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Large Deviations in Quantum Spin Chains

By

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Abstract

In this note, we review our recent work on large deviation principles in quantum spin chains.

§ 1. Large deviation principle

While a lot of things are known about the large deviation for classical systems, our knowledge on large deviations in quantum systems is still very restricted. The large deviation principle of the distribution of particle number for equilibrium states was shown for free fermion system in [1], and for dilute gas in [2]. Large deviation results in quantum spin system for observables that depend only on one site were established in high temperature KMS-states, in [3], using cluster expansion techniques. In [4], large deviation upper bounds were proven for general observables, for KMS-states in dimension one. Furthermore, it was shown that a state in one dimension, which satisfies a certain factorization property satisfies a large deviation upper bound [5]. This factorization property is satisfied by KMS-states as well as C^*-finitely correlated states. It was also shown in [5] that the distributions of the ergodic averages of a one-site observable with respect to an ergodic C^*-finitely correlated state satisfy full large deviation principle. In [6], we showed full large deviation principle for KMS-states and C^*-finitely correlated states on a quantum spin chain, with respect to general local observables.
The infinite spin chain with one site algebra $M_d(\mathbb{C})$ is given by the UHF C^*-algebra

$$\mathfrak{A}_{\mathbb{Z}} := \overline{\bigotimes_{\mathbb{Z}} M_d(\mathbb{C})}^{C^*},$$

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which is the C^*- inductive limit of the local algebras

$$\left\{\mathfrak{A}_\Lambda := \bigotimes_\Lambda M_d(\mathbb{C}) | \quad \Lambda \subset \mathbb{Z}, \quad |\Lambda| < \infty\right\}.$$

For any subset S of $\mathbb{Z}$, we identify $\mathfrak{A}_S := \overline{\bigotimes_S M_d(\mathbb{C})}^{C^*}$ with a subalgebra of $\mathfrak{A}_\mathbb{Z}$ under the natural inclusion. Here, $|\Lambda|$ denotes the number of elements in Λ. We use a notation $\mathcal{S}_0 := \{\Lambda \subset \mathbb{Z} : |\Lambda| < +\infty\}$. We also use a notation $\mathfrak{A}_n := \mathfrak{A}_{[-n,n]}$. The algebra of local observables is defined by

$$\mathfrak{A}_{loc} := \cup_{\Lambda \in \mathcal{S}_0} \mathfrak{A}_\Lambda.$$

Let $\gamma_j,\ j \in \mathbb{Z}$ be the j-lattice translation. A state ω is called translation-invariant if $\omega \circ \gamma_j = \omega$ for all $j \in \mathbb{Z}$. We denote the set of all translation-invariant states by $S_\gamma(\mathfrak{A})$. An interaction is a map Φ from $\mathcal{S}_0$ to $\mathfrak{A}_\mathbb{Z}$ such that $\Phi(X) = \Phi(X)^* \in \mathfrak{A}_X$ for any finite $X \subset \mathbb{Z}$. In this note, we will always assume that Φ is a finite range translation-invariant interaction, i.e., there exists $r \in \mathbb{N}$ such that

$$\Phi(X) = 0, \quad if \quad diam(X) > r,$$

and Φ is invariant under γ,

$$\Phi(X + j) = \gamma_j\left(\Phi(X)\right), \quad \forall j \in \mathbb{Z}, \quad \forall X \in \mathcal{S}_0.$$

We define the mean energy of Φ by

$$A_\Phi := \sum_{X \ni 0} \Phi(X).$$

For finite $\Lambda \subset \mathbb{Z}$, we define the local Hamiltonian

$$H_\Phi(\Lambda) := \sum_{I \subset \Lambda} \Phi(I).$$

Next we introduce KMS states. Let Ψ be a translation-invariant finite range interaction, and define the local Hamiltonian associated with a finite subset $\Lambda \subset \mathbb{Z}$ by

$$H_\Psi(\Lambda) := \sum_{I \subset \Lambda} \Psi(I).$$

It is known that there exists a strongly continuous one parameter group of $*$-automorphisms τ_Ψ on $\mathfrak{A}_\mathbb{Z}$, such that

$$\lim_{\Lambda \nearrow \mathbb{Z}} \left\| \tau_\Psi^t(A) - e^{itH_\Psi(\Lambda)} A e^{-itH_\Psi(\Lambda)} \right\| = 0, \quad \forall t \in \mathbb{R}, \quad \forall A \in \mathfrak{A}_\mathbb{Z}.$$

The equilibrium state corresponding to the interaction Ψ is characterized by the KMS condition. A state ω over $\mathfrak{A}_{\mathbb{Z}}$ is called a (τ_Ψ, β)-KMS state, if

$$\omega(A\tau_\Psi^{i\beta}(B)) = \omega(BA),$$

holds for any pair (A, B) of entire analytic elements for τ_Ψ. It is known that one dimensional quantum spin system has a unique (τ_Ψ, β)-KMS state.
Now we introduce the probability distribution of space average of local observables, with respect to ω_Ψ. The distribution of space average of local Hamiltonian, $\frac{1}{2n+1}H_\Phi([-n, n])$ with respect to a state ω_Ψ is given by the probability measure

$$\mu_n(B) := \omega_\Psi\left(1_B\left(\frac{1}{2n+1}H_\Phi([-n, n])\right)\right), \quad B \in \mathcal{B},$$

where $\mathcal{B}$ denotes the Borel sets of $\mathbb{R}$ and $1_B(\frac{1}{2n+1}H_\Phi([-n, n])) \in \mathfrak{A}_{[-n,n]}$ is the spectral projection of $\frac{1}{2n+1}H_\Phi([-n, n])$ corresponding to the set B.
Let $I : \mathbb{R} \to [0, \infty]$ be a lower semicontinuous mapping. We say that $\{\mu_n\}$ satisfies a large deviation principle with rate function I iff

$$-\inf_{x\in\Gamma^o} I(x) \leq \lim_{n\to\infty} \inf \frac{1}{n} \log \mu_n(\Gamma) \leq \lim_{n\to\infty} \sup \frac{1}{n} \log \mu_n(\Gamma) \leq -\inf_{x\in\bar{\Gamma}} I(x).$$

Furthermore, I is said to be a good rate function if all the level sets $\{x : I(x) \leq \alpha\}$, $\alpha \in [0, \infty)$ are compact subsets of $\mathbb{R}$(see [7]).
In [6], the full large deviation principle was shown for any kind of local observable Φ, in KMS-states generated by any finite range translation invariant interaction Ψ:

Theorem 1.1 (Ogata[6]). *Let Ψ be a translation-invariant finite range interaction, τ_Ψ associated C^*-dynamics and ω_Ψ a (τ_Ψ, β)-KMS state. Furthermore, let Φ be another translation-invariant finite range interaction and μ_n the distribution of $\frac{1}{2n+1}H_\Phi([-n, n])$ with respect to ω_Ψ. Then the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies large deviation principle with a good rate function I.*

In the proof, the existence and differentiability of the logarithmic moment generating function

$$f_\Phi(\alpha) := \lim_{N\to\infty} \frac{1}{2N+1} \log \omega_\Psi(e^{\alpha H_\Phi[-N,N]})$$

was proven. In fact, it is shown in the proof that $f_\Phi(\alpha)$ has an analytic extension at a neighborhood of the real line. Then by the Gärtner-Elllis Theorem, the large deviation principle holds. The the rate function $I(x)$ of large deviation principle is given by the Legendre transform of f_Φ:

$$I(x) = \sup_{\alpha\in\mathbb{R}} (\alpha x - f_\Phi(\alpha)).$$

The main method in [6] is the non-commutative Ruelle transfer operator method. The non-commutative Ruelle transfer operator was introduced by H. Araki [8] to study the Gibbs state in one dimensional quantum spin systems. It was generalized in [9] and [10]. In [6], we constructed a family of transfer operators $\{L_\alpha\}_{\alpha\in\mathbb{R}}$ acting on some Banach space F_θ which satisfies the following properties.:

1. L_α has an isolated eigenvalue $\lambda(\alpha) > 0$ which is equal to the spectral radius of L_α.The rest of the spectrum is included in a disc with radius strictly smaller than $\lambda(\alpha)$,
2. $\lambda(\alpha) = f(\alpha) := \lim_{n\to\infty} \frac{1}{n} \log \mu_n \left(e^{\alpha n x}\right)$,
3. $\mathbb{R} \ni \alpha \mapsto L_\alpha \in B(F_\theta)$ has an analytic extension.

By the regular perturbation theory, we can conclude that $\lambda(\alpha) = f_\Phi(\alpha)$ is differentiable. An alternative proof was introduced in [11].

§ 2. The rate function

The next problem is to characterize the rate function $I(x)$. In [12], $I(x)$ was characterized in terms of mean relative entropy associated with Φ. Let $\mathfrak{B}_n$ be an abelian subalgebra of $\mathfrak{A}_n$ including the local Hamiltonian $H_\Phi[-n,n]$. For any state ψ over $\mathfrak{A}$, There exists a unique density matrix $D_{\psi|_{\mathfrak{B}_n}} \in \mathfrak{B}_n$, satisfying

$$\psi(A) = Tr_{[-n,n]} D_{\psi|_{\mathfrak{B}_n}} A, \quad \forall A \in \mathfrak{B}_n.$$

The relative entropy between φ_1 and φ_2 with respect to $\mathfrak{B}_n$ is defined by

$$S(\varphi_1|_{\mathfrak{B}_n}, \varphi_2|_{\mathfrak{B}_n}) = Tr_{[-n,n]} D_{\varphi_1|_{\mathfrak{B}_n}} \left(\log D_{\varphi_1|_{\mathfrak{B}_n}} - \log D_{\varphi_2|_{\mathfrak{B}_n}}\right).$$

We define the mean relative entropy $S_{M,\Phi}(\varphi_1, \varphi_2)$ with respect to Φ between translation invariant states φ_1, φ_2 by

$$S_{M,\Phi}(\varphi_1, \varphi_2) := \liminf_{n\to\infty} \frac{1}{2n+1} S(\varphi_1|_{\mathfrak{B}_n}, \varphi_2|_{\mathfrak{B}_n}). \tag{2.1}$$

In terms of $S_{M,\Phi}$, we can characterize $I(x)$:

Theorem 2.1 (Ogata-ReyBellet[12]). *Let $x \in \mathbb{R}$. If the rate function $I(x)$ in Theorem 1.1 takes finite value, (i.e., $I(x) < +\infty$,) then*

$$I(x) = \inf_{\varphi\in S_\gamma(\mathfrak{A}), \varphi(A_\Phi)=x} S_{M,\Phi}(\varphi|_{\mathfrak{B}_n}, \omega_\Psi|_{\mathfrak{B}_n}). \tag{2.2}$$

Furthermore, there exists $\alpha = \alpha(x) \in \mathbb{R}$ such that the limit

$$\omega_x(Q) := \lim_{n_1,n_2\to\infty} \frac{\omega_\Psi(e^{\frac{\alpha}{2}H_\Phi[-n_1,n_2]}Qe^{\frac{\alpha}{2}H_\Phi[-n_1,n_2]})}{\omega_\Psi(e^{\alpha H_\Phi[-n_1,n_2]})}, \quad \forall Q \in \mathfrak{A}, \tag{2.3}$$

exists and defines a translation invariant state satisfying

$$\omega_x(A_\Phi) = x. \tag{2.4}$$

For this ω_α, the limit

$$\lim_{n\to\infty} \frac{1}{2n+1} S(\omega_x|_{\mathfrak{B}_n}, \omega_\Psi|_{\mathfrak{B}_n}) = S_{M,\Phi}(\omega_\alpha, \omega_\Psi) \tag{2.5}$$

exists and attains the infimum of (2.2):

$$I(x) = S_{M,\Phi}(\omega_\alpha, \omega_\Psi). \tag{2.6}$$

We call this ω_x, a tilted state.

§ 3. Examples

In this section, we consider special cases, namely, one site interaction and classical spin chain.

One site interaction Suppose that the interactions are one-site, i.e., $\Phi(I), \Psi(I) \neq 0$ only if $I = \{n\}$ for some $n \in \mathbb{Z}$. Then, the moment generating function is given by

$$\begin{aligned} f_\Phi(\alpha) &= \lim_{n\to\infty} \frac{1}{2n+1} \log \omega_\Psi \left(e^{\alpha H_\Phi[-n,n]} \right) \\ &= \log Tr_{\{0\}} \left(e^{-\beta\Psi(0)} e^{\alpha\Phi(0)} \right) - \log Tr_{\{0\}} \left(e^{-\beta\Psi(0)} \right). \end{aligned}$$

Furthermore, the tilted state is given by a product state:

$$\omega_x(Q) := \lim_{n_1,n_2\to\infty} \frac{\omega_\Psi(e^{\frac{\alpha}{2}H_\Phi[-n_1,n_2]}Qe^{\frac{\alpha}{2}H_\Phi[-n_1,n_2]})}{\omega_\Psi(e^{\alpha H_\Phi[-n_1,n_2]})} = \bigotimes_{n\in\mathbb{Z}} \varphi_x(Q).$$

Here, φ_x is a state over $M_d(\mathbb{C})$ given by

$$\varphi_x(A) = \frac{Tr_{\{0\}} \left(e^{-\beta\Psi(0)} e^{\frac{\alpha}{2}\Phi(0)} A e^{\frac{\alpha}{2}\Phi(0)} \right)}{Tr_{\{0\}} \left(e^{-\beta\Psi(0)} \right)}, \quad A \in M_d(\mathbb{C})$$

$$\varphi_x(\Phi(0)) = x.$$

Classical Spin chain Suppose the interactions Φ and Ψ are given by elements of abelian subalgebra of $\mathfrak{A}$. More precisely, let $\mathcal{A}$ be an abelian subalgebra of $M_d(\mathbb{C})$ and assume that $\Phi(I), \Psi(I) \in \otimes_I \mathcal{A}$, for any I. This corresponds to a classical spin chain. In this case, the moment generating function is given by

$$\begin{aligned}
f_\Phi(\alpha) &= \lim_{n\to\infty} \frac{1}{2n+1} \log \omega_\Psi \left(e^{\alpha H_\Phi[-n,n]} \right) \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \begin{pmatrix} \log Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n]} e^{\alpha H_\Phi[-n,n]} \right) \\ - \log Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n]} \right) \end{pmatrix} \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \begin{pmatrix} \log Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n] + \alpha H_\Phi[-n,n]} \right) \\ - \log Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n]} \right) \end{pmatrix}.
\end{aligned}$$

Note that this is a difference of free energies. Therefore, it turns out that in classical case, the moment generating functions are given by the difference of free energies. Furthermore, the tilted state turns out to be a Gibbs state for another interaction:

$$\begin{aligned}
\omega_x(Q) &:= \lim_{n_1,n_2\to\infty} \frac{\omega_\Psi(e^{\frac{\alpha}{2} H_\Phi[-n_1,n_2]} Q e^{\frac{\alpha}{2} H_\Phi[-n_1,n_2]})}{\omega_\Psi(e^{\alpha H_\Phi[-n_1,n_2]})} \\
&= \lim_{n\to\infty} \frac{Tr_{[-n,n]} \left(e^{\beta H_\Psi[-n,n]} e^{\frac{\alpha}{2} H_\Phi[-n,n]} Q e^{\frac{\alpha}{2} H_\Phi[-n,n]} \right)}{Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n]} e^{\alpha H_\Phi[-n,n]} \right)} \\
&= \lim_{n\to\infty} \frac{Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n] + \alpha H_\Phi[-n,n]} Q \right)}{Tr_{[-n,n]} \left(e^{-\beta H_\Psi[-n,n] + \alpha H_\Phi[-n,n]} \right)}.
\end{aligned}$$

Next let us consider the mean relative entropy with respect to Φ. The abelian subalgebra $\mathcal{B}_n = \otimes_{[-n,n]} \mathcal{A}$ includs the local Hamiltonian $H_\Phi[-n,n]$. For a translation invariant state φ, we denote by φ^{cl} the restriction of φ to the abelian subalgebra $\bigotimes_{\mathbb{Z}} \mathcal{A}$. Then we have

$$\begin{aligned}
S_{M,\Phi}(\varphi, \omega_\Psi) &= \liminf_{n\to\infty} \frac{1}{2n+1} S(\varphi|_{\mathfrak{B}_n}, \omega_\Psi|_{\mathfrak{B}_n}) \\
&= \lim_{n\to\infty} \frac{1}{2n+1} S(\varphi^{cl}, \omega_\Psi) = S_{cl,M}(\varphi^{cl}, \omega_\Psi).
\end{aligned}$$

Here, we denote by $S_{cl,M}$ the mean relative entropy of the classical spin chain. Hence we obtain the well-known formula for classical spin chain:

$$I(x) = \inf_{\varphi \in S_\gamma(\mathfrak{A}), \varphi(A_\Phi) = x} S_{cl,M}(\varphi, \omega_\Psi).$$

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