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On the existence of ground states for the Pauli-Fierz model with a variable mass

By

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Abstract

The purpose of this paper is to review [9]. The existence of ground states of the Pauli-Fierz model with a variable mass is considered. This paper presents the outline of the proof of it under the infrared regularity condition.

§ 1. Introduction

The Pauli-Fierz model describes a minimal interaction between a low energy electron and a quantized radiation field, where the electron is governed by a Schrödinger operator. The Pauli-Fierz Hamiltonian is the physical quantity corresponding to the energy of the system and is realized as a self-adjoint operator on a certain Hilbert space and its bottom of the spectrum is called the ground state energy. An eigenvector associated with the ground state energy is called a ground state, if it exists.

The existence of ground states of the Pauli-Fierz Hamiltonian is investigated in [1, 2, 4, 8, 10, 12]. In [2, 8], the infrared regularity condition is not assumed. In [4, 8], the existence of ground states is shown for arbitrary values of coupling constants. The uniqueness of the ground state of the Pauli-Fierz Hamiltonian is proven in [11].

The Pauli-Fierz Hamiltonian with a variable mass is considered in this paper. It is derived from the analogy of the Nelson model on a pseudo Riemannian manifold [5, 6, 7]. Under the infrared regularity condition, this Hamiltonian has ground states for all values of a coupling constant when a variable mass decays sufficiently fast.

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§ 2. Definition of the Pauli-Fierz model

§ 2.1. Hilbert space of states

We consider the Hilbert space of states of total system as

$$\mathcal{H} := \mathcal{H}_P \otimes \mathcal{F},$$

where

$$\mathcal{H}_P := L^2(\mathbb{R}^3)$$

describes state space of one electron and $\mathcal{F}$ is the boson Fock space over $L^2(\mathbb{R}^3; \mathbb{C}^2)$ defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2) \right].$$

Here $\bigotimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2)$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^3; \mathbb{C}^2)$ with $\bigotimes_s^0 L^2(\mathbb{R}^3; \mathbb{C}^2) = \mathbb{C}$. The inner product on $\mathcal{F}$ is given by

$$(\Psi,\, \Phi)_{\mathcal{F}} = \overline{\Psi^{(0)}}\Phi^{(0)} + \sum_{n=1}^{\infty} \int_{\mathbb{R}^{3n}} \overline{\Psi^{(n)}(k_1, \cdots, k_n)} \Phi^{(n)}(k_1, \cdots, k_n) dk_1 \cdots dk_n. \tag{2.1}$$

The Hilbert space $\mathcal{H}$ can be identified with

$$\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx \cong L^2(\mathbb{R}^3) \oplus \left[\bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{3+3n}; \mathbb{C}^2) \right]. \tag{2.2}$$

Here $L^2_{\mathrm{sym}}(\mathbb{R}^{3+3n}; \mathbb{C}^2)$ is the set of $L^2(\mathbb{R}^{3+3n}; \mathbb{C}^2)$-functions such that

$$f(x, k_1, \cdots, k_n) = f(x, k_{\sigma(1)}, \cdots, k_{\sigma(n)})$$

for an arbitrary permutation σ.

Let T be a densely defined closable operator on $L^2(\mathbb{R}^3; \mathbb{C}^2)$. Then $\Gamma(T)$ and $d\Gamma(T)$ are defined by

$$\Gamma(T) := \oplus_{n=0}^{\infty} \otimes^n T, \qquad d\Gamma(T) := \oplus_{n=0}^{\infty} \otimes^n T^{(n)}, \tag{2.3}$$

where $\otimes^0 T = 1$, $T^{(n)} := \overline{\sum_{k=1}^n 1 \otimes \cdots 1 \otimes \overset{kth}{T} \otimes 1 \cdots \otimes 1}$ and $T^{(0)} = 0$. The number operator is defined by

$$N := d\Gamma(1).$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ smeared by $f \in L^2(\mathbb{R}^3;\mathbb{C}^2)$ on $\mathcal{F}$ are defined by

$$D\left(a^\dagger(f)\right) = \left\{\Psi \in \mathcal{F} \;\Big|\; \sum_{n=1}^{\infty} n \left\| S_n(f \otimes \Psi^{(n-1)}) \right\|^2 < \infty \right\}, \tag{2.4}$$

$$\left(a^\dagger(f)\Psi\right)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), n \geq 1, \quad \left(a^\dagger(f)\Psi\right)^{(0)} = 0, \tag{2.5}$$

$$a(f) = (a^\dagger(\overline{f}))^*, \tag{2.6}$$

where S_n denotes the symmetrization operator of degree n and $D(T)$ the domain of T. $\Omega := (1, 0, 0, \cdots) \in \mathcal{F}$ is called the Fock vacuum. Let

$$(a(k)\,\Psi)^{(n)}(k_1, \cdots, k_n) := \sqrt{n+1}\, \Psi^{(n+1)}(k, k_1, \cdots, k_n) \tag{2.7}$$

for $\Psi \in D(N^{1/2})$. Then for almost every k, $a(k)\Psi \in \mathcal{F}$.

§ 2.2. Definition of the Pauli-Fierz model

Let v be a multiplication operator on $L^2(\mathbb{R}^3)$. We introduce assumptions on v.

Assumption 1.

(1) $\sigma_P(-\Delta + v) \subset (0, \infty)$*;*

(2) $v(x) \leq \text{const.} \langle x \rangle^{-\beta}$ *with* $\beta > 3$*, where* $\langle x \rangle = \sqrt{1+|x|^2}$.

Here $\sigma_P(T)$ *denotes the set of eigenvalues of* T*.*

Then there exists a unique function $\Psi(k, x)$ such that for $k \neq 0$,

$$(-\Delta_x + v(x))\, \Psi(k, x) = |k|^2 \Psi(k, x) \tag{2.8}$$

and $\Psi(k, x)$ satisfies the Lippman-Schwinger equation:

$$\Psi(k, x) = e^{ikx} - \frac{1}{4\pi} \int \frac{e^{i|k||x-y|} v(y)}{|x-y|} \Psi(k, y) dy. \tag{2.9}$$

We will use the regularity properties of $\Psi(k, x)$ below to show the existence of ground states.

Lemma 2.1. *Suppose Assumption 1. Then*

(a)

$$|\, \Psi(k, x) - e^{ikx} \,| \leq \text{const.}\ \langle x \rangle^{-1} \tag{2.10}$$

holds.
(b) $\Psi(k,x)$ *is continuously differentiable in* x *for each fixed* k *but* $k \neq 0$ *and*

$$\begin{aligned}(2.11)\quad & \frac{\partial}{\partial x_\mu}\Psi(k,x) - ik_\mu e^{ikx} \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{e^{i|k||x-y|}(x_\mu - y_\mu)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x_\mu - y_\mu)}{|x-y|^2}\right)v(y)\Psi(k,y)dy.\end{aligned}$$

In particular, for any compact set D *but* $0 \notin D$, $\sup_{k\in D,x}\left|\frac{\partial \Psi}{\partial x_\mu}(k,x)\right| < \infty$.
(c) For $k \neq 0$ *and* $k + h \neq 0$,

$$\frac{1}{|h|}|\Psi(k+h,x) - \Psi(k,x)| \leq \text{const.}(\,1 + |x|\,), \tag{2.12}$$

$$\frac{1}{|h|}\left|\frac{\partial}{\partial x_\nu}\Psi(k+h,x) - \frac{\partial}{\partial x_\nu}\Psi(k,x)\right| \leq \text{const.}(\,1 + |k| + |x| + |k|\,|x|\,) \tag{2.13}$$

hold, and $\Psi(k,x)$ *and* $\frac{\partial}{\partial x_\nu}\Psi(k,x)$ *are differentiable in* $k \in \mathbb{R}^3 \setminus \{0\}$ *for each fixed* x.

Let us introduce the dispersion relation and the quantized radiation field with a variable mass v.

Definition 2.2. The dispersion relation with a variable mass is given by

$$\hat{\omega} := \sqrt{-\Delta + v} \tag{2.14}$$

on $L^2(\mathbb{R}^3;\mathbb{C}^2)$, where v is called a variable mass. The free Hamiltonian is defined by the second quantization of $\hat{\omega}$:

$$H_{\mathrm{f}} = d\Gamma(\hat{\omega}). \tag{2.15}$$

Let $m \geq 0$ and $\hat{\omega}_m := \sqrt{-\Delta + v + m^2}$. We set

$$H_{\mathrm{f}}(m) = d\Gamma(\hat{\omega}_m).$$

In order to define the quantized radiation field, we introduce a cutoff functions: $\hat{\varphi}_j^\mu$, $j = 1, 2$, $\mu = 1, 2, 3$.

Assumption 2.

(1) *The support of* $\hat{\varphi}_j^\mu$ *is compact;*

(2) $\hat{\varphi}_j^\mu$ *is differentiable and the derivative function is bounded;*

(3) (infrared regularity condition)
It holds that

$$\int_{\mathbb{R}^3}\frac{|\hat{\varphi}_j^\mu(k)|^{2p}}{|k|^{5p}}dk < \infty \quad \textit{for all} \quad 0 < p < 1. \tag{2.16}$$

Let the test function $\rho_x^\mu = (\rho_x^{\mu,1}, \rho_x^{\mu,2}) \in L^2(\mathbb{R}^3;\mathbb{C}^2) \in L^2(\mathbb{R}^3;\mathbb{C}^2)$ be such that

$$\rho_x^{\mu,j}(y) := (2\pi)^{-3/2} \int \overline{\Psi(k,x)}\Psi(k,y)\hat{\varphi}_j^\mu(k)dk.$$

The quantized radiation field with a variable mass is given by

$$A_\mu(x) := \frac{1}{\sqrt{2}}\left(a^\dagger\left(\hat{\omega}^{-1/2}\rho_x^\mu\right) + a\left(\overline{\hat{\omega}^{-1/2}\rho_x^\mu}\right)\right), \quad \mu = 1,2,3, \tag{2.17}$$

for each $x \in \mathbb{R}^3$.

Definition 2.3. Let V be a multiplication operator, and V_+ and V_- the positive part and the negative part of V, respectively. Then the quadratic form q_m^V is defined by

$$\begin{aligned} q_m^V(\Psi,\Phi) = &\frac{1}{2}\sum_{\mu=1}^{3}\left((p_\mu + \sqrt{\alpha}A_\mu)\Psi, (p_\mu + \sqrt{\alpha}A_\mu)\Phi\right) \\ &+ \left(H_{\mathrm{f}}^{1/2}(m)\Psi,\, H_{\mathrm{f}}^{1/2}(m)\Phi\right) + \left(V_+^{1/2}\Psi,\, V_+^{1/2}\Phi\right) - \left(V_-^{1/2}\Psi,\, V_-^{1/2}\Phi\right) \end{aligned} \tag{2.18}$$

with the form domain

$$Q(q_m^V) = D(|p|) \cap D(H_{\mathrm{f}}^{1/2}(m)) \cap D(|V|^{1/2}). \tag{2.19}$$

Here α is a coupling constant. When $m = 0$, we denote q^V for q_0^V.

§ 2.3. Generalized Fourier transformation

By [14], under Assumption 1, the generalized Fourier transformation is defined by

$$f \mapsto \mathcal{F}f(\cdot) := (2\pi)^{-3/2}\mathrm{l.i.m.}\int f(x)\overline{\Psi(\cdot,x)}dx, \tag{2.20}$$

which is a unitary transformation on $L^2(\mathbb{R}^3)$. By $1 \otimes \Gamma(\mathcal{F}) : \mathcal{H} \to \mathcal{H}$, the quadratic form q_m^V is transformed as

$$\begin{aligned} \hat{q}_m^V(\Psi,\,\Phi) &= q_m^V(1 \otimes \Gamma(\mathcal{F})\,\Psi,\, 1 \otimes \Gamma(\mathcal{F})\,\Phi) \\ &= \frac{1}{2}\sum_{\mu=1}^{3}\left((p_\mu + \sqrt{\alpha}\hat{A}_\mu)\Psi,\,(p_\mu + \sqrt{\alpha}\hat{A}_\mu)\Phi\right) + \left(\hat{H}_{\mathrm{f}}^{1/2}(m)\Psi,\,\hat{H}_{\mathrm{f}}^{1/2}(m)\Phi\right) \\ &\quad + \left(V_+^{1/2}\Psi,\,V_+^{1/2}\Phi\right) - \left(V_-^{1/2}\Psi,\,V_-^{1/2}\Phi\right) \end{aligned} \tag{2.21}$$

with the form domain

$$Q(\hat{q}_m^V) = D(|p|) \cap D(\hat{H}_{\mathrm{f}}^{1/2}(m)) \cap D(|V|^{1/2}). \tag{2.22}$$

Here

$$\hat{A}_\mu(x) := \frac{1}{\sqrt{2}} \sum_{j=1,2} \left(a^\dagger \left(\overline{\frac{\hat{\varphi}_j^\mu \Psi(\cdot, x)}{\sqrt{\omega}}} \right) + a \left(\frac{\hat{\varphi}_j^\mu \Psi(\cdot, x)}{\sqrt{\omega}} \right) \right), \quad \omega(k) = |k|, \tag{2.23}$$

and

$$\hat{H}_{\mathrm{f}}(m) := d\Gamma(\omega_m), \quad \omega_m(k) := \sqrt{k^2 + m^2}. \tag{2.24}$$

We introduce following assumptions on V:

Assumption 3.

(1) *V is a measurable function and for almost every $x \in \mathbb{R}^3$, $-\infty < V(x) < \infty$;*

(2) *For all $\epsilon > 0$, there exists a positive constant C_ϵ such that for $\Psi \in D(|p|)$,*

$$\| V_-^{1/2} \Psi \|^2 \leq \epsilon \| \, |p| \Psi \|^2 + C_\epsilon \| \Psi \|^2; \tag{2.25}$$

(3) *$Q(\hat{q}_m^V)$ is dense.*

Proposition 2.4. *Suppose Assumptions 1, 2 and 3. Then there exists the unique self-adjoint operator $\hat{H}_m^V$ such that $Q(\hat{q}_m^V) = D(|\hat{H}_m^V|^{1/2})$ and for all Ψ and $\Phi \in Q(\hat{q}_m^V)$,*

$$\hat{q}_m^V(\Psi, \Phi) - E^V(m)(\Psi, \Phi) = \Big((\hat{H}_m^V - E^V(m))^{1/2} \Psi, (\hat{H}_m^V - E^V(m))^{1/2} \Phi \Big).$$

Here we denote the ground state energy of $\hat{q}_m^V$ by

$$E^V(m) := \inf_{\Psi \in Q(\hat{q}_m^V), \|\Psi\|=1} \hat{q}_m^V(\Psi, \Psi). \tag{2.26}$$

Formally, the Pauli-Fierz Hamiltonian H_m^V is given by

$$H_m^V := \frac{1}{2} \sum_{\mu,\nu} \left(p_\mu + \sqrt{\alpha} A_\mu \right) a_{\mu\nu} \left(p_\nu + \sqrt{\alpha} A_\nu \right) + H_{\mathrm{f}}(m) + V. \tag{2.27}$$

Here $\{a_{\mu,\nu}\}_{\mu,\nu=1,2,3} = \{a_{\mu,\nu}(x)\}_{\mu,\nu=1,2,3}$ is positive definite. We consider only the case of $a_{\mu\nu}(x) = \delta_{\mu,\nu}$ for simplicity.

§ 3. Binding condition

We introduce functions ϕ_R and $\tilde{\phi}_R$ below. Let $\phi \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \leq \phi(x) \leq 1$ and

$$\phi(x) = \begin{cases} 1 \text{ if } |x| < 1, \\ 0 \text{ if } |x| > 2. \end{cases}$$

Let $\tilde{\phi} \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \le \tilde{\phi}(x) \le 1$ and

$$\phi(x)^2 + \tilde{\phi}(x)^2 = 1.$$

We set for $R > 0$,

$$\phi_R(x) := \phi(x/R), \qquad \tilde{\phi}_R(x) := \phi(x/R). \tag{3.1}$$

Let

$$E^V(R,\, m) = \inf_{\|\, \tilde{\phi}_R \Psi \,\| = 1, \Psi \in D(\hat{H}^V_m)} (\, \tilde{\phi}_R \Psi,\, \hat{H}^V_m \tilde{\phi}_R \Psi \,). \tag{3.2}$$

$\lim_{R\to\infty} E^V(R,\, m) - E^V(m)$ formally describes ionization energy by definition, it is expected that positive ionization energy yields ground state.

Assumption 4 (Binding condition)**.**

$$E^V(m) < \lim_{R\to\infty} E^V(R,\, m). \tag{3.3}$$

§ 4. Massive case

The existence of ground states in the case of $m > 0$ is considered in this section.

Theorem 4.1. *Let $m > 0$. Suppose Assumptions 1-4. Then ground states of $\hat{H}^V_m$ exist for all values of a coupling constant.*

Outline of Proof. Let $\{\Psi^j\}_j \subset Q(\hat{q}^V_m)$ be a sequence such that weakly converges to 0. It suffices to show that

$$\liminf_{j\to\infty} \hat{q}^V_m(\Psi^j,\, \Psi^j) > E^V(m). \tag{4.1}$$

We can suppose that $\sup_j \hat{q}^V_m(\Psi^j, \Psi^j) < \infty$. Let ϕ_R and $\tilde{\phi}_R$ be in (3.1).

$$\begin{aligned} \hat{q}^V_m(\,\Psi^j,\, \Psi^j\,) = {} & \hat{q}^V_m(\,\Psi^j_R,\, \Psi^j_R\,) + \hat{q}^V_m(\,\tilde{\Psi}^j_R,\, \tilde{\Psi}^j_R\,) \\ & -\frac{1}{2}\|\,(|\nabla\phi_R| \otimes 1)\Psi^j\,\|^2 - \frac{1}{2}\|\,(|\nabla\tilde{\phi}_R| \otimes 1)\Psi^j\,\|^2. \end{aligned} \tag{4.2}$$

holds. Here $\Psi^j_R = \phi_R\Psi^j$ and $\tilde{\Psi}^j_R = \tilde{\phi}_R\Psi^j$. Let j_1 and j_2 be nonnegative, smooth functions on $\mathbb{R}^3$ such that

$$j_1(k) = \begin{cases} 1 \text{ if } |k| < 1, \\ 0 \text{ if } |k| > 2 \end{cases} \quad \text{and} \quad j_1(k)^2 + j_2(k)^2 = 1. \tag{4.3}$$

We set $\hat{j}_{l,P} = j_l(-i\nabla_k/P)$, $l = 1, 2$, and

$$\hat{j}_P\Psi = \hat{j}_{1,P}\Psi \oplus \hat{j}_{2,P}\Psi, \tag{4.4}$$

for $\Psi \in L^2(\mathbb{R}^3; \mathbb{C}^2)$. Let us define the isometric operator from $\mathcal{F}$ to $\mathcal{F} \otimes \mathcal{F}$ by

$$\begin{aligned} &d\check{\Gamma}(\hat{j}_P)a^\dagger(h_1)\cdots a^\dagger(h_n)\Omega \\ &= a^\dagger(\hat{j}_{1,P}h_1)\cdots a^\dagger(\hat{j}_{1,P}h_n)\Omega \oplus a^\dagger(\hat{j}_{2,P}h_1)\cdots a^\dagger(\hat{j}_{2,P}h_n)\Omega. \end{aligned} \tag{4.5}$$

By the localization argument (see [8]), it holds that

$$\liminf_{j\to\infty} \hat{q}_m^V(\Psi_R^j, \Psi_R^j) \geq (E^V(m) + m) \liminf_{j\to\infty} \|\Psi_R^j\|^2 + o_R(P^0) \tag{4.6}$$

and

$$\hat{q}_m^V(\, \tilde{\Psi}_R^j,\, \tilde{\Psi}_R^j\,) \geq E_{R,m}^V \,\|\, \tilde{\Psi}_R^j \,\|^2 + o(R^0). \tag{4.7}$$

Here $o_R(P^0)$ goes to zero as $P \to \infty$ for each fixed $R > 0$. By (4.2), (4.6) and (4.7), we can see that

$$\liminf_{j\to\infty} \hat{q}_m^V(\Psi^j, \Psi^j) \geq E^V(m) + \min\{\, m,\, E^V(R,m) - E^V(m)\,\}. \tag{4.8}$$

By the binding condition, we obtain (4.1). □

§ 5. The case of $m = 0$

Throughout in this section, we suppose Assumptions 1, 2, 3 and Assumption 4 with $m = 0$. Φ_m denotes the normalized ground state of $\hat{H}_m^V$. Similarly to the case of $v = 0$, the following lemma holds.

Lemma 5.1. *Let $\{m_j\}_{j=1}^\infty$ be a sequence converging to 0. Then*

$$\lim_{j\to\infty} E^V(m_j) = E^V(0)$$

and for sufficiently small $0 < m$, the binding condition holds.

The pull through formula below leads to a photon number bound (Lemma 5.3 and Corollary 5.4) and a photon derivative bound (Lemma 5.6).

Lemma 5.2 (Pull through formula)**.** *Let $f \in D(\omega_m)$. Then $a(f)\Phi_m \in Q(\hat{q}_m^V)$ and for all $\eta \in Q(\hat{q}_m^V)$,*

$$\begin{aligned} &\hat{q}_m^V(\,\eta,\, a(f)\,\Phi_m\,) - E^V(m)(\,\eta,\, a(f)\,\Phi_m\,) \\ &= -\sqrt{\alpha}\,(\,\eta,\, (\overline{f},\, \overline{G}) \cdot (\,p + \sqrt{\alpha}\hat{A}\,)\,\Phi_m\,) + \frac{i\sqrt{\alpha}}{2}(\,\eta, (\overline{f},\, \nabla_x \cdot \overline{G}\,)\,\Phi_m\,) - (\,\eta, a(\omega_m f)\,\Phi_m\,). \end{aligned} \tag{5.1}$$

holds. Here

$$G_j^\mu(k,x) := \frac{\hat{\varphi}_j^\mu(k)\Psi(k,x)}{\sqrt{2\omega(k)}}.$$

Lemma 5.3. *Let $\theta = (\theta_1,\, \theta_2) \in L^\infty(\mathbb{R}^3; \mathbb{R}^2)$. Then*

$$\| \, d\Gamma(\, \theta^2 \,)^{1/2} \, \Phi_m \, \|^2 \le C\alpha \sum_{\mu,j} \int \frac{\hat{\varphi}_j^\mu(k)^2 \, \theta_j(k)^2}{\omega(k)\, \omega_m(k)^2} \, dk, \tag{5.2}$$

where C is a constant independent of α and sufficiently small m.

Outline of proof of Lemma 5.3. Inserting $\eta = a(f)\Phi_m$ into (5.2), we have

$$\begin{aligned} (\, a(f)\, \Phi_m,\, a(\omega_m f)\, \Phi_m \,) \le & -\sqrt{\alpha} \left(a(f)\, \Phi_m,\, (\overline{f}, \overline{G}) \cdot (\, p + \sqrt{\alpha}\hat{A} \,)\, \Phi_m \right) \\ & + \frac{i\sqrt{\alpha}}{2} \left(a(f)\, \Phi_m, (\overline{f},\, \nabla_x \cdot \overline{G} \,)\, \Phi_m \,\right). \end{aligned} \tag{5.3}$$

Let $f := \omega_m \theta g_i$. Here $\{g_i\}_{i=1}^\infty$ is a complete orthonormal system such that each $g_i \in D(\omega_m^{1/2})$. Note that

$$\begin{aligned} & \sum_{i=1}^\infty \left(a(\omega_m^{-1/2}\, \theta\, g_i)\, \Phi_m,\, a(\omega_m^{1/2}\, \theta\, g_i)\, \Phi_m \right) \\ & = \sum_{j=1,2} \int_{\mathbb{R}^3} \theta_j(k)^2 \| \, a_j(k)\Phi_m \, \|^2 \, dk = \left\| \, d\Gamma(\, \theta^2 \,)^{1/2} \, \Phi_m \, \right\|^2. \end{aligned} \tag{5.4}$$

Then by (5.3) and (5.4),

$$\begin{aligned} & \| \, d\Gamma(\, \theta^2 \,)^{1/2} \, \Phi_m \, \|^2 \\ & \le \; 2\alpha \int_{\mathbb{R}^3} \omega_m(k)^{-2} \, \| \, \theta(k)G(k) \cdot (p + \sqrt{\alpha}\hat{A})\, \Phi_m \, \|^2 \, dk \\ & \quad + \frac{\alpha}{2} \int_{\mathbb{R}^3} \omega_m(k)^{-2} \, \| \, \theta(k)\nabla_x \cdot G(k)\, \Phi_m \, \|^2 \, dk. \end{aligned} \tag{5.5}$$

can be estimated. Since $\Psi(k,\, x)$ and $\hat{\varphi}(k)\frac{\partial}{\partial x_\mu}\Psi(k,\, x)$ are bounded in k and x, we can see that the lemma follows. □

From Lemma 5.3, we can see that following facts hold.

Corollary 5.4. *It holds that*

(1) $\sup_{m<m_0} \|N^{1/2}\Phi_m\| < \infty,$

(2) $\operatorname{supp} \Phi_m^{(n)}(x,\cdot) \subset \Pi_{k=1}^n \left[\cup_{j,\mu} \operatorname{supp} \hat{\varphi}_j^\mu \right].$

We can show the spatial exponentially decay of Φ_m for many external potentials. See [13].

Assumption 5.

(1) *For sufficiently large* $|x|$, $V(x) > \text{const.}|x|^{2n}$.

(2) $\liminf_{|x|\to\infty} V(x) > \inf \sigma(H_p)$ *and for all* $t > 0$, $e^{-tH_P} : L^2 \to L^\infty$ *with*

$$\|e^{-tH_p} f\|_{L^\infty(\mathbb{R}^3)} \leq \text{const.}\|f\|_{L^2(\mathbb{R}^3)},$$

where $H_P = -\frac{1}{2}\Delta + V$.

Theorem 5.5. *Suppose Assumption 5. Then for some* c *and* $m_0 > 0$,

$$\sup_{0<m<m_0} \| \exp(c|x|)\Phi_m \| < \infty. \tag{5.6}$$

holds.

Outline of Proof. Since $\Phi_m = e^{tE} e^{-t\hat{H}_m^V}\Phi_m$, by the functional integral representation of $e^{-t\hat{H}_m^V}$, we can see that for all $t \geq 0$,

$$\|\Phi_m(x)\|_{\mathcal{F}} \leq C e^{tE^V(m)} \mathbb{E}^x \left[e^{-\int_0^t V(B_s)ds} \right] \tag{5.7}$$

holds. Here $(B_t)_{t\geq 0}$ denotes Brownian motion starting from x. C is a constant independent of x and m.

$$e^{t(x)E^V(m)} \mathbb{E}^x \left[e^{-\int_0^{t(x)} V(B_s)ds} \right] \leq C_1 \exp(-C_2|x|^{n+1}) \tag{5.8}$$

and

$$e^{t'(x)E^V(m)} \mathbb{E}^x \left[e^{-\int_0^{t'(x)} V(B_s)ds} \right] \leq C_1' \exp(-C_2'|x|) \tag{5.9}$$

hold. Here $t(x) = |x|^{1-n}$, $t'(x) = \beta|x|$. (5.8) and (5.9) are called Carmona's estimate [3]. By (5.7), (5.8) and (5.9), the theorem can be proven.

Lemma 5.6. *Suppose Assumption 5. Let* $1 \leq p < 2$. *Then*
(a) $\Phi_m^{(n)} \in H^1(\mathbb{R}^{3+3n})$ *for all* $n \geq 0$;
(b) $\{\|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)}\}_{0<m\leq m_0}$ *is bounded, where* m_0 *is sufficiently small number and* Ω *is any measurable and bounded set in* $\mathbb{R}^{3+3n}$.
Here $W^{1,p}(\Omega)$ *is the Sobolev space.*

Outline of proof. Let $f = \omega_m^{-1/2}\theta g_i$. By the pull through formula with $f(x)$ replaced by $f(x+h) - f(x)$, similarly to the proof of Lemma 5.3, we can see that for almost every k and sufficiently small h,

$$\begin{aligned}(5.10)\ & \| \, |h|^{-1}(a(k+h)-a(k))\,\Phi_m \,\|^2 \\ & \le \frac{\text{const.}}{\omega_m(k)^2}\Big(\sum_{\mu=1}^{3}(\, \| \, |h|^{-1}(\delta_h G_\mu)(k)\Phi_m \,\|^2 + \| \, | \, |h|^{-1}(\nabla_x \delta_h G_\mu)(k) \, | \, \Phi_m \,\|^2 \,) \\ & \quad + \| \, |h|^{-1} \, (\nabla_x \cdot \delta_h G)(k) \, \Phi_m \,\|^2 + \sum_{j,\mu} \frac{\hat{\varphi}_j^\mu(k+h)^2}{\omega_m(k+h)^2\,\omega(k+h)} \frac{|\omega(k+h)-\omega(k)|^2}{|h|^2} \Big).\end{aligned}$$

Here $(\delta_h G_\mu)(k) = G_\mu(k+h,x) - G_\mu(k,x)$. By Lemma 2.1 (c) and Assumption 5,

$$\begin{aligned}(5.11)\qquad & \| \, |h|^{-1}(a(k+h)-a(k))\,\Phi_m \,\|^2 \\ & \le C\omega_m(k)^{-2}\sum_{\nu,j}\left((1+|k|^{-3})\hat{\varphi}_j^\nu(k)^2 + |k|^{-1}\sum_\lambda | \, \partial_\lambda \hat{\varphi}_j^\nu(k) \, |^2 \right)\end{aligned}$$

holds for almost every k and sufficiently small $|h|$. Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Thus by Alaoglu theorem, for almost every k, there exists the sequence $\{h_l(k)\}_{l=1}^\infty$ depending on k so that

$$\lim_{l\to\infty} h_l(k) = 0$$

and $|h_l(k)|^{-1}(a(k-|h_l(k)|e_\mu) - a(k))\Phi_m$ weakly converges to some vector $v_\mu(k)$:

$$v_\mu(k) := \text{w-}\lim_{l\to\infty} |h_l(k)|^{-1}(a(k-|h_l(k)|e_\mu)-a(k))\Phi_m.$$

It can be proven that $v_\mu^{(n)}(k)(x,k_1,\cdots,k_n)$ is the weak derivative $\Phi_m^{(n+1)}(x,k,k_1,\cdots,k_n)$ with respect to k_μ. Thus by (5.11), (a) and (b) are proven directly. □

Theorem 5.7. *Let $m=0$. Suppose Assumption 5. Then ground states of $\hat{H}^V$ exist for all values of a coupling constant.*

By Lemmas 5.3, 5.6 and Theorem 5.5, Theorem 5.7 can be proven similarly to [8, Theorem2.1].

§ 6. Remarks on infrared cutoffs

We assumed the infrared regularity condition, but in the case of $v=0$, we can show the existence of ground states of $\hat{H}$ without the infrared regularity condition. In the case of $v \ne 0$,

$$(6.1)\quad \Psi(k,x) - e^{ikx} = \sum_{n=1}^{\infty}\left(\frac{1}{4\pi}\right)^n \int_{\mathbb{R}^3} \frac{e^{i|k|\sum_{j=1}^n |y_j - y_{j-1}|}\Pi_{j=1}^n v(y_j)}{\Pi_{j=1}^n |y_j - y_{j-1}|}\, dy_1\cdots dy_n$$

and

$$\nabla_x \Psi(k,x) - ike^{ikx} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \Big(\frac{e^{i|k||x-y|}(x-y)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x-y)}{|x-y|} \Big) v(y)\,\Psi(k,y)\,dy$$

hold. Here $y_0 := x$. The right hand side of (6.1) is not $O(|k|)$, $(k \to 0)$. This is the reason that we assumed the infrared regularity condition. To see this, let us consider the case of $v = 0$. Set $v = 0$ and $\hat{\varphi}_j^\mu(k) = \chi_\Lambda(k) e_j^\mu(k)$, $j = 1,2$, $\mu = 1,2,3$, where χ_Λ is the characteristic function of the set $\{k|\ |k| < \Lambda\}$ and $e_1(k)$ and $e_2(k)$, $k \in \mathbb{R}^3 \setminus \{0\}$ are polarization vectors given by

$$e_1(k) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad e_2(k) := \frac{k \times e_1(k)}{|k|}. \tag{6.2}$$

Note that the infrared regularity condition is not assumed in this case. Define the unitary operator U as

$$U := \exp[i\sqrt{\alpha} x \cdot \hat{A}(0)]. \tag{6.3}$$

Put

$$\tilde{\hat{q}}_m^V(\Psi,\, \Phi) := \hat{q}_m^V(U\Psi,\, U\Phi) \tag{6.4}$$

and

$$\tilde{\hat{A}}(x) := \hat{A}(x) - \hat{A}(0). \tag{6.5}$$

Then

$$\begin{aligned} &\tilde{\hat{q}}_m^V(\Psi,\, a(f)\tilde{\Phi}_m) - E^V(m)(\Psi,\, a(f)\tilde{\Phi}_m) \\ &= -\sqrt{\alpha}(\Psi,\, (\overline{f},\, \overline{\tilde{G}})(p + \sqrt{\alpha}\tilde{A})\tilde{\Phi}_m) - (\Psi,\, a(\omega_m f)\tilde{\Phi}_m) + i(\Psi,\, (\overline{f},\, \omega_m w)\tilde{\Phi}_m) \end{aligned} \tag{6.6}$$

follows. Here $\tilde{\Phi}_m = U\Phi_m$, $w_j := \frac{\chi_\Lambda(k) e_j(k)\cdot x}{\sqrt{\omega(k)}}$ and $\tilde{G}_j^\mu := \frac{\chi_\Lambda(k) e_j^\mu(k)(e^{ikx}-1)}{\sqrt{2\omega(k)}}$. Similarly to the proof of Theorem 5.3, we have

$$\|\, a(k)\tilde{\Phi}_m \,\|^2 \le \text{const.}\omega(k)^{-2} \Big\{ \|\, \tilde{G}\tilde{\Phi}_m \|^2 + \|\, |\nabla_x \tilde{G}| \tilde{\Phi}_m \|^2 + \omega(k)^2 \|\, w\tilde{\Phi}_m \|^2 \Big\} \chi_\Lambda(k). \tag{6.7}$$

Since $|e^{ikx} - 1| \le |k||x|$ and $|\nabla_x e^{ikx}| = |k|$, by the exponential decay of $\tilde{\Phi}_m$, it holds that

$$\|\, d\Gamma(\, \theta^2\,)^{1/2}\, \tilde{\Phi}_m \,\|^2 \le C\alpha \sum_j \int \frac{\chi_\Lambda(k)\, \theta_j(k)^2}{\omega(k)}\, dk. \tag{6.8}$$

Here C is a constant independent of α and m for sufficiently small m. Also by (6.6), for almost every k and sufficiently small h,

$$\begin{aligned}(6.9)\qquad & \| (a(k+h)-a(k))\tilde{\Phi}_m \|^2 \\ & \leq \frac{\text{const.}}{\omega(k)^2}\Big\{ \| |\delta_h \tilde{G}|\tilde{\Phi}_m \|^2 + \| |\nabla_x \delta_h \tilde{G}|\tilde{\Phi}_m\|^2 + \omega(k)^2 \|\delta_h(\omega w)\tilde{\Phi}_m\|^2 \\ & \quad + \frac{1}{|k+h|} \| |x|\tilde{\Phi}_m \|^2 |\omega(k+h)-\omega(k)| \chi_\Lambda(k+h) \Big\}\end{aligned}$$

can be proven. Since $|e^{ikx}-1| \leq |k||x|$ and $|\nabla_x e^{ikx}| = |k|$, by Assumption 5, we can see that

$$\begin{aligned}(6.10)\qquad & |h|^{-1} \| (a(k+h)-a(k))\tilde{\Phi}_m \|^2 \\ & \leq \text{const.} \left\{ \frac{1}{|k|(k_1^2+k_2^2)} + \frac{1}{|k-h|((k_1-h_1)^2+(k_2-h_2)^2}\right\}\end{aligned}$$

holds. This inequality implies that $\{\| \tilde{\Phi}_m^{(n)} \|_{W^{1,p}(\Omega)}\}_{0<m\leq m_0}$ with $1 \leq p < 2$ is bounded, where Ω is a bounded set and m_0 a sufficiently small number. Therefore in this case, the existence of ground states can be proven without the infrared regularity condition. Key inequalities are

$$(6.11)\qquad |e^{ikx}-1| \leq |k||x|$$

and

$$(6.12)\qquad |\nabla_x e^{ikx}| = |k|.$$

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