutions includes Umemura's algebraic solution $y = \sqrt{t}$ for the parameters $\alpha + \beta = 0, \gamma + \delta = 1/2$. Some of our solutions also include one of the Riccati solutions[11],[6].

2 Isomonodromic deformation equations of P_{VI}

There are two types of isomonodromic deformation equations of P_{VI}. One is Garnier-Okamoto's second order single type equation [10] and the other is Jimbo-Miwa's 2×2 matrix type equation [7]. In this paper we use Jimbo-Miwa's equation.

2.1 Jimbo-Miwa's form

In this section we recall Jimbo-Miwa's form [7] for isomonodromic deformation equation in terms of 2×2 matrices. We denote the following system by L_{VI}^{JM}.

$$\frac{\partial \Psi(x,t)}{\partial x} = A(x,t)\Psi(x,t), \quad A(x,t) = \sum_{j=0,1,t} \frac{A_j}{x-j} = \begin{pmatrix} a_{11}(x,t) & a_{12}(x,t) \\ a_{21}(x,t) & a_{22}(x,t) \end{pmatrix}, \tag{2.1}$$

$$A_j = \begin{pmatrix} z_j + \theta_j & -u_j z_j \\ u_j^{-1}(z_j + \theta_j) & -z_j \end{pmatrix} \quad (j = 0, 1, t),$$

$$\frac{\partial \Psi(x,t)}{\partial t} = B(x,t)\Psi(x,t), \quad B(x,t) = -\frac{A_t}{x-t}. \tag{2.2}$$

We define A_∞, y and z as follows:

$$A_\infty = -\sum_{j=0,1,t} A_j = \begin{pmatrix} \frac{1}{2}(\theta_\infty - \sum_{j=0,1,t}\theta_j) & 0 \\ 0 & -\frac{1}{2}(\theta_\infty + \sum_{j=0,1,t}\theta_j) \end{pmatrix}, \tag{2.3}$$

$$a_{12}(x,t) = -\sum_{j=0,1,t} \frac{u_j z_j}{x-j} = \frac{k(x-y)}{x(x-1)(x-t)}, \quad z = a_{11}(y,t) = \sum_{j=0,1,t} \frac{z_j + \theta_j}{y-j}, \tag{2.4}$$

where y, z, z_j, u_j and k are functions of t, and $\theta_j (j = 0, 1, t, \infty)$ are parameters. In what follows, instead of θ_j, we mainly use the parameters $\alpha_i (i = 0, 1, 2, 3, 4)$ defined by the following relations:

$$\theta_0 = \alpha_4, \quad \theta_1 = \alpha_3, \quad \theta_t = \alpha_0, \quad \theta_\infty = 1 - \alpha_1 \quad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1). \tag{2.5}$$

From the integrability condition of (2.1) and (2.2), we have

$$
\begin{aligned}
t(t-1)\frac{dy}{dt} &= 2zy(y-1)(y-t)-\alpha_4(y-1)(y-t)-\alpha_3 y(y-t) \\
&- (\alpha_0-1)y(y-1), \qquad (2.6) \\
t(t-1)\frac{dz}{dt} &= \left(-3y^2+2(1+t)y-t\right)z^2+\Big[(2y-1-t)\alpha_4+(2y-t)\alpha_3 \\
&+ (2y-1)(\alpha_0-1)\Big]z-\alpha_2(\alpha_1+\alpha_2). \qquad (2.7)
\end{aligned}
$$

Eliminating z, we have (1.1) with

$$\alpha=\frac{\alpha_1^2}{2}=\frac{(1-\theta_\infty)^2}{2},\quad \beta=\frac{-\alpha_4^2}{2}=\frac{-1}{2}\theta_0^2,\quad \gamma=\frac{\alpha_3^2}{2}=\frac{1}{2}\theta_1^2,\quad \delta=\frac{1-\alpha_0^2}{2}=\frac{1-\theta_t^2}{2}. \qquad (2.8)$$

The system of equations (2.6) and (2.7) can be written as a Hamiltonian system with the polynomial Hamiltonian H_{VI} given by

$$
\begin{aligned}
t(t-1)H_{VI} &= y(y-1)(y-t)z^2-\Big[\alpha_4(y-1)(y-t)+\alpha_3 y(y-t)+(\alpha_0-1)y(y-1)\Big]z \\
&+ \alpha_2(\alpha_1+\alpha_2)(y-t). \qquad (2.9)
\end{aligned}
$$

Remark 1 *This polynomial Hamiltonian system is the same as Garnier-Okamoto's Hamiltonian system. Putting $\psi={}^t(\psi_1,\psi_2)$ and eliminating ψ_2 from (2.1), we have the same second order single equation as Garnier-Okamoto's equation [10].*

2.2 Normalized Jimbo-Miwa's form

In this section, we give the normalized Jimbo-Miwa's isomonodromic deformation equation whose linear monodromy belongs to SL(2,**C**). We denote this system by L_{VI}^{JM}.

$$
\begin{aligned}
\frac{\partial\bar\Psi(x,t)}{\partial x} &= \bar A(x,t)\bar\Psi(x,t),\quad \bar A(x,t)=\sum_{j=0,1,t}\frac{\bar A_j}{x-j}=\begin{pmatrix}\bar a_{11}(x,t) & \bar a_{12}(x,t)\\ \bar a_{21}(x,t) & \bar a_{22}(x,t)\end{pmatrix}, \qquad (2.10)\\
\bar A_j &= \begin{pmatrix}\bar z_j+\frac{\theta_j}{2} & -\bar u_j\bar z_j\\ \bar u_j^{-1}(\bar z_j+\theta_j) & -\bar z_j-\frac{\theta_j}{2}\end{pmatrix}\quad (j=0,1,t),\\
\frac{\partial\bar\Psi(x,t)}{\partial t} &= \bar B(x,t)\bar\Psi(x,t),\quad \bar B(x,t)=-\frac{\bar A_t}{x-t}. \qquad (2.11)
\end{aligned}
$$

We define $\bar A_\infty, y$ and $\bar z$ as follows:

$$
\begin{aligned}
\bar A_\infty &= -\sum_{j=0,1,t}\bar A_j=\begin{pmatrix}\frac{\theta_\infty}{2} & 0\\ 0 & -\frac{\theta_\infty}{2}\end{pmatrix},\quad \bar a_{12}(x,t)=\frac{\bar k(x-y)}{x(x-1)(x-t)}, \qquad (2.12)\\
\bar z &= \bar a_{11}(y,t)=\sum_{j=0,1,t}\frac{\bar z_j+\frac{\theta_j}{2}}{y-j}, \qquad (2.13)
\end{aligned}
$$

where $y, \bar{z}, \bar{z}_j, \bar{u}_j, \bar{k}$ are functions of t, and θ_j, θ_∞ are parameters. Hereinafter we use α_i which are defined by (2.5). From the integrability condition of (2.10) and (2.11), we have

$$t(t-1)\frac{dy}{dt} = 2y(y-1)(y-t)\bar{z} + y(y-1), \tag{2.14}$$

$$\begin{aligned} t(t-1)\frac{d\bar{z}}{dt} &= \left[-3y^2 + 2(1+t)y - t\right]\bar{z}^2 - (2y-1)\bar{z} \\ &+ \left[-\frac{1-\alpha_1^2}{4} - \frac{\alpha_4^2}{4}\cdot\frac{t}{y^2} + \frac{\alpha_3^2}{4}\cdot\frac{t-1}{(y-1)^2} - \frac{\alpha_0^2}{4}\cdot\frac{t(t-1)}{(y-t)^2}\right]. \end{aligned} \tag{2.15}$$

Eliminating $\bar{z}$, we again obtain (1.1) with (2.8). The system of equations (2.14) and (2.15) is a rational Hamiltonian system with the Hamiltonian $\bar{H}_{VI}$ defined by

$$\begin{aligned} t(t-1)\bar{H}_{VI} &= y(y-1)(y-t)\bar{z}^2 + y(y-1)\bar{z} \\ &- \left[\frac{1-\alpha_1^2}{-4}y + \frac{\alpha_4^2}{4}\cdot\frac{t}{y} - \frac{\alpha_3^2}{4}\cdot\frac{t-1}{(y-1)} + \frac{\alpha_0^2}{4}\cdot\frac{t(t-1)}{(y-t)}\right]. \end{aligned} \tag{2.16}$$

Remark 2 *There is the following canonical transformation between L_{VI}^{JM} and $\bar{L}_{VI}^{JM}$ which keeps y invariant:*

$$dz \wedge dy - dH_{VI} \wedge dt = d\bar{z} \wedge dy - d\bar{H}_{VI} \wedge dt. \tag{2.17}$$

3 Meromorphic solutions around the fixed singularities

In this section we will classify all of the meromorphic solutions around a fixed singularity. We consider a solution of (2.14) and (2.15) (and that of (2.6) and (2.7) simultaneously) around $t = 0$:

$$y(t) = t^l \sum_{i=0}^{\infty} a_i t^i, \quad \bar{z}(t) = t^m \sum_{i=0}^{\infty} b_i t^i, \quad z(t) = t^n \sum_{i=0}^{\infty} c_i t^i \ (l, m, n \in \boldsymbol{Z}). \tag{3.1}$$

Theorem 3 *For generic values of parameters, the sixth Painlevé equation has the following four meromorphic solutions around $t = 0$:*

$$\text{(0-I)}: \quad y(t) = \frac{\alpha_4}{\alpha_4 - \alpha_0}t + \frac{\alpha_0\alpha_4\left[-1 - \alpha_1^2 + \alpha_3^2 + (\alpha_4 - \alpha_0)^2\right]}{2\left[1 - (\alpha_4 - \alpha_0)^2\right](\alpha_4 - \alpha_0)^2}t^2 + O(t^3), \tag{3.2}$$

$$\bar{z}(t) = \frac{1 - \alpha_1^2 + \alpha_3^2 - (\alpha_4 - \alpha_0)^2}{4\left[1 - (\alpha_4 - \alpha_0)^2\right]} + O(t), \tag{3.3}$$

$$z(t) = \frac{\alpha_4 - \alpha_0}{t} + O(t^0), \tag{3.4}$$

$$\text{(0-II)}: \quad y(t) = \frac{\alpha_4}{\alpha_4 + \alpha_0}t + \frac{-\alpha_0\alpha_4\left[1 + \alpha_1^2 - \alpha_3^2 - (\alpha_4 + \alpha_0)^2\right]}{2\left[1 - (\alpha_4 + \alpha_0)^2\right](\alpha_4 + \alpha_0)^2}t^2 + O(t^3), \tag{3.5}$$

$$\bar{z}(t) = \frac{1 - \alpha_1^2 + \alpha_3^2 - (\alpha_4 + \alpha_0)^2}{4\left[1 - (\alpha_4 + \alpha_0)^2\right]} + O(t), \tag{3.6}$$

$$z(t) = \frac{\alpha_2(\alpha_1+\alpha_2)}{1-\alpha_4-\alpha_0} + O(t), \tag{3.7}$$

$$\text{(0-III)}: \quad y(t) = \frac{\alpha_1+\alpha_3}{\alpha_1} + \frac{-\alpha_3\left[1+\alpha_4^2-\alpha_0^2-(\alpha_1+\alpha_3)^2\right]}{2\alpha_1\left[1-(\alpha_1+\alpha_3)^2\right]}t + O(t^2), \tag{3.8}$$

$$\bar{z}(t) = \frac{-\alpha_1}{2(\alpha_1+\alpha_3)} + O(t), \quad z(t) = \frac{-\alpha_1\alpha_2}{\alpha_1+\alpha_3} + O(t), \tag{3.9}$$

$$\text{(0-IV)}: \quad y(t) = \frac{\alpha_1-\alpha_3}{\alpha_1} + \frac{\alpha_3\left[1+\alpha_4^2-\alpha_0^2-(\alpha_1-\alpha_3)^2\right]}{2\alpha_1\left[1-(\alpha_1-\alpha_3)^2\right]}t + O(t^2), \tag{3.10}$$

$$\bar{z}(t) = \frac{-\alpha_1}{2(\alpha_1-\alpha_3)} + O(t), \quad z(t) = \frac{-\alpha_1(\alpha_1+\alpha_2)}{\alpha_1-\alpha_3} + O(t). \tag{3.11}$$

These solutions satisfy the system (2.6), (2.7) and (2.14), (2.15) and they are convergent since (2.6) and (2.7) are of Briot-Bouqet type at $t=0$ [1]. We gave a proof of the convergence for the fifth Painlevé transcendents in [5]. For generic values of parameters, there are no meromorphic solutions around $t=0$ except for these four solutions. The solution (0-I) exists for $\alpha_4-\alpha_0 \notin \boldsymbol{Z}$.

Let σ_1 and σ_2 be the Bäcklund transformations for the P_{VI} defined as follows:

x	α_0	α_1	α_2	α_3	α_4	y	z	t
$\sigma_1(x)$	α_0	α_1	α_2	α_4	α_3	$1-y$	$-z$	$1-t$
$\sigma_2(x)$	α_0	α_4	α_2	α_3	α_1	$\frac{1}{y}$	$-y(yz+\alpha_2)$	$\frac{1}{t}$

If we let σ_1 and σ_2 act on the solutions (0-I), (0-II), (0-III) and (0-IV), we then obtain the meromorphic solutions of the system (2.6), (2.7) and (2.14), (2.15) which are meromorphic around $t=1$ and $t=\infty$.

Theorem 4 *The sixth Painlevé equation has the following meromorphic solutions around $t=1$ and $t=\infty$.*
(1) Around $t=1$,

$$\text{(1-I)}: \quad y(t) = 1 + \frac{\alpha_3}{\alpha_0-\alpha_3}(1-t) + \frac{\alpha_0\alpha_3\left[-1-\alpha_1^2+\alpha_4^2+(\alpha_0-\alpha_3)^2\right]}{2\left[1-(\alpha_0-\alpha_3)^2\right](\alpha_0-\alpha_3)^2}(1-t)^2 + O((1-t)^3), \tag{3.12}$$

$$\bar{z}(t) = \frac{1-\alpha_1^2+\alpha_4^2-(\alpha_0-\alpha_3)^2}{4\left[1-(\alpha_0-\alpha_3)^2\right]} + O((1-t)), \tag{3.13}$$

$$z(t) = \frac{\alpha_0-\alpha_3}{1-t} + O((1-t)^0), \tag{3.14}$$

$$\text{(1-II)}: \quad y(t) = 1 + \frac{-\alpha_3}{\alpha_0+\alpha_3}(1-t) + \frac{\alpha_0\alpha_3\left[1+\alpha_1^2-\alpha_4^2-(\alpha_0+\alpha_3)^2\right]}{2\left[1-(\alpha_0+\alpha_3)^2\right](\alpha_0+\alpha_3)^2}(1-t)^2 + O((1-t)^3), \tag{3.15}$$

$$\bar{z}(t) = \frac{1-\alpha_1^2+\alpha_4^2-(\alpha_0+\alpha_3)^2}{4\left[1-(\alpha_0+\alpha_3)^2\right]} + O((1-t)), \tag{3.16}$$

$$z(t) = \frac{\alpha_2(\alpha_1+\alpha_2)}{\alpha_0+\alpha_3-1} + O((1-t)), \tag{3.17}$$

$$(1\text{-III}): \quad y(t) = -\frac{\alpha_4}{\alpha_1} + \frac{\alpha_4\left[1+\alpha_3^2-\alpha_0^2-(\alpha_4+\alpha_1)^2\right]}{2\alpha_1\left[1-(\alpha_4+\alpha_1)^2\right]}(1-t) + O((1-t)^2), \tag{3.18}$$

$$\bar{z}(t) = \frac{\alpha_1}{2(\alpha_1+\alpha_4)} + O((1-t)), \quad z(t) = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_4} + O((1-t)), \tag{3.19}$$

$$(1\text{-IV}): \quad y(t) = \frac{\alpha_4}{\alpha_1} + \frac{-\alpha_4\left[1+\alpha_3^2-\alpha_0^2-(\alpha_4-\alpha_1)^2\right]}{2\alpha_1\left[1-(\alpha_4-\alpha_1)^2\right]}(1-t) + O\left((1-t)^2\right), \tag{3.20}$$

$$\bar{z}(t) = \frac{-\alpha_1}{2(\alpha_4-\alpha_1)} + O((1-t)), \; z(t) = \frac{\alpha_1(\alpha_2+\alpha_4)}{\alpha_1-\alpha_4} + O((1-t)). \tag{3.21}$$

(2) Around $t=\infty$,

$$(\infty\text{-I}): \quad y(t) = \frac{\alpha_1-\alpha_0}{\alpha_1}t + \frac{\alpha_1\left[(1+\alpha_3^2+\alpha_4^2-\alpha_1^2-(\alpha_0-\alpha_1)^2)\left(\frac{\alpha_0}{\alpha_1}\right)^2+1+\alpha_0^2\right]}{2\alpha_0\left[1-(\alpha_0-\alpha_1)^2\right]} + O((t^{-1})), \tag{3.22}$$

$$\bar{z}(t) = \frac{-\alpha_1}{2(\alpha_1-\alpha_0)}\frac{1}{t} + O(t^{-2}), \quad z(t) = -\frac{\alpha_1(\alpha_1+\alpha_2)}{(\alpha_1-\alpha_0)t} + O(t^{-2}), \tag{3.23}$$

$$(\infty\text{-II}): \quad y(t) = \frac{\alpha_1+\alpha_0}{\alpha_1}t + \frac{-\alpha_1\left[(1+\alpha_3^2+\alpha_4^2-\alpha_1^2-(\alpha_0+\alpha_1)^2)\left(\frac{\alpha_0}{\alpha_1}\right)^2+1+\alpha_0^2\right]}{2\alpha_0\left[1-(\alpha_0+\alpha_1)^2\right]} + O((t^{-1}), \tag{3.24}$$

$$\bar{z}(t) = \frac{-\alpha_1}{2(\alpha_1+\alpha_0)}\frac{1}{t} + O(t^{-2}), \quad z(t) = -\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_0}\cdot\frac{1}{t} + O(t^{-2}), \tag{3.25}$$

$$(\infty\text{-III}): \quad y(t) = \frac{\alpha_4}{\alpha_4+\alpha_3} + \frac{-\alpha_3\alpha_4\left[-1+\alpha_0^2-\alpha_1^2+(\alpha_3+\alpha_4)^2\right]}{2\left[1-(\alpha_3+\alpha_4)^2\right](\alpha_3+\alpha_4)^2}\frac{1}{t} + O(t^{-2}), \tag{3.26}$$

$$\bar{z}(t) = \frac{1-\alpha_1^2+\alpha_0^2-(\alpha_3+\alpha_4)^2}{4\left[1-(\alpha_3+\alpha_4)^2\right]}\frac{1}{t} + O(t^{-2}), \tag{3.27}$$

$$z(t) = \frac{\alpha_2(\alpha_1+\alpha_2)}{1-\alpha_3-\alpha_4}\cdot\frac{1}{t} + O(t^{-2}), \tag{3.28}$$

$$(\infty\text{-IV}): \quad y(t) = \frac{\alpha_4}{\alpha_4-\alpha_3} + \frac{\alpha_3\alpha_4\left[-1+\alpha_0^2-\alpha_1^2+(\alpha_3-\alpha_4)^2\right]}{2\left[1-(\alpha_3-\alpha_4)^2\right](\alpha_3-\alpha_4)^2}\frac{1}{t} + O(t^{-2}), \tag{3.29}$$

$$\bar{z}(t) = \frac{1-\alpha_1^2+\alpha_0^2-(\alpha_3-\alpha_4)^2}{4\left[1-(\alpha_3-\alpha_4)^2\right]}\frac{1}{t} + O(t^{-2}), \tag{3.30}$$

$$z(t) = \alpha_4-\alpha_3+O(t^{-1}). \tag{3.31}$$

Observation 5 *If we assume the meromorphy of a solution around $t=0$ and $t=1$, $y(t)$ and $\bar{z}(t)$ inevitably become holomorphic there.*

Thorem 6 *These twelve meromorphic solutions are invariant under the action of the Bäcklund transformation group.*

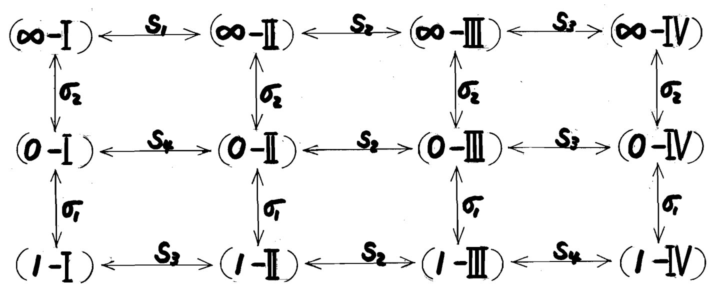

Figure 1: The Bäcklund transformations of the twelve solutions

4 The linear monodromy for the solution (0-I)

For a solution of the sixth Painlevé equation, let $M_j (j = 0, t, 1, \infty)$ be the monodromy matrices of the equation (2.10) along the path around $x = j$ shown in Figure 2.

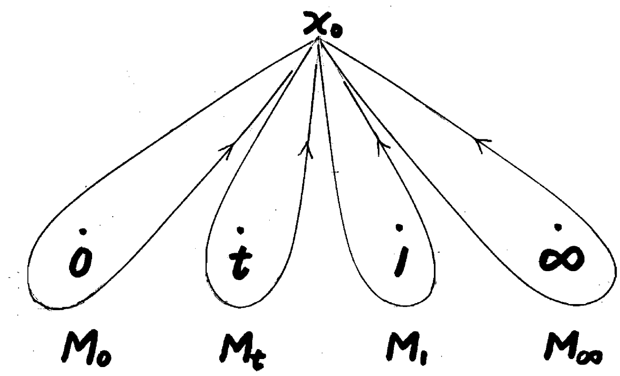

Figure 2: The paths going around regular singular points with the base point x_0.

Note that $M_j (j = 0, t, 1, \infty)$ satisfy

$$M_\infty M_1 M_t M_0 = I_2. \tag{4.1}$$

We can then calculate the linear monodromy $\{M_0, M_t, M_1, M_\infty\}$ explicitly for the solution (0-I) by the method given in [8].

Theorem 7 *The linear monodromy of* (2.10) *for the solution (0-I) is as follows [6]:*

$$M_0 = \begin{pmatrix} e^{-\pi i \alpha_4} & 0 \\ 0 & e^{\pi i \alpha_4} \end{pmatrix}, \quad M_t = \begin{pmatrix} e^{\pi i \alpha_0} & 0 \\ 0 & e^{-\pi i \alpha_0} \end{pmatrix}, \tag{4.2}$$

$$M_1 = \Gamma_{01}^{-1} \begin{pmatrix} e^{-\pi i \alpha_3} & 0 \\ 0 & e^{\pi i \alpha_3} \end{pmatrix} \Gamma_{01}, \quad M_\infty = \Gamma_{0\infty}^{-1} \begin{pmatrix} -e^{\pi i \alpha_1} & 0 \\ 0 & -e^{-\pi i \alpha_1} \end{pmatrix} \Gamma_{0\infty}. \tag{4.3}$$

where

$$\Gamma_{01} = \begin{pmatrix} \frac{\Gamma(1+\alpha_0-\alpha_4)\Gamma(\alpha_3)}{\Gamma(1-\alpha_1-\alpha_2-\alpha_4)\Gamma(1-\alpha_2-\alpha_4)} & \frac{\Gamma(1+\alpha_4-\alpha_0)\Gamma(\alpha_3)}{\Gamma(1-\alpha_0-\alpha_1-\alpha_2)\Gamma(1-\alpha_0-\alpha_2)} \\ \frac{\Gamma(1+\alpha_0-\alpha_4)\Gamma(-\alpha_3)}{\Gamma(\alpha_0+\alpha_1+\alpha_2)\Gamma(\alpha_0+\alpha_2)} & \frac{\Gamma(1+\alpha_4-\alpha_0)\Gamma(-\alpha_3)}{\Gamma(\alpha_1+\alpha_2+\alpha_4)\Gamma(\alpha_2+\alpha_4)} \end{pmatrix}, \tag{4.4}$$

$$\Gamma_{0\infty} = \begin{pmatrix} \frac{e^{(\alpha_0+\alpha_1+\alpha_2)\pi i}\Gamma(1+\alpha_0-\alpha_4)\Gamma(-\alpha_1)}{\Gamma(\alpha_0+\alpha_2)\Gamma(1-\alpha_1-\alpha_2-\alpha_4)} & \frac{e^{(\alpha_1+\alpha_2+\alpha_4)\pi i}\Gamma(1+\alpha_4-\alpha_0)\Gamma(-\alpha_1)}{\Gamma(\alpha_2+\alpha_4)\Gamma(1-\alpha_0-\alpha_1-\alpha_2)} \\ \frac{e^{(\alpha_0+\alpha_2)\pi i}\Gamma(1+\alpha_0-\alpha_4)\Gamma(\alpha_1)}{\Gamma(\alpha_0+\alpha_1+\alpha_2)\Gamma(1-\alpha_2-\alpha_4)} & \frac{e^{(\alpha_2+\alpha_4)\pi i}\Gamma(1+\alpha_4-\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_1+\alpha_2+\alpha_4)\Gamma(1-\alpha_0-\alpha_2)} \end{pmatrix}. \tag{4.5}$$

We remark that $\alpha_0 - \alpha_4 \notin \boldsymbol{Z}$ if the solution (0-I) exists.

In a similar way, we can calculate the linear monodromy explicitly for all of the twelve solutions in Theorem 3 and Theorem 4.

Theorem 8 *The twelve solutions in Theorem 3 and Theorem 4 are monodromy solvable.*

If we define p_j=trM_j and p_{jk}=trM_jM_k for the monodromy matrices, they satisfy the following relation which is called Fricke's cubic surface:

$$\begin{aligned} &p_{01}p_{1t}p_{t0} + p_{01}^2 + p_{1t}^2 + p_{t0}^2 - (p_0p_1 + p_tp_\infty)p_{01} - (p_1p_t + p_0p_\infty)p_{1t} \\ &-(p_0p_t + p_1p_\infty)p_{t0} + p_0^2 + p_1^2 + p_t^2 + p_\infty^2 + p_0p_1p_tp_\infty - 4 = 0. \end{aligned} \tag{4.6}$$

For the monodromy matrices (4.2) and (4.3), we have

$$p_0 = 2\cos\alpha_4\pi, \quad p_t = 2\cos\alpha_0\pi, \quad p_1 = 2\cos\alpha_3\pi, \quad p_\infty = -2\cos\alpha_1\pi, \tag{4.7}$$

$$\begin{aligned} p_{01} &= \frac{2}{\sin(\alpha_4-\alpha_0)\pi\sin\alpha_3\pi}\Big[\cos(\alpha_3+\alpha_4)\pi\sin(\alpha_1+\alpha_2+\alpha_4)\pi\sin(\alpha_2+\alpha_4)\pi \\ &- \cos(\alpha_3-\alpha_4)\pi\sin(\alpha_0+\alpha_1+\alpha_2)\pi\sin(\alpha_0+\alpha_2)\pi\Big], \end{aligned} \tag{4.8}$$

$$\begin{aligned} p_{1t} &= \frac{2}{\sin(\alpha_4-\alpha_0)\pi\sin\alpha_3\pi}\Big[\cos(\alpha_0-\alpha_3)\pi\sin(\alpha_1+\alpha_2+\alpha_4)\pi\sin(\alpha_2+\alpha_4)\pi \\ &\quad -\cos(\alpha_0+\alpha_3)\pi\sin(\alpha_0+\alpha_1+\alpha_2)\pi\sin(\alpha_0+\alpha_2)\pi\Big], \end{aligned} \tag{4.9}$$

$$p_{t0} = 2\cos(\alpha_0-\alpha_4)\pi. \tag{4.10}$$

Remark 9 *Twenty-seven lines are included in Fricke's cubic surface. Twelve solutions in Theorem 3 and Theorem 4 pass through the intersetion points of 24 lines among them. This is pointed out by K. Iwasaki.*

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