SINGULAR PARTIAL DIFFERENTIAL EQUATIONS WITH RESONANCE AND SMALL DENOMINATORS

広島大学・理学研究科　　吉野　正史 (Masafumi Yoshino) [1]
Graduate School of Science
Hiroshima University

Keywords: divergence, resonance, small denominators
MSJC: Primary 35C10; Secondary 37F50, 37G05

Abstract: We study divergence of formal power series solutions of a singular nonlinear system of partial differential equations from the viewpoints of a resonance and a small divisor problem. The results are applied to the transforming equation in the normal form theory of a singular vector field.

1. Introduction

The object of this note is to study divergence phenomena of a system of semilinear equations appearing in the normal form theory of vector fields. We are interested in the divergence caused by a resonance or small denominators of a transforming equation. We will study the problem from the viewpoint of the theory of functional equations. For the study of small denominators, we refer [3], [5] and the references therein. We will present a simple criterion for nonlinear terms that guarantees the convergence or the divergence of a formal solution.

This paper is organized as follows. In Section 2 we state the criterion for the convergence which is different from the Poincaré condition or the Diophantine condition. In Section 3 we give examples with divergent formal solutions. Section 4 is devoted to the discussion and the future problems. The results in Sections 2 and 3 will be published elsewhere with complete proofs.

2. Convergence Criterions

Let $x = {}^t(x_1, \ldots, x_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable in $\mathbb{C}^n$. We set $\frac{\partial}{\partial x_j} = \partial_{x_j}$, $(j \geq 1)$. For an n-square constant matrix Λ, we denote by

[1]e-mail: yoshino@math.sci.hiroshima-u.ac.jp,
Supported by Grant-in-Aid for Scientific Research (No. 16654028), Ministry of Education, Science and Culture, Japan.

L_Λ the Lie derivative of the linear vector field $\Lambda x \cdot \partial_x$

$$L_\Lambda v = [\Lambda x, v] = \langle \Lambda x, \partial_x v\rangle - \Lambda v, \tag{2.1}$$

where $\langle \Lambda x, \partial_x v\rangle = \sum_{j=1}^n (\Lambda x)_j (\partial/\partial x_j) v$, with $(\Lambda x)_j$ being the j-th component of Λx. We consider the system of equations

$$L_\Lambda u = R(u(x)), \tag{2.2}$$

where $u = {}^t(u_1, u_2, \dots, u_n)$ is an unknown vector function, and

$$R(x) = {}^t(R_1(x), R_2(x), \dots, R_n(x))$$

is holomorphic in some neighborhood of $x = 0 \in \mathbb{C}^n$ such that $R(x) = O(|x|^2)$ when $|x| \to 0$. The equation (2.2) appears as a linearizing equation of a singular vector field. Because we can always reduce Λ to a Jordan normal form by a linear change of unknown functions, we may assume that Λ is put in a Jordan normal form. Moreover we assume that there exists $\exists\ \tau_0$, $0 \le \tau_0 \le \pi$ such that

$$\text{every component of } e^{-i\tau_0}\Lambda \text{ is a real number.} \tag{2.3}$$

It follows that if λ_j $(j = 1, 2, \dots, n)$ are the eigenvalues of Λ with multiplicity, then we have

$$\exists\ \tau_0,\ 0 \le \tau_0 \le \pi,\ e^{-i\tau_0}\lambda_j \in \mathbb{R} \quad (j = 1, 2, \dots, n), \tag{2.4}$$

where $\mathbb{R}$ is the set of real numbers. If we set $u(x) = x + v(x)$, $v(x) = O(|x|^2)$, then v satisfies the system of semilinear equations

$$L_\Lambda v = R(x + v(x)). \tag{2.5}$$

Let $\mathbb{Z}_+$ be the set of nonnegative integers, and let $\mathbb{Z}_+^n(k)$ $(k \ge 0)$ be the set of multi-integers $\gamma = {}^t(\gamma_1, \gamma_2, \dots, \gamma_n)$ such that $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n \ge k$. For $\gamma \in \mathbb{Z}_+^n$, we set $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. For $k \ge 0$ and $n \ge 1$, we denote by $\mathbb{C}_k^n[[x]]$ the set of formal power series $\sum_{|\eta| \ge k} u_\eta x^\eta$ $(u_\eta \in \mathbb{C}^n)$. We also denote the set of convergent n-vector power series which vanish up to the $k-1$-th derivatives by $\mathbb{C}_k^n[x]$. We decompose $\Lambda = \Lambda_S + \Lambda_N$, where Λ_S and Λ_N are the semi-simple and the nilpotent parts of Λ, respectively. We denote by L_{Λ_S} the Lie derivative of the linear vector field $\Lambda_S x \cdot \partial_x$.

For a formal power series $f(x) = \sum_\gamma f_\gamma x^\gamma$, we define the majorant of f, $M(f)(x)$ by

$$M(f)(x) := \sum_\gamma |f_\gamma| x^\gamma. \tag{2.6}$$

For formal power series with real coefficients $a(x) = \sum_\gamma a_\gamma x^\gamma$ and $b(x) = \sum_\gamma b_\gamma x^\gamma$, we define $a \ll b$ if

$$a_\gamma \le b_\gamma \quad \text{for all } \gamma \in \mathbb{Z}_+^n.$$

We similarly define

$$(f_1(x), f_2(x), \dots, f_n(x)) \ll (g_1(x), g_2(x), \dots, g_n(x))$$

if $f_j(x) \ll g_j(x)$ for $j = 1, 2, \dots, n$.

Let $c > 0$ be a constant. Let $\mathcal{A}_+$ (resp. $\mathcal{A}_-$) be the set of $g(x) = (g_1(x), \dots, g_n(x)) \in \mathbb{C}_2^n[x]$ such that

$$(L_{\Lambda_S} - c)M(g) \gg 0 \quad (\text{ resp. } \ (L_{\Lambda_S} + c)M(g) \ll 0) \tag{2.7}$$

and that $g(x)$ is a finite sum of the functions $f = {}^t(f_1, f_2, \dots, f_n) \in \mathbb{C}_2^n[x]$ with the expansion at the origin

$$f_j(x) = x_\nu \sum_\gamma f_{j,\gamma} x^\gamma, \quad f_{j,\gamma} \in \mathbb{C}, \quad j = 1, 2, \dots, n, \tag{2.8}$$

where ν is such that the j-th and the ν-th components of Λ_S belong to the same Jordan block of Λ_N. We can prove that $\mathcal{A}_\pm$ are linear spaces. Then we have

Theorem 2.1. *Suppose that (2.3) holds. Let $R(x) \in \mathcal{A}_\pm$. Assume that every component of $R(x)$ is a polynomial of x with degree $< c+1$ if $\Lambda_N \neq 0$. Then the equation (2.5) has a holomorphic solution in some neighborhood of $x = 0$.*

Remark 2.2. We cannot weaken the assumptions of Theorem 2.1 in general. In fact, we encounter divergence caused by small denominators and a Jordan block. More precisely, we have

(a) If $c = 0$, then Theorem 2.1 does not hold in general because of small denominators. Namely, there exists R such that $L_{\Lambda_S} M(R) \gg 0$ (resp. $L_{\Lambda_S} M(R) \ll 0$) and (2.5) has a formal power series solution which does not converge in any neighborhood of the origin.

(b) The conditions $L_{\Lambda_S} M(R) \ll 0$ or $L_{\Lambda_S} M(R) \gg 0$ is necessary in general in order that Theorem 2.1 holds. In fact, there exists R which satisfies neither $L_{\Lambda_S} M(R) \ll 0$ nor $L_{\Lambda_S} M(R) \gg 0$ such that (2.5) has a formal power series solution which does not converge in any neighborhood of the origin.

(c) The condition that R is a polynomial in the case $\Lambda_N \neq 0$ is necessary in general. In fact, if $\Lambda_N \neq 0$, then there exists $R \in \mathcal{A}_+$ (resp. $R \in \mathcal{A}_-$) which is not a polynomial such that (2.5) has a formal power series solution which does not converge in any neighborhood of the origin.

Finally we note that because (2.5) has a resonance in general, the uniqueness of a solution in Theorem 2.1 does not hold in general.

3. Divergence and Diophantine Phenomena

We study divergence caused by small denominators and the presence of a Jordan block. We first consider in $x \in \mathbb{C}^2$

$$L_\Lambda u = R(x+u), \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -\tau \end{pmatrix}, \tag{3.1}$$

where $\tau > 0$ is a Liouville number chosen later and $u = O(|x|^2)$. Then we have

Theorem 3.1. *For every $R_1(x)$ holomorphic in some neighborhood of the origin, there exist a Liouville number $\tau > 0$ and a holomorphic perturbation $R_2(x) \not\equiv 0$ such that $L_{\Lambda_S} M(R_2) \gg 0$ or $L_{\Lambda_S} M(R_2) \ll 0$ holds and that the unique formal power series solution of (3.1) for $R = R_1 + R_2$ diverges in any neighborhood of the origin.*

Remark 3.2. We know that there exists a Liouville number τ such that for almost all R, (3.1) admits no analytic solution in any neighborhood of the origin. (cf. [3]) Our result implies that for every R_1 there exists a Liouville number τ and a special class of nonlinear perturbations R_2 such that (3.1) with $R = R_1 + R_2$ admits no analytic solution in any neighborhood of the origin. We remark that Theorem 3.1 can be generalized to the case of n-independent variables.

Next we study the divergence caused by the presence of a nontrivial Jordan block even if a Diophantine condition is verified. We consider in $x \in \mathbb{C}^3$

$$L_\Lambda u = R(x+u), \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\tau & -1 \\ 0 & 0 & -\tau \end{pmatrix}, \tag{3.2}$$

where $\tau > 0$ is an irrational number and $u = O(|x|^2)$. Then we have

Theorem 3.3. Let $c > 0$. For every irrational number $\tau > 0$ there exists $R \in \mathcal{A}_+$ (resp. $R \in \mathcal{A}_-$), $R \not\equiv 0$ such that R is not a polynomial and that (3.2) has no analytic solution in any neighborhood of the origin.

Theorem 3.3 shows that if $\Lambda_N \neq 0$, the condition that R is a polynomial in Theorem 2.1 is necessary in general. Theorem 3.3 can be generalized to the case of n-independent variables.

4. Discussions and problems

The main subject in this paper is the study of the divergence phenomena of the system of singular partial differential equations which appears in geometry when the Poincaré condition or the Diophantine condition is not satisfied. We expect that Theorems 3.1 and 3.3 can be extended to the case of more general nonlinear terms R with n-independent variables. (cf. [2], [3]). The divergence phenomenon caused by the presence of a nontrivial Jordan block in the linear part without the Poincaré condition is not fully understood.

Next it is interesting to study the divergent solutions from the analytical and geometrical points of view. It is natural to guess that one can give an analytical meaning to a divergent solution via the Borel-Laplace resummation method or more general resummation methods. Then one may discuss the geometrical meaning of the resummed divergent solutions. It is also an interesting question whether number theoretical aspects enter in the resummation argument. Concerning Theorem 2.1 we hope that we can gain a better understanding of divergence phenomena through the study of the criterions which guarantee the convergence of formal solutions that are different from the Poincaré condition and the Diophantine condition. Finally, we remark that the study of a divergence phenomenon caused by the existence of a zero eigenvalue is also an interesting problem. (cf. [1]).

Acknowledgements. The results of this note were reported in the conference, *Algebraic, Analytic and Geometric Aspects of Complex Differential Equations and their Deformations. Painlevé Hierarchies* held at RIMS Kyoto University from May 15 to May 20, 2006. The author would like to thank the organizers of the conference and Prof. Y. Takei for inviting him to this conference.

References

[1] Cicogna, G., Walcher, S.: Convergence of normal form transformations: the role of symmetries. Acta Appl. Math. **70**, no. 1-3, 95–111 (2002).

[2] Hibino, M.: Formal Gevrey theory for singular first order semi-linear partial differential equations. Osaka J. Math. **41** , no. 1, 159–191 (2004).

[3] Pérez Marco R.: Total convergence or small divergence in small divisors, Commun. Math. Phys. **223**:3, 451–464 (2001).

[4] Schmidt W.: Diophantine approximation. Lect. Notes in Mathematics. Springer Verlag, **785**. Berlin - Heidelberg - New York (1980).

[5] Stolovitch L.: Singular complete integrability. Publ. Math. I.H.E.S. **91**, 134–210 (2000).

Present adress: Graduate School of Sciences, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan.