ON THE GALOIS REDUCIBILITY OF A GERM OF QUASI-HOMOGENEOUS FOLIATION

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1. Introduction

From recent developments in non linear differential Galois theory, we have now three equivalent definitions for the Galois reducibility of a codimension one foliation defined by a germ of holomorphic one-form ω:

- the first one is related to Godbillon-Vey sequences: there exists a finite sequence of lenght at most three of meromorphic one forms ω_0, ω_1, and ω_2 such that ω_0 is an equation of the foliation and

$$d\omega_0 = \omega_0 \wedge \omega_1,\ d\omega_1 = \omega_0 \wedge \omega_2,\ d\omega_2 = \omega_1 \wedge \omega_2.$$

- the second one is related to the existence of first integrals for the foliation with a particular type of transcendence which belongs to a Darboux or Liouville or Riccati type differential extension;

- the last one refers to the existence of a proper Galois D-groupoid for the foliation. This notion has been introduced by B. Malgrange in [9], (see also [10]): consider the groupoid $J_k^*\Delta$ of invertible jets of order k of maps from a polydisc Δ to itself. A D-groupoid is the projective limit on k of subvarieties Y_k of $J_k^*\Delta$ which are (strict) subgroupoids of $J_k^*\Delta$ outside a codimension one analytic set (this condition will allow us to deal with singularities). Each D-groupoid admits a D-Lie algebra obtained by the linearization of its equations along the identity solutions. The Galois D-groupoid Gal$\mathcal{F}$ of a foliation $\mathcal{F}$ is the smallest one whose D-algebra contains the Lie algebra of tangents vector fields to $\mathcal{F}$. It is a proper one if it doesn't coincide with the whole groupoid Aut$\mathcal{F}$ of the germs of diffeomorphisms which keep invariant the foliation.

This last point of view is related -maybe equivalent- to the one developed by H. Umemura in [18]. The equivalence of the two first points of view has been proved by G. Casale in [3]. The equivalence between the two last ones was described by B. Malgrange in manuscripted notes, and has been extensively proved by G. Casale in [5] with some different arguments. In particular, the transverse rank of Gal$\mathcal{F}$ (i.e. the order of its transverse local expression) is also the minimal lenght of a Godbillon-Vey sequence for $\mathcal{F}$ and characterizes the type of transcendence for first integrals, namely of Liouvillian type for the transverse rank two, and Riccati type for the transverse rank three.

The aim of this note is to present criterions for the Galois reducibility of a germ of foliation in the following context: $\mathcal{F}$ is defined by a vector field $X = X_h + \cdots$ where the

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"initial" hamiltonian vector field

$$X_h = \frac{\partial h}{\partial y}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial y}$$

is quasi-homogeneous with to respect to $R = p_1 x\frac{\partial}{\partial x} + p_2 y\frac{\partial}{\partial y}$ (p_1, p_2 positive integers): $R(h) = \delta h$, $\delta = \deg_R(h)$. The dots mean terms of higher quasihomogeneous degree. We furthermore require that h has an isolated singularity –with Milnor number $\mu = \dim_{\mathbb{C}}\mathcal{O}_2/(\partial h/\partial x, \partial h/\partial y)$– and that X still keep invariant the analytic set $h = 0$. Therefore, X is a logarithmic vector field for the polar set $h = 0$, and we have:

$$X = aX_h + bR,\ a \in \mathcal{O}_2, b \in \mathcal{O}_2,\ a(0) = 1$$

with $deg_R(bR) > \deg_R(X_h)$. The restriction to this class of foliation is motivated by the two following reasons:
- the desingularization of these foliations by blowing up's is "simple": it is similar to the one of the quasi-homogeneous function h: the exceptional divisor is only a chain of projective lines and all the strict transforms of all the irreducible components of h –except the axis if they appear in h– meet the same "principal" projective line C.
- in this class of foliations, we have at our disposal *formal normal forms* which give us complete formal invariants: see [16].

This will allow us to give two different types of criterions for the Galois reducibility of $\mathcal{F}$: a geometric one which is related to the holonomy of the principal component C of the desingularized foliation, and an algorithmic one which directly holds on the normalized formal equation of the foliation. Such formal criterion has been previously developed in the so-called "cuspidal" case ($h = y^2 - x^3$) in [8]. We shall furthermore describe the relationship between these two criterions, and we shall conclude with some open questions. An extended version of this note with complete proofs is available in [14].

2. A geometric criterion for the Galois reducibility.

We first remark that Malgrange's definition of a D-Galois closure not only holds for foliations, but it still makes sense for any discrete or continous dynamical system. For example, we can consider the Galois envelope of a subgroup G of the group Diff($\mathbb{C}$,0) of germs of diffeomophism at the origin of $\mathbb{C}$: this is the smallest D-groupoid on a disc around the origin of $\mathbb{C}$ such that the elements of G are solutions of this D-groupoid. Notice that the complete list of D-groupoids on the disc is known: see [4]. All strict subgroupoid of the maximal one $\mathrm{Aut}(\Delta)$ are generated by differential equations of order at most three. We shall call Liouvillian diffeomorphism (resp. Riccati type diffeomorphism) a diffeomorphism whose Galois closure has order at most two (resp. three).

One can easily check that the proper Galois closure of a holomorphic foliation induces a proper one for any holonomy group of a leaf of $\mathcal{F}$, or for a leaf of the desingularized foliation $\widetilde{F}$ (for the processus of desingularization in this context, see [17] and [12]). Thus the holonomy group $\mathrm{Hol}_C(\widetilde{F})$ of the principal component C of the exceptional divisor of $\widetilde{F}$ has a proper Galois closure of finite rank. In our class of foliations, the converse holds:

Theorem 1. *The Galois groupoid of the germ of quasi-homogeneous foliation $\mathcal{F}$ is a proper one if and only if the Galois envelope of* $\mathrm{Hol}_C(\widetilde{F})$ *is a proper one.*

The main argument in the proof of this theorem is an extension of the equation which defines the Galois closure of $\mathrm{Hol}_C(\widetilde{F})$ to the whole exceptional divisor. This is possible, since the elements of the holonomy group of C are solutions of this equation and therefore keep it invariant.

This theorem reduces the initial problem to the determination of the Galois closure of a subgroup G of Diff($\mathbb{C}$,0). We shall now describe this closure. From [4], the Galois closure of one diffeomorphism h is known: if h is formally linearizable, h admits a proper Galois closure if and only if h is analytically linearizable. If h is a resonant diffeomorphism, it admits a proper Galois closure of Liouville type (resp. Riccati type) if and only if its analytic invariant has a very specific form, called *unitary* or *binary*: in the description of Martinet-Ramis [11], the cocycles only are ramification of homographies for the binary case and alternately identities or ramification of homographies for the unitary case. A subgroup G of Diff($\mathbb{C}$,0) is an *exceptional* one if the subgroup G_1 of its elements tangent to the identy is a monogeneous one. Among the non linearizable groups, the non exceptional groups are exactly the rigid ones: their formal class coincides with the analytic one: see [6]. One should say that an exceptional group is a unitary or binary one if G_1 can be generated by one element whose analytic invariant is unitary or binary.

Theorem 2. *The only subgroups of* Diff*($\mathbb{C}$,0) which have a proper Galois closure are:*
(1) the analytically linearizable groups;
(2) the non exceptional solvable groups;
(3) the exceptional unitary groups;
(4) the exceptional binary groups.
Furthermore, the rank of their D-envelope is at most one in case (1), at most two in cases (2) and (3), and at most three in case (4).

To give an idea of the proof, we can examine the case (2): we know that if G is a solvable group, there exists a formal vector field θ which is invariant by each element of the group up to a multiplicative constant (see [15]). If furthermore the group G is a non exceptional one, this vector field is convergent. This invariance relation gives us a differential equation satified by the element g. We have to derivate it in order to make disappear the multiplicative constants, and to obtain the second order equation of the D-groupoid satisfied by all the elements of G. The case (1) is similar, and the cases (3) and (4) reduce to the study of G. Casale for monogeneous groups. In order to prove that this list is complete, the key point is the following claim: any subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ with a proper Galois closure is a solvable one. This claim only holds in the local situation. Indeed, it is related to the following fact: any proper Lie sub-algebra of the Lie algebra of one variable vector fields is a finite dimensional one, whose dimension is at most three. Furthermore, if each vector field vanishes at the same point, as in the local context, this Lie algebra is a two dimensional and solvable one.

3. A formal criterion for the Galois reducibility.

Let $X = X_h + \cdots$, $\mathcal{F}_X$ the foliation defined by X and F_h its "initial part" defined by X_h. Recall the normal forms for $\mathcal{F}_X$, i.e. a representative of X under a formal change of coordinate, up to a multiplication by a unity (for details, see [16]). If we want to construct a formal conjugacy between $\mathcal{F}_X$ and its initial part $\mathcal{F}_h$, we find obstructions in the cokernel of the derivation X_h. One can prove that, in the quasi-homogeneous context, $\mathrm{coker}(X_h)$ is a free module of rank μ over the ring $\mathbb{C}[[h]]$ of the first integrals of X_h. This is the key point to obtain the following prenormal forms:

Theorem 3. *Let $a_1, \cdots a_\mu$ be a monomial basis of $\mathcal{O}_2/Jac(h)$, where $Jac(h)$ is the jacobian ideal generated by the partial derivatives of h. There exists an element $(d_1, \cdots d_\mu)$ of $\mathbb{C}[[h]]^\mu$, a formal diffeomorphism Φ which conjugates up to a unity the vector field X to the formal vector field*

$$Y = X_h + \sum_{k=1}^{\mu} d_k(h) a_k R.$$

In the previous step, there is no unicity of the prenormal form Y. One can prove that the set of prenormal forms for $\mathcal{F}_X$ is the orbit of one of them under the action of a final reduction group of transformations of the following type: $\Phi = \exp b \cdot R$, with a formal coefficient b in $\mathcal{I}$: $b = b(h)$. Such transformations satisfy the relation $h \circ \Phi = \varphi \circ h$ for a one variable formal diffeomorphism φ. In order to study the action of this group on the prenormal forms, it is convenient to introduce a modified expression of them. We shall make use of the two following remarks:

i- Setting $\alpha = h^{-\delta_0/\delta}$, he have $[\alpha X_h, R] = 0$. The introduction of this multivalued coefficient will allow us to work with an abelian basis of logarithmic vector fields.

ii- Setting $r_i = \frac{\deg(\alpha a_i)}{\delta}$ we have $R(\alpha a_i h^{-r_i}) = 0$. This will allow us to work with coefficients which are constants for R.

Multiplying Y with α, and grouping coefficients in order to transform coefficients a_i in constants f_i for R we obtain the following "adapted" prenormal forms:

$$X_\alpha + \sum_{i=1}^{\mu} f_i \delta_i(h) R \tag{1}$$

with $X_\alpha = \alpha X_h$, $f_i = \alpha a_i h^{-r_i}$ and $\delta_i = d_i(h) h^{r_i}$. By these two tricks, any element Φ of the final reduction group keep invariant X_α and the coefficients f_i. Therefore we can immediatly check that the action of Φ on the adapted prenormal forms reduces to the action of φ on the one variable vector fields

$$\theta_i(z) = d_i(z) z^{r_i+1} \frac{d}{dz}.$$

Since $r_i = p_i/\delta$ for a positive integer p_i, we can uniformize these vector fields setting $t = z^{1/\delta}$ in

$$\theta_i(t) = \delta^{-1} d_i(t^\delta) t^{p_i+1} \frac{d}{dt}.$$

We obtain:

Corollary 4. *The collection $\mathcal{L}(\mathcal{F})$ of these μ vector fields $\theta_i(t)$ up to a common conjugacy is a complete formal invariant of $\mathcal{F}$.*

Furthermore, we may choose φ -and therefore Φ- in such a way that one of the vector fields θ_i is normalized under its usual normal form

$$\delta^{-1}\frac{t^{q_i+1}}{1+\lambda t^{q_i}}\frac{d}{dt}, \text{ with } q_i = \delta k_i + p_i$$

where k_i is the multiplicity of each series d_i. Going back to the non adapted prenormal forms, we obtain the final normal forms X_N in which one of the coefficients $d_i(h)$ is a rational function of h: $d_i(h) = \frac{h^m}{1+\lambda h^{m+n}}$. Notice that as soon as $\mu \geq 2$, these final models are *formal* vector fields. We know from [1] that there are examples of analytic vector fields with divergent such normal forms. This reference also allows us to conjecture that they always are transversally k-summable models.

We now make use of this algorithmic invariant $\mathcal{L}(\mathcal{F})$ to characterize the Galois reducible foliations. We first restrict our study to the non exceptional foliations, i.e. foliations whose holonomy group of its principal component is a non exceptional one.

Theorem 5. *Consider a non exceptional quasi-homogeneous foliation $\mathcal{F}$. The Galois groupoid of $\mathcal{F}$ is proper if and only if the explicit invariant $\mathcal{L}(\mathcal{F})$ generates a finite dimensional Lie algebra.*
In this case, $\mathcal{L}(\mathcal{F})$ is always of dimension one, and the foliation is at most Liouvillian (its transverse rank is at most two).

It is easy to check that if $\mathcal{L}(\mathcal{F})$ generates a finite dimensional Lie algebra of one variable vector fields, the dimension of this one is at most three, here at most two since all the vector fields vanishes at the origin, and finally at most one since it doesn't contain any element of order one. Therefore, the final normal form X_N here becomes a rational one, and defines an analytic foliation $\mathcal{F}_N$. Setting $\delta_i(h) = c_i\delta(h)$, one can easily check that $\omega_0 = \omega_N$ (dual logarithmic form of X_N) and $\omega_1 = \delta'(h)/h$ (the logarithmic derivative of δ) defines a Godbillon-Vey sequence of lenght two for the foliation $\mathcal{F}_N$, which therefore is a Liouvillian one. Since, in the non exceptional case, the foliation $\mathcal{F}$ is analytically conjugated to $\mathcal{F}_N$, $\mathcal{F}$ is also a Liouvillian foliation. On the converse, if $\mathcal{F}$ is a Galois reducible foliation, one can prove that the vector fields θ_i are all solutions of a linear differential equation induced by the Godbillon-Vey sequence.

In the exceptional cases, which are always formally Liouvillian, we can only characterize the analytic Galois reducibility with the geometric previous criterion. In this case, the foliation will be a Liouvillian one for unitary invariant, or of Riccati type, (transverse rank three) for binary invariants. Notice that in the local context, the foliations which are of Riccati type but non Liouvillian are very rare: they reduce to the class of exceptional foliations with binary non unitary invariants.

4. Comparison between the geometric and algorithmic criterions.

The first criterion involves a geometric transcendental invariant –a holonomy group– which cannot be computed in the general situation. On the other hand, the second criterion involves an algorithmic invariant $\mathcal{L}(\mathcal{F})$, which does not have a geometric meaning. It is a remarkable fact that, for Galois reducible non exceptional foliations, we can compute the first one from the second one.

In order to do this, it is more appropriate to introduce the *relative holonomy* of $\mathcal{F}$ with respect to its initial part $\mathcal{F}_h$ (see also [13]): one can define the holonomy h_i, $i = 1, \cdots \mu$ of any "horizontal" family of evanescent cycles in the fibers of h. If θ is a generator of the one dimensional Lie algebra $\mathcal{L}(\mathcal{F})$, one can prove that

$$h_i = \exp[T_i]\theta, \ i = 1, \cdots \mu$$

where the T_i's are periods of a relatively closed one form induced by the dual normal form of $\mathcal{F}$ on each horizontal family of evanescent cycles.

Notice that the relative holonomy depends on the choice of the initial part $\mathcal{F}_h$ of the foliation $\mathcal{F}$. This initial part is a particular case of a notion introduced by N. Corral in [7] (see also [2] for the dicritical case). In our quasi-homogeneous case, this initial foliation is unique up to conjucacy. It is not clear that we still can define a *unique* initial part for any germ of foliation.

5. Open problems.

Here are some problems directly motivated by the present study:

(1) Find the relation between geometric and algorithmic invariants of a quasi-homogeneous foliation outside the Galois reducible case. In the general case, this transcendental relation will not reduce to the integration of a one-variable vector field. We shall probably have to introduce Campbell-Hausdorff type formulae.
(2) Prove the k-summability of the final normal forms in the quasi-homogeneous case, and find the geometric or dynamical meaning of the order k.
(3) Construct formal normal forms for any non dicritical germ of foliation, having in mind the present motivations: a good representative of a holomorphic foliation may allow us to determine its Galois closure, and to compute its geometric invariants. Nevertheless we agree divergent models in order to get the previous conditions.
(4) What about the dicritical case (the projective line of the desingularization are not yet invariants components)?
(5) (suggested by B. Malgrange) Does the general study of codimension one germs of foliation reduces to the dimension two case?
(6) Develop a similar study for an algebraic foliation on the projective plane, near an algebraic invariant set.

References

[1] M. Canalis-Durand, R. Schaefke - *Divergence and summability of normal forms of systems of differential equations with nilpotent linear part.* Ann. Fac. Sci. Toulouse Math. (6) 13 (2004), no. 4, 493–513.

[2] F. Cano, N. Corral - *Dicritical logarithmic foliations,* Publ. Mat. (Barcelona) no 50-1 (2006) 87–105.

[3] G. Casale - *Suites de Godbillon-Vey et intégrales premières,* C.R. Acad. Sci. Paris 335 (2002).

[4] G. Casale - *D-enveloppe d'un difféomorphisme de* $(\mathbb{C}, 0)$. Ann. Fac. Sci. Toulouse Math. (6) 13 (2004), no. 4, 515–538.

[5] G. Casale - *Feuilletages singuliers de codimension un, groupoïde de Galois et intégrales premières,* Ann. Inst. Fourier 56 (2006), no. 3, 735–779.

[6] D. Cerveau and R. Moussu - *Groupes d'automorphismes de* $(\mathbb{C}, 0)$ *et équations différentielles* $y\,dy + \cdots = 0$. Bull. Soc. math. France **116** (1988), 459–488.

[7] N. Corral - *Sur la topologie des courbes polaires de certains feuilletages singuliers,* Ann. Inst. Fourier 53 (2003), no. 3, 787–814.

[8] F. Loray and R. Meziani - *Classification de certains feuilletages associés à un cusp.* Bol. Soc. Bras. Math. **25** (1994), 93–106.

[9] B. Malgrange - *Le groupoïde de Galois d'un feuilletage,* Monographie **38**, vol. **2** de l'Enseignement Mathématiques (2001).

[10] B. Malgrange - *On the non linear Galois theory,* Chinese Ann. Math. ser B, **23, 2**, (2002)

[11] J. Martinet and J.P. Ramis - *Classification analytique des équations différentielles non linéaires résonnantes du pemier ordre.* Ann. Sci. Ecole Norm. Sup., t.**16** (1983), 571–621.

[12] J.F. Mattei and R. Moussu - *Holonomie et intégrales premières,* Ann. Sci. Ecole Norm. Sup., t.**13** (1980), 469–523.

[13] R. Moussu - *Holonomie évanescente des équations différentielles dégénérées transverses.* Singularities and Dynamical Systems, S.N. Pnevmatikos ed., Elsevier Sc. Publisher (1985) 161–173

[14] E. Paul - *The Galoisian envelope of a germ of foliation: the quasi-homogeneous case,* Preprint Labo E. Picard, Institut de Mathématiques de Toulouse. arXiv math.DS/0612280

[15] E. Paul - *Feuilletages holomorphes singuliers à holonomie résoluble,* J. reine angew. Math. **514** (1999), 9-70.

[16] E. Paul - *Formal normal form for a perturbation of a quasi-homogeneous hamiltonian vector field.* Journal of Dynamical and Control Systems **10** (2004) 545-575.

[17] A. Seidenberg - *Reduction of singularities ofthe differential equation* $Ady = Bdx$, Amer. J. Math. **90** (1968)

[18] H. Umemura - *Galois theory of algebraic and differential equation,* Nagoya Math. J. vol 144, (1996), 1-58 *Differential Galois theory of infinite dimension,* Nagoya Math. J. vol 144, (1996), 59-135.

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