# Charge Transport in Random Media and Boltzmann Limits for Single Particle and Manybody Models

By

## THOMAS CHEN*

## Abstract

Based on three lectures given at the conference "Renormalization Group Methods in the Mathematical Sciences" at RIMS, Kyoto University, in September 2009, we survey the results in [13, 14, 15] addressing the kinetic scaling limits and effective Boltzmann equations for the weakly disordered Anderson model. Moreover, we present related results for ideal Fermi gases in random media, based on a joint work with I. Sasaki, [16], and for Fermi gases in random media with dynamical Hartree-Fock interactions, based on a collaboration with I. Rodnianski, [17].

## § 1. Introduction

In this article, we survey the results in [13, 14, 15, 16, 17] addressing the transport properties of charged quantum mechanical particles (electrons) in random media, such as semiconductors. The main questions in this research area address the mathematically rigorous understanding of electric conductivity or insulation, from first principles in quantum mechanics. The presentation is structured as follows.

In Section 2, we discuss the Boltzmann limit for the quantum dynamics of an electron in a random medium. A widely used model to study such a system is the Anderson model, with Hamiltonian $H_\omega = -\Delta + \eta\, \omega_x$ acting on $\ell^2(\mathbb{Z}^d)$, where $(\Delta f)(x) = \sum_{|y-x|=1} f(y)$ is the nearest neighbor Laplacian on $\mathbb{Z}^d$, and $\{\omega_x\}$ is an i.i.d. random field of random variables which act as multiplication operators.

*Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA.
e-mail: `tc@math.utexas.edu`

The absence of electron transport (*Anderson localization*, electric insulators) for large $|\eta| \gg 1$, is nowadays mathematically well-understood, [2, 33]. On the other hand, the weak disorder regime, $|\eta| \ll 1$, poses some very prominent open problems. Only in dimension $d = 1$, it is known that Anderson localization occurs for all values of $|\eta| > 0$, [12]. In $d = 2$, it is conjectured that even for small $|\eta| \ll 1$, Anderson localization persists. In $d \geq 3$, it is conjectured that there exists a component of absolutely continuous spectrum, associated to delocalized states and electric conduction. For a kinetic scaling determined by macroscopic time and space coordinates $(T, X) = \eta^2(t, x)$, it is proven that, as $\eta \to 0$, the semiclassical dynamics is determined by a linear Boltzmann equation in the seminal works [26, 34, 51]; see also [13, 14, 15]. It is proven in the breakthrough work [27, 28, 30, 29] that for $d \geq 3$, the dynamics predicted by the Anderson model is diffusive, in a scaling limit that corresponds to large but finite microscopic times. It is expected that diffusive transport holds for all times, thus explaining electric conductivity, and the delocalization of electron wave functions in the relevant energy regimes. We also refer to the important related works [3, 9, 10, 11, 21, 25, 37, 38, 39, 40, 43, 44, 45, 47, 48, 50].

In Section 3, we discuss the results from [14] (which contains a joint result with L. Erdős). It is proven for the same kinetic scaling limit as in Section 2 that the macroscopic dynamics is determined by a linear Boltzmann equation, in higher mean $L^r$ with respect to the randomness, for any finite $r \geq 1$. This significantly improves the mode of convergence in [15] and in [26, 34] where convergence of the mean was established. The complexity and number (superfactorial versus factorial) of Feynman graphs entering this analysis is significantly higher than in the works above.

In Section 4, we discuss lower bounds on the localization lengths for the Anderson model. In the important work [49], C. Shubin, W. Schlag, and T. Wolff proved for the Anderson model with small Gaussian or Bernoulli randomness that with probability one, the localization length of eigenstates is bounded below by $O(\lambda^{-2})$ in $d = 1$, and by $O(\lambda^{-2+\delta})$ in $d = 2$, outside a small exceptional energy range. A very interesting related question was studied by J. Bourgain in [10] where the fact is established that with large probability, the weakly disordered Anderson model on $\mathbb{Z}^2$ with a random potential decaying like $|x|^{-\sigma}$ with $\sigma > \frac{1}{2}$ exhibits purely absolutely continuous (a.c.) spectrum and scattering. Similar problems were studied in [32, 20, 47].

We discuss results from [14]; it is proven for decay exponents $0 < \sigma \leq \frac{1}{2}$ that if $\sigma = \frac{1}{2}$ (a *marginal* renormalization problem), the localization length of eigenfunctions is bounded below by $2^{\lambda^{-\frac{1}{4}+\eta}}$, while for $0 < \sigma < \frac{1}{2}$ (a *relevant* renormalization problem), the lower bound is $\lambda^{-\frac{2-\eta}{1-2\sigma}}$, for any $\eta > 0$. These estimates "interpolate" between the lower bound $\lambda^{-2+\eta}$ of Schlag-Shubin-Wolff for $\sigma = 0$, and $\infty$ corresponding to pure a.c. spectrum for $\sigma > \frac{1}{2}$ as established by Bourgain (see also [11]). In particular, we discuss

how the localization conjecture for the weakly disordered Anderson model in $d = 2$ can, in this setting, be interpreted in the context of renormalization group theory.

In Section 5, we discuss the dynamics of an ideal Fermi gas in a random medium, based on joint work with I. Sasaki, [16]. The Anderson model neglects the repulsion between the electrons due to Coulomb interactions, and the Pauli principle. In [16], we consider some basic aspects of the question about the extent to which manybody effects influence the predictions of the Anderson model at small disorders. In [16], we analyze the dynamics of an ideal (i.e., $\lambda = 0$) homogenous Fermi gas in a weak random potential. We derive the kinetic scaling limit for the momentum distribution function with a translation invariant initial state and prove that it is determined by a linear Boltzmann equation. We prove that if the initial state is quasifree, then the time evolved state, averaged over the randomness is not quasifree, but has a quasifree kinetic limit. We show that the momentum distributions determined by the Gibbs states of a free fermion field are stationary solutions of the linear Boltzmann equation; this includes the limit of zero temperature. We note that recently, important results on the persistence of localization in fermionic manybody models at strong disorders (a topic which is not addressed here) have been established in [4, 18, 19].

In section 6, we discuss the joint work [17] with I. Rodnianski, which investigates the dynamics of a Fermi gas in a random medium where the particle interactions between the fermions are modeled in dynamical Hartree-Fock theory. We derive Boltzmann equations in kinetic scaling limits for scaling regimes determined by different ratios between the strengths of the randomness, and of the particle interactions. Central to this work is the development of methods to control the *nonlinear* self-interactions of the quantum field, combined with Feynman graph expansion methods to govern the randomness.

## Acknowledgment

## §2.   Kinetic limit for the weakly disordered Anderson model

In this section, we address the derivation of linear Boltzmann equations as the kinetic scaling limit of the Schrodinger dynamics in the weakly disordered Anderson model on the lattice, $\mathbb{Z}^3$, [13]. The results are closely related to the pioneering work of L. Erdös and H.-T. Yau, [34], in which the weak coupling and kinetic limit has been derived for a random Schrödinger equation in the continuum $\mathbb{R}^d$, $d = 2, 3$, for a Gaussian random potential, globally in macroscopic time. The corresponding local in macroscopic time result was first proved by H. Spohn [51]. In [13], the results of [34] are extended to the lattice, and to non-Gaussian randomness.

### §2.1.   The Anderson Model

Let $\Lambda_L := [-L, L]^d \cup \mathbb{Z}^d$ be a box with side length $L \ll 1$ which we will eventually send to infinity. Let $\Lambda_L^* := \frac{1}{L}\Lambda_L$ denote the associated dual lattice.

The *Anderson model* on $\Lambda_L$ is defined by the Hamiltonian

$$(2.1) \qquad\qquad H_\omega \ = \ \Delta \ + \ \eta V_\omega$$

acting on the Hilbert space $\ell^2(\Lambda_L)$. For concreteness, we shall assume periodic boundary conditions. Here, $\Delta$ denotes the centered nearest neighbor Laplacian,

$$\Delta\psi(x) = \sum_{|x-y|=1} \psi(y) \, .$$

Its spectrum is given by $\mathrm{spec}\,\Delta = [-2d, 2d] \cap \frac{1}{L^d}\mathbb{Z}$. Furthermore, $V_\omega$ denotes a random potential which acts as a multiplication operator in position space, $(V_\omega\psi)(x) = \omega_x\psi(x)$, $x \in \Lambda_L$, where $\{\omega_x\}_{x\in\Lambda_L}$ is a field of centered i.i.d. random variables. That is, $\mathbb{E}[\omega_x] = 0$, $\mathbb{E}[\omega_x^2] = 1$. The parameter $\eta > 0$ accounts for the disorder strength. We assume the moment bounds

$$(2.2) \qquad\qquad \mathbb{E}[\omega_x^{2m}] =: \tilde{c}_{2m} \leq (2m)!\, c_\omega \ \ , \ \ \tilde{c}_2 = 1 \ , \ \forall x \in \mathbb{Z}^3 \ , \ \forall m \geq 1 \ ,$$

hold where the constant $c_\omega < \infty$ is independent of $m$. For any $L < \infty$, $H_\omega$ is almost surely selfadjoint on $\ell^2(\Lambda_L)$.

For the Fourier transform, we use the convention

$$(2.3) \qquad\qquad \widehat{f}(p) := \sum_{x\in\Lambda_L} e^{-2\pi i p\cdot x}\, f(x) \, ,$$

where $p \in \Lambda_L^*$, and

$$(2.4) \qquad\qquad f(x) = \frac{1}{L^d} \sum_{p\in\Lambda_L^*} e^{2\pi i p\cdot x}\, \widehat{f}(p)$$

for its inverse. For brevity, we will use the notation

$$(2.5) \qquad \int dp \equiv \frac{1}{L^d} \sum_{p \in \Lambda_L^*}$$

in the sequel, which recovers its usual meaning in the thermodynamic limit $L \to \infty$.

The nearest neighbor lattice laplacian is a Fourier multiplication operator

$$\widehat{\Delta \psi}(k) = e_\Delta(k)\, \widehat{\psi}(k)$$

whose symbol

$$e_\Delta(k) = \sum_{i=1}^{d} 2\cos(2\pi k_i)$$

is the kinetic energy of the quantum mechanical electron.

In the Anderson model, the dynamics of an electron in a random medium is determined by the *Schrödinger equation* on $\ell^2(\Lambda_L)$,

$$i\, \partial_t\, \psi_t(x) = H_\omega\, \psi_t(x)$$

$$\psi_0 \in \ell^2(\Lambda_L)$$

The solution to this Cauchy problem is given by the unitary flow generated by $H_\omega$, that is, $\psi_t = e^{-itH_\omega}\, \psi_0$.

## § 2.2.   Kinetic scaling limits for small disorders $\eta \ll 1$

We will now discuss [13] (which contains a joint result with Erdös and Yau), which generalizes the results in [26, 34] to the lattice and non-Gaussian case. It is shown that the kinetic scaling limit of the quantum dynamics in the weakly disordered Anderson model is governed by a linear Boltzmann equation. A key technical problem is posed by frequency space resolvent integrals which are singular on overlapping, non-convex surfaces. Its solution involves considerations related to restriction theory in Harmonic Analysis, [13, 27, 28, 29, 53]; see also [39, 40].

We consider the Wigner transform of $\psi_t = e^{-itH_\omega}\psi_0$

$$W_t(x, v) = 8 \sum_{\substack{y, z \in \Lambda_L \\ y+z=2x}} \overline{\psi_t(y)}\, \psi_t(z)\, e^{2\pi i v(y-z)},$$

and the associated rescaled Wigner transform

$$W_T^{(\eta)}(X, V) = \left(\frac{1}{\eta^2}\right)^3 W_{T/\eta^2}(X/\eta^2, V)$$

according to the kinetic scaling $(T, X) = \eta^2(t, x)$. Then, the Boltzmann limit holds in the sense of weak convergence in distribution.

**Theorem 2.1.**    *[34, 13] For $\mu > 0$, let*

$$(2.6) \qquad \phi_0^\mu(x) \; := \; \mu^{\frac{3}{2}} \, \frac{h(\mu x) \, e^{2\pi i \frac{S(\mu x)}{\mu}}}{\|h\|_{\ell^2(\mu\mathbb{Z}^3)}} \; ,$$

*with $h, S \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ of Schwartz class, and $\|h\|_{L^2(\mathbb{R}^3)} = 1$. Assume $L$ sufficiently large that $\phi_0^\mu\big|_{\Lambda_L} = \phi_0^\mu$. Let $\phi_t^\mu$ be the solution of the random Schrödinger equation*

$$(2.7) \qquad i\,\partial_t \phi_t^\mu \; = \; H_\omega \, \phi_t^\mu$$

*on $\ell^2(\Lambda_L)$ with initial condition $\phi_0^\mu$, and let*

$$(2.8) \qquad W_T^{(\mu)}(X, V) \; := \; W_{\phi_{\mu^{-1}T}^\mu}^\mu(X, V)$$

*denote the corresponding rescaled Wigner transform.*
     *Choosing*

$$(2.9) \qquad \mu \; = \; \eta^2 \, ,$$

*where $\eta$ is the coupling constant in (2.1), it follows that*

$$(2.10) \qquad \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}\big[ \, \langle \, J \, , W_T^{(\eta^2)} \, \rangle \, \big] \; = \; \langle \, J \, , F_T \, \rangle \, ,$$

*where $F_T(X, V)$ solves the linear Boltzmann equation*

$$\partial_T F_T(X, V) + 2\pi \, \nabla_V e_\Delta(V) \cdot \nabla_X F_T(X, V)$$
$$(2.11) \qquad\qquad = \int_{\mathbb{T}^3} dU \, \sigma(U, V) \, [F_T(X, U) - F_T(X, V)]$$

*with initial condition*

$$F_0(X, V) = w - \lim_{\mu \to 0} W_{\phi_0^\mu}^\mu$$
$$(2.12) \qquad\qquad = |h(X)|^2 \, \delta(V - \nabla S(X)) \, ,$$

*and where*

$$\sigma(U, V) \; := \; 2\pi \, \delta(e_\Delta(U) - e_\Delta(V))$$

*denotes the collision kernel.*

## §2.3.   Main ingredients of the proof

As a starting point, we represent the solution $\psi_t$ of the random Schrödinger equation as a truncated resolvent expansion,

$$\psi_t \; = \; \frac{1}{2\pi i} \int_{\mathbb{R}+i\epsilon} dz \, e^{-itz} \, \frac{1}{H_\omega - z} \, \psi_0$$

$$\psi_t \; = \; \sum_{n=0}^{N} \psi_t^{(n)} \; + \; R_t^{(N)}$$

$$\psi_t^{(n)} \; = \; \frac{(-\eta)^n}{2\pi i} \, e^{\epsilon t} \int_{\mathbb{R}} dE \, e^{-itE} \left( \frac{1}{\Delta - E - i\epsilon} \, V_\omega \right)^n \frac{1}{\Delta - E - i\epsilon} \, \psi_0$$

This induces a decomposition of the Wigner transform into

$$W_t \; = \; \sum_{n,\widetilde{n}=0}^{N+1} W_t^{(n,\widetilde{n})} \, ,$$

and we consider its pairing with a rescaled test function, $\langle \, W_t \, , \, J_{\eta^2} \, \rangle$, where we define $J_\mu(x,v) := \mu^{-3} J(\mu x, v)$.

Taking the expectation with respect to the random potential, one obtains an expansion of the form

$$\mathbb{E}[\, \langle \, W_t \, , \, J_{\eta^2} \, \rangle \,] = \sum_{n,\widetilde{n}=0}^{N+1} \mathbb{E}[\, \langle \, W_t^{(n,\widetilde{n})} \, , \, J_{\eta^2} \, \rangle \,]$$

$$= \sum_{n,\widetilde{n}=0}^{N+1} \sum_{\pi \in \Gamma_{n,\widetilde{n}}} \mathrm{Amp}_{J_{\eta^2}}(\pi)$$

where the resulting terms are organized by use of *Feynman graphs*. By $\Gamma_{n,\widetilde{n}}$, we are denoting the set of *Feynman graphs* on $n + \widetilde{n}$ vertices corresponding to copies of $V_\omega$, and one distinguished vertex corresponding to $J_{\eta^2}$, see the next section.

The number of graphs is given by $|\Gamma_{n,\widetilde{n}}| \sim (n + \widetilde{n})!$

### 2.3.1. Graph expansion

The elements of the set of *Feynman graphs* $\Gamma_{n,\widetilde{n}}$, with $n + \widetilde{n} \in 2\mathbb{N}$, are defined as follows. We consider two horizontal solid lines, which we refer to as *particle lines*, joined by a distinguished vertex which we refer to as the $J$-vertex (corresponding to the the integration against the rescaled test function $J_{\eta^2}$. On the line on its left, we introduce $n$ vertices, and on the line on its right, we insert $\widetilde{n}$ vertices. We refer to those vertices as *interaction vertices*, and enumerate them from 1 to $2\bar{n}$ starting from the left. The edges between the interaction vertices are referred to as *propagator lines*. We label them by the momentum variables $u_0, ..., u_{2\bar{n}+1}$, increasingly indexed starting from the left. To the $j$-th propagator line, we associate the resolvent $\frac{1}{E(u_j) - \alpha - i\epsilon}$ if $0 \leq j \leq n$, and $\frac{1}{E(u_j) - \widetilde{\alpha} + i\epsilon}$ if $n + 1 \leq j \leq 2\bar{n} + 1$. To the $\ell$-th interaction vertex (adjacent to the edges labeled by $u_{\ell-1}$ and $u_\ell$), we associate the random potential $\widehat{V}_\omega(u_\ell - u_{\ell-1})$, where $1 \leq \ell \leq 2\bar{n} + 1$.

A *contraction graph* associated to the above pair of particle lines joined by the $\rho_0$-vertex, and decorated by $n + \widetilde{n}$ interaction vertices, is the graph obtained by pairwise connecting interaction vertices by dashed *contraction lines*. We denote the set of all such contraction graphs by $\Gamma_{n,\widetilde{n}}$; it contains

$$(2.13) \qquad |\Gamma_{n,\widetilde{n}}| = (2\bar{n} - 1)(2\bar{n} - 3) \cdots 3 \cdot 1 = \frac{(2\bar{n})!}{\bar{n}! 2^{\bar{n}}} = O(\bar{n}!)$$

elements.

If in a given graph $\pi \in \Gamma_{n,\widetilde{n}}$, the $\ell$-th and the $\ell'$-th vertex are joined by a contraction line, we write

$$(2.14) \qquad \qquad \ell \sim_\pi \ell',$$

and we associate the delta distribution

$$(2.15) \qquad \delta(u_\ell - u_{\ell-1} - (u_{\ell'} - u_{\ell'-1})) = \mathbb{E}\big[\widehat{V}_\omega(u_\ell - u_{\ell-1})\,\widehat{V}_\omega(u_{\ell'} - u_{\ell'-1})\big]$$

to this contraction line.
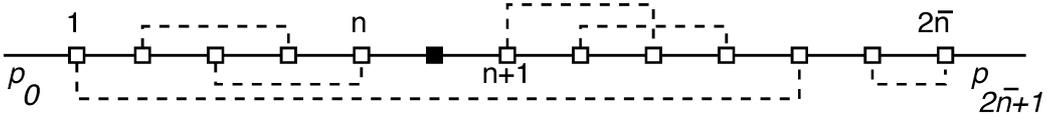


Figure 1. An example of a Feynman graph, $\pi \in \Gamma_{n,\widetilde{n}}$, with $n = 5$, $\widetilde{n} = 7$. The distinguished vertex is filled.

We consider the following classification of Feynman graphs, [34].

- A subgraph consisting of one propagator line adjacent to a pair of vertices $\ell$ and $\ell + 1$, and a contraction line connecting them, i.e., $\ell \sim_\pi \ell + 1$, where both $\ell$, $\ell + 1$ are either $\leq n$ or $\geq n + 1$, is called an *immediate recollision*.

- The graph $\pi \in \Gamma_{n,n}$ (i.e., $n = \widetilde{n} = \bar{n}$) with $\ell \sim_\pi 2n - \ell$ for all $\ell = 1, \ldots, n$, is called a *basic ladder* diagram. The contraction lines are called *rungs* of the ladder. We note that a rung contraction always has the form $\ell \sim_\pi \ell'$ with $\ell \leq n$ and $\ell' \geq n+1$. Moreover, in a basic ladder diagram one always has that if $\ell_1 \sim_\pi \ell_1'$ and $\ell_2 \sim_\pi \ell_2'$ with $\ell_1 < \ell_2$, then $\ell_2' < \ell_1'$.

- A diagram $\pi \in \Gamma_{n,\widetilde{n}}$ is called a *decorated ladder* if any contraction is either an immediate recollision, or a rung contraction $\ell_j \sim_\pi \ell_j'$ with $\ell_j \leq n$ and $\ell_j' \geq n$ for $j = 1, \ldots, k$, and $\ell_1 < \cdots < \ell_k$, $\ell_1' > \cdots > \ell_k'$. Evidently, a basic ladder diagram is the special case of a decorated ladder which contains no immediate recollisions (so that necessarily, $n = \widetilde{n}$).

- A diagram $\pi \in \Gamma_{n,\tilde{n}}$ is called *crossing* if there is a pair of contractions $\ell \sim_\pi \ell'$, $j \sim_\pi j'$, with $\ell < \ell'$ and $j < j'$, such that $\ell < j$.

- A diagram $\pi \in \Gamma_{n,\tilde{n}}$ is called *nesting* if there is a subdiagram with $\ell \sim_\pi \ell + 2k$, with $k \geq 1$, and either $\ell \geq n+1$ or $\ell + 2k \leq n$, with $j \sim_\pi j+1$ for $j = \ell+1, \ell+3, \ldots, \ell + 2k - 1$. The latter corresponds to a progression of $k-1$ immediate recollisions.

We note that any diagram that is not a decorated ladder contains at least a crossing or a nesting subdiagram.

To each Feynman graph, $\pi \in \Gamma_{m,n}$, we associate its Feynman amplitude, given by

$$\mathrm{Amp}_{J_{\eta^2}}(\pi) = \frac{\eta^{2\bar{n}}}{(2\pi)^2} \; e^{2\epsilon t} \int_{\mathbb{R}^2} dE \; dE' \; e^{-it(E-E')}$$

$$\int du_0 \cdots du_{2\bar{n}+1} \; \overline{\widehat{\psi}_0(u_0)} \; \widehat{\psi}_0(u_{2\bar{n}+1}) \; \frac{1}{\eta^6} \; \widehat{J}\left(\frac{u_{n+1}-u_n}{\eta^2}, \frac{u_{n+1}+u_n}{2}\right)$$

$$(2.16) \qquad \delta_\pi(\underline{u}) \left[\prod_{j=0}^{n} \frac{1}{e_\Delta(u_j) - E - i\epsilon}\right] \left[\prod_{\ell=n+1}^{2\bar{n}+1} \frac{1}{e_\Delta(u_\ell) - E' + i\epsilon}\right]$$

for $\pi \in \Pi_{m,n}$ and $\bar{n} = \frac{m+n}{2}$ (zero if $m+n \notin 2\mathbb{N}$). We set

$$(2.17) \qquad \qquad \epsilon = \frac{1}{t} = \frac{\eta^2}{T}$$

so that the overall exponential factor $e^{2\epsilon t}$ remains bounded. Here, $\delta_\pi$ denotes the product of the delta distributions associated to all contractions between random potentials in $\pi$.

To prove the theorem, the following estimates are crucial:

- *A priori bound:* One first verifies that for every Feynman graph $\pi$, one obtains the a priori bound

$$\left|\mathrm{Amp}_{J_{\eta^2}}(\pi)\right| \leq \left(\log \frac{1}{\eta}\right)^3 \left(c\, T \log \frac{1}{\eta}\right)^{\bar{n}}$$

To obtain this bound, one chooses a suitable spanning tree $T_\pi$ with $\bar{n} + 2$ edges for every given Feynman graph $\pi \in \Gamma_{n,\tilde{n}}$. The edges contained in $T_\pi$ are called tree edges, and accordingly, the momentum variables and resolvents supported on them are called tree momenta and tree resolvents, respectively. The edges in $\pi$ not contained in $T_\pi$ are referred to as loop edges (because adding them to $T_\pi$ produces loop subdiagrams), and correspondingly, they carry loop momenta and loop resolvents. A spanning tree $T_\pi$ is admissible if it contains all contraction lines, and precisely one edge adjacent to the distinguished vertex.

Associated to an admissible choice of $T_\pi$, one integrates out all delta distributions using the tree momenta, whereby the tree momenta are substituted by linear combinations of loop momenta. Next, one applies the $L^\infty$-bound to the resolvents supported on $\bar{n}$ edges of $T_\pi$,

$$(2.18) \qquad \left\| \frac{1}{e_\Delta(\,\cdot\,) - E + i\epsilon} \right\|_{L^\infty(\mathbb{T}^d)} \lesssim \frac{1}{\epsilon}$$

and $L^1$-estimates on the $\bar{n}$ loop resolvents,

$$(2.19) \qquad \left\| \frac{1}{e_\Delta(\,\cdot\,) - E + i\epsilon} \right\|_{L^1(\mathbb{T}^d)} \lesssim \log\frac{1}{\epsilon} \,.$$

Moreover, the integrals over the spectral parameters $E, E'$ can be controlled by applying

$$(2.20) \qquad \left\| \int dE \, \frac{1}{|e_\Delta(\,\cdot\,) - E + i\epsilon|} \right\|_{L^\infty(\mathbb{T}^d)} \lesssim \log\frac{1}{\epsilon}$$

to the two remaining tree resolvents. The a priori bound then follows.

• *Dominant diagrams:* The dominant contributions to the expansion are obtained from decorated ladder diagrams, where

$$\left| \mathrm{Amp}_{J_{\eta^2}}(\pi_{ladder}) \right| \leq \frac{(c\,T)^{\bar{n}}}{\sqrt{\bar{n}!}}$$

is *summable* in $\bar{n}$, uniformly in $\eta$.

The scaling limit of decorated ladder diagrams gives the solution $F_T(X, V)$ of the linear Boltzmann equation.

• *Nesting and crossing diagrams:* For every Feynman graph $\pi$ that contains a crossing or a nesting diagram, one obtains the upper bound

$$\left| \mathrm{Amp}_{J_{\eta^2}}(\pi_{crossing/nesting}) \right| \leq \eta^{\frac{2}{5}} \underbrace{\left( \log\frac{1}{\eta} \right)^3 \left( c\,T \log\frac{1}{\eta} \right)^{\bar{n}}}_{\text{a priori bound}}$$

where the gain of a factor $\eta^{\frac{2}{5}}$ over the a priori bound is crucial. The number of graphs exhibiting a crossing or a nesting is $O(\bar{n}!)$.

Choosing the truncation of the resolvent expansion at

$$N \approx \frac{\log\frac{1}{\eta}}{\log\log\frac{1}{\eta}} \,,$$

one obtains that

$$\sum_{m,n=1}^{N} \sum_{\Gamma_{m,n}} \left| \mathrm{Amp}_{J_{\eta^2}}(\pi_{crossing/nesting}) \right| \lesssim \eta^\delta$$

for some $\delta > 0$.

• *Terms involving the remainder term $R_t^{(N)}$:* If $n$ and/or $\widetilde{n} = N + 1$, the corresponding term in the expansion for the Wigner transform involves the remainder term $R_t^{(N)}$ of the resolvent expansion. It can be proven that the sum of these contributions are also bounded by $\lesssim \eta^\delta$ for the given choice of $N$, [34, 13]. We will not discuss the fairly technical proof of this result in this survey.

In conclusion, collecting all of the above estimates, and letting $\eta \to 0$, one obtains the asserted Boltzmann limit.                    $\square$

### 2.3.2.  Crossing estimates

The most difficult part in the analysis is the proof of smallness of the Feynman amplitudes associated with crossing diagrams. Amplitudes of nesting graphs, on the other hand, can be straightforwardly controlled by use of analyticity arguments. For every crossing diagram, it is possible to choose a spanning tree for which the associated bound on the Feynman amplitude contains a factor of the form

$$(2.21) \qquad \int_{\mathbb{T}^3 \times \mathbb{T}^3} dp\, dq \; \frac{1}{|e_\Delta(p) - E - i\epsilon|} \frac{1}{|e_\Delta(q) - E' + i\epsilon|} \frac{1}{|e_\Delta(p + q - u) - E - i\epsilon|}$$

which can be trivally bounded by

$$(2.22) \qquad\qquad\qquad\qquad \lesssim \frac{(\log \frac{1}{\epsilon})^2}{\epsilon}$$

Here, $p, q$ are loop momenta which appear only on the subgraph associated to this expression, and $u$ is a linear combination of loop momenta in $\pi$ independent of $p, q$. The bound 2.22 is insufficient for our purposes because it does not improve on the a priori bound.

To improve on this bound, we observe that the singular integrand in (2.21) is concentrated on the intersection of the $\epsilon$-tubular neighborhoods of isoenergy surfaces

$$(2.23) \qquad\qquad\qquad \Sigma_E = \left\{ p \in \mathbb{T}^3 \,|\, e_\Delta(p) = E \right\}.$$

The idea is to exploit the smallness of the intersection measure, in order to improve on the trivial bound (2.22).

As a matter of fact, improving (2.22) by a factor $\epsilon^\delta$ for any arbitrary $\delta > 0$ suffices for our purposes.

We remark that in the case of the continuum, $\mathbb{R}^d$, the surfaces $\Sigma_E$ are spheres, and it is easy to control the size of their $\epsilon$-thickened intersections, [34].

*The geometry of level surfaces for lattice models*

In the case of lattice models, the geometry of the isoenergy surfaces is necessarily more complicated than in the continuum case. The prototypical situation is given by the case of the nearest neighbor laplacian which acts as a Fourier multiplication operator with symbol

$$(2.24) \qquad e_\Delta(p) \; = \; 2\cos(2\pi p_1) \, + \, 2\cos(2\pi p_2) \, + \, 2\cos(2\pi p_3)$$

in $d = 3$. The surface $\Sigma_E = \{p \in \mathbb{T}^3 \,|\, e_\Delta(p) = E\}$ is non-convex, and exhibits *lines of vanishing Gauss curvature.*

Naturally, one might ask if this problem can be circumvented by a different choice of the kinetic energy operator than the nearest neighbor Laplacian $\Delta$. However, this problem cannot be avoided by a different choice of the kinetic energy operator, due to the topology of $(\mathbb{Z}^d)^* \cong \mathbb{T}^d$. The reason is that at best, the kinetic energy $e_\Delta :$ $\mathbb{T}^d \to [-2d, 2d]$ is a (perfect) Morse function (which is the case for the nearest neighbor laplacian), in case of which the Morse inequalities enforce a transition of $\Sigma_E$ between a topological sphere and a surface of genus at least 3, depending on the paramter $E$. This is because the Betti numbers $b_p$ of $\mathbb{T}^3$ are $b_0 = 1 = b_3$, and $b_1 = 3 = b_2$.

In [27], Erdös and Salmhofer have obtained a gain of $\epsilon^{\frac{1}{4}}$ upon the a priori bound (2.22) via a direct parametrization of level surfaces in $3D$. Their result is obtained from a very involved analysis and holds in greater generality than only for (2.24).

In [13], the author has independently provided an improvement of $\epsilon^{\frac{1}{5}}$ upon (2.22) by a short argument related to restriction estimates in Harmonic Analysis, involving dimensional reduction. It holds for a smaller class of energy functions than those admitted in [27], in that it exploits the fact that the addititive structure of $e_\Delta(p)$ (2.24) with respect to coordinate components.

For this discussion, we let $d = 3$. The argument proceeds as follows. We rewrite the 3-dimensional integral into a parametrized 2-D integral,

$$\int_{\mathbb{T} \times \mathbb{T}} dp_3 \, dq_3 \int_{\mathbb{T}^2 \times \mathbb{T}^2} d\underline{p} \, d\underline{q} \; \frac{1}{|\widetilde{e_\Delta}(\underline{p}) - E(p_3) - i\epsilon|}$$
$$\frac{1}{|\widetilde{e_\Delta}(\underline{q}) - E'(q_3) + i\epsilon|} \frac{1}{|\widetilde{e_\Delta}(\underline{p} + \underline{q} - \underline{u}) - E(p_3 + q_3 - u_3) - i\epsilon|}$$

where $p = (p_1, p_2, p_3)$ and $\underline{p} := (p_1, p_2)$, and similarly for $q$ and $\underline{q}$. Moreover,

$$(2.25) \qquad \widetilde{e_\Delta}(\underline{p}) \; := \; 2\cos(2\pi p_1) + 2\cos(2\pi p_2)$$

and

$$(2.26) \qquad E(p_3) \; := \; E - 2\cos(2\pi p_3) \, ,$$

et cetera. We note that there exists exactly one critical value $E_{crit} = 0$, for which the isoenergy curve $\{p \in \mathbb{T}^2 | \widetilde{e_\Delta}(p) = E_{crit}\}$ is a union of straight lines, and thus has vanishing curvature. The idea is to exploit the fact that for a $(p_3, q_3)$-set of large measure, the level lines of the 2-dimensional problem parametrized by $\underline{p}, \underline{q}$ have sufficiently a curvature sufficiently bounded away from zero.

Let, for brevity, $E_1 := E(p_3)$, $E_2 := E(q_3)$, and $E_3 := E(p_3 + q_3 - u_3)$.

We focus on the most singular part of

$$(2.27) \qquad \widetilde{R}_i(\underline{p}) := \frac{1}{|\widetilde{e_\Delta}(\underline{p}) - E_i - i\epsilon|}$$

which can be estimated by

$$(2.28) \qquad R_i^\nu(\underline{p}) := \chi_i^\nu(\underline{p}) \frac{1}{|\widetilde{e_\Delta}(\underline{p}) - E_i - i\epsilon|} \lesssim \frac{\nu}{\epsilon} \delta_i^\nu(\underline{p})$$

where

$$(2.29) \qquad \chi_i^\nu(\underline{p}) := \chi_{\{\underline{p}| \, |\widetilde{e_\Delta}(\underline{p}) - E_i| < \nu\}}(\underline{p})$$

and

$$(2.30) \qquad \delta_i^\nu(\underline{p}) := \frac{1}{\nu} \chi_i^\nu(\underline{p})$$

for some choice of $\nu > 0$ which remains to be optimized, and $\chi$ a smooth characteristic function. Then,

$$(2.31) \qquad (\delta_i^\nu)^\vee(\underline{x}) = \int d\underline{p}\, e^{-2\pi i \underline{p} \cdot \underline{x}} \, \delta_i^\nu(\underline{p})$$

has a decay

$$(2.32) \qquad |(\delta_i^\nu)^\vee(\underline{x})| \lesssim \tau^{-1/2} |x|^{-1/2} e^{-\nu |x|}$$

for $|E_i - E_{crit}| > \tau$ where $E_{crit}$ is the critical energy value for which the level line has zero curvature. This curvature induced decay in $\underline{x}$-space is related to restriction problems in Harmonic Analysis, and is proven by use of stationary phase arguments.

If $|E_i - E_{crit}| > \tau$ for $i = 1, 2, 3$, the contribution to the crossing integral can be estimated by

$$(2.33) \qquad \left\langle R_1^\nu * R_2^\nu, R_3^\nu \right\rangle \lesssim (\frac{\nu}{\epsilon})^3 \sum_{\underline{x}} (\delta_1^\nu)^\vee(\underline{x})\, (\delta_2^\nu)^\vee(\underline{x})\, (\delta_3^\nu)^\vee(\underline{x})$$

$$(2.34) \qquad \lesssim (\frac{\nu}{\epsilon})^3 \sum_{0 < |\underline{x}| \lesssim \frac{1}{\nu}} \tau^{-3/2} |\underline{x}|^{-3/2}$$

$$(2.35) \qquad = (\frac{\nu}{\epsilon})^3 \tau^{-3/2} \nu^{-1/2} \,.$$

On the other hand, the measure of the set of $(p_3, q_3)$-values for which there exists at least one index $i \in \{1, 2, 3\}$ such that $|E_i - E_{crit}| \leq \tau$ is small, of order $O(\tau^{1/2})$ (for the choice of the cosine function, as it appears in $\widetilde{e_\Delta}$). On this $(p_3, q_3)$-set, the level lines have small curvature $\leq \tau$, and we only get

$$(2.36) \qquad \left\langle R_1^\nu * R_2^\nu \,,\, R_3^\nu \right\rangle \leq \frac{1}{\epsilon}(\log \frac{1}{\epsilon})^2 \,.$$

Finally, for the non-singular contribution where $|\widetilde{e_\Delta}(\underline{p}) - E_i| > C\nu$ for at least one value of $i$, we get a contribution of size

$$(2.37) \qquad \left| \left\langle \widetilde{R}_1 * \widetilde{R}_2 \,,\, \widetilde{R}_3 \right\rangle - \left\langle R_1^\nu * R_2^\nu \,,\, R_3^\nu \right\rangle \right| \lesssim \frac{1}{\nu}(\log \frac{1}{\epsilon})^2$$

to (2.21).

Combining the bounds for the two $(p_3, q_3)$-regions discussed above, we find the upper bound

$$(2.38) \quad \int dp_3 \, dq_3 \left\langle \widetilde{R}_1 * \widetilde{R}_2 \,,\, \widetilde{R}_3 \right\rangle \lesssim (\frac{\nu}{\epsilon})^3 \tau^{-3/2} \nu^{-1/2} + (\frac{1}{\nu} + \frac{\tau^{1/2}}{\epsilon})(\log \frac{1}{\epsilon})^2$$

and choosing $\nu = \epsilon^{4/5}$ and $\tau = \epsilon^{2/5}$, one obtains

$$\int_{\mathbb{T}^3 \times \mathbb{T}^3} dp \, dq \, \frac{1}{|e_\Delta(p) - E - i\epsilon|} \frac{1}{|e_\Delta(q) - E' + i\epsilon|} \frac{1}{|e_\Delta(p + q - u) - E - i\epsilon|} \lesssim \frac{(\log \frac{1}{\epsilon})^2}{\epsilon^{4/5}} \,,$$

which improves (2.22) by a factor $\epsilon^{1/5} = O(\eta^{2/5})$. $\qquad\qquad\qquad\qquad\square$

## § 3. Convergence in higher mean

In the previous section, we have discussed the fact that the *dynamics on the average* in the weakly disordered Anderson model is determined by a linear Boltzmann equation. In this section, we want to address the following problem:

*Key question: Does the average dynamics reflect the typical dynamics for a generic realization of the random potential ?*

As we will see, the answer is yes.

### § 3.1. Statement of the main results

For the analysis in this section, we assume that the initial condition $\widehat{\psi_0}$ is again of WKB type (2.6), and, in addition, that the singularities of its Fourier transform are "not too pathological". The latter is expressed by the requirement that the Fourier transform of the WKB initial condition (2.6) satisfies a *concentration of singularity condition*:

$$(3.1) \qquad \widehat{\phi_0^\mu}(k) \; = \; f_\infty^\mu(k) \, + \, f_{sing}^\mu(k) \, ,$$

where

$$(3.2) \qquad \| \, f_\infty^\mu \, \|_{L^\infty(\mathbb{T}^3)} \; < \; c \, ,$$

and

$$(3.3) \qquad \| \, |f_{sing}^\mu| * |f_{sing}^\mu| \, \|_{L^2(\mathbb{T}^3)} \; = \; \| \, |f_{sing}^\mu|^\vee \, \|_{\ell^4(\mathbb{Z}^3)}^2 \; \leq \; c' \, \mu^{\frac{4}{5}}$$

for finite, positive constants $c$, $c'$ independent of $\mu$. This condition imposes a restriction on the possible choices of the phase function $S$ in (2.6).

The following simple, but physically important examples of $\widehat{\phi_0^\mu}$ satisfy (3.1) - (3.3).

**3.1.1. Example** Let $S(X) = pX$ for $X \in \mathrm{supp}\{\, h \,\}$, and $p \in \mathbb{T}^3$. Then,

$$(3.4) \qquad \widehat{\phi_0^\mu}(k) \; = \; \frac{\mu^{-\frac{3}{2}} \widehat{h}(\mu^{-1}(k - p))}{\|h\|_{\ell^2(\mu\mathbb{Z}^3)}} \; =: \; \delta_\mu(k - p) \, .$$

Since $h$ is of Schwartz class, $\delta_\mu$ is a smooth bump function concentrated on a ball of radius $O(\mu)$, with $\|\delta_\mu\|_{L^2(\mathbb{T}^3)} = 1$. Accordingly, we find

$$(3.5) \qquad (|\delta_\mu| * |\delta_\mu|)(k) \; \approx \; \chi(|k| < c\mu) \, ,$$

and

$$(3.6) \qquad \| \, |\delta_\mu| * |\delta_\mu| \, \|_{L^2(\mathbb{T}^3)} \; = \; \| \, |\delta_\mu|^\vee \, \|^2_{\ell^4(\mathbb{Z}^3)} \; \leq \; c \, \mu^{\frac{3}{2}} \, .$$

Hence, (3.1) - (3.3) is satisfied, with $f^\mu_\infty = 0$. We remark that in this example, $p \in \mathbb{T}^3$ corresponds to the velocity of the macroscopic initial condition $F_0(X, V)$ in (2.12) for the linear Boltzmann evolution.

**3.1.2. Example** As a small generalization of the previous case, we may likewise assume for $S$ that for every $k \in \mathbb{T}^3$, there are finitely many solutions $X_j(k)$ of $\nabla_X S(X_j(k)) = k$, and that $X_j(\,\cdot\,) \in C^1(\mathrm{supp}\{\,h\,\})$ for each $j$. Moreover, we assume that $|\det \mathrm{Hess}\, S(X)| > c$ uniformly on $\mathrm{supp}\{\,h\,\}$. Then, by stationary phase arguments, [53], one finds that

$$(3.7) \qquad \widehat{\phi}^\mu_0(k) \; = \; f^\mu_\infty(k) + f^\mu_{sing}(k) \quad , \quad \|f^\mu_\infty\|_{L^\infty(\mathbb{T}^3)} \; < \; c$$

with

$$(3.8) \qquad f^\mu_{sing}(k) \; = \; \sum_j c_j \, \delta^{(j)}_\mu(k - \nabla_X S(X_j(k))) \, ,$$

for constants $c_j$ independent of $\mu$, and smooth bump functions $\delta^{(j)}_\mu$ similar to (3.4). One again obtains $\| \, |f^\mu_{sing}|^\vee \, \|^2_{\ell^4(\mathbb{Z}^3)} \leq c\mu^{\frac{3}{2}}$, which verifies that (3.1) - (3.3) holds. $\nabla S$ determines the velocity distribution of the macroscopic initial condition $F_0(X, V)$ in (2.12).

We may now state the main result of [14].

**Theorem 3.1.** *(T. C., [14]) Assume that the Fourier transform of (2.6), $\widehat{\phi}^\mu_0$, satisfies the concentration of singularity property (3.1) - (3.3). Then, for*

$$(3.9) \qquad \qquad \mu \; = \; \eta^2 \, ,$$

*and for any fixed, finite $r \in 2\mathbb{N}$, any $T > 0$, and for any Schwartz class function $J$, the estimate*

$$(3.10) \quad \lim_{L \to \infty} \left( \mathbb{E}\Big[ \, \Big| \Big\langle J, W^{(\eta^2)}_T \Big\rangle - \mathbb{E}\big[ \Big\langle J, W^{(\eta^2)}_T \Big\rangle \big] \Big|^r \Big] \right)^{\frac{1}{r}} \; \leq \; c(r, T)\eta^{\frac{1}{300r}}$$

*holds for $\eta$ sufficiently small, and a finite constant $c(r, T)$ that does not depend on $\eta$. Consequently,*

$$(3.11) \qquad \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}\Big[ \, \Big| \Big\langle J, W^{(\eta^2)}_T \Big\rangle - \Big\langle J, F_T \Big\rangle \Big|^r \Big] \; = \; 0$$

*for any $1 \leq r < \infty$ (i.e. convergence in $r$-th mean), and any $T \in \mathbb{R}_+$.*

## §3.2. Main ingredients of the proof

Our starting point again is the truncated resolvent expansion

$$\psi_t = \frac{1}{2\pi i} \int_{\mathbb{R}+i\epsilon} dz\, e^{-itz} \frac{1}{H_\omega - z} \psi_0$$

$$\psi_t = \sum_{n=0}^{N} \psi_t^{(n)} + R_t^{(N)}$$

$$\psi_t^{(n)} = \frac{(-\eta)^n}{2\pi i} e^{\epsilon t} \int_{\mathbb{R}} dE\, e^{-itE} \left( \frac{1}{\Delta - E - i\epsilon} V_\omega \right)^n \frac{1}{\Delta - E - i\epsilon} \psi_0 \,.$$

Accordingly, it induces the decomposition of the Wigner transform into

$$W_t = \sum_{n_1,n_2=0}^{N+1} W_{t;n_1,n_2} \,.$$

To determine higher order moments, we let $r \in 2\mathbb{N}$. Then, clearly,

$$\left( \mathbb{E}\left[ \left( \langle J_{\eta^2}, W_{\phi_t} \rangle - \mathbb{E}\langle J_{\eta^2}, W_{\phi_t} \rangle \right)^r \right] \right)^{\frac{1}{r}}$$

(3.12)
$$\leq C N \sum_{n_1,n_2=0}^{N+1} \left( \mathbb{E}\left[ \left| \langle J_{\eta^2}, W_{t;n_1,n_2} \rangle - \mathbb{E}\langle J_{\eta^2}, W_{t;n_1,n_2} \rangle \right|^r \right] \right)^{\frac{1}{r}} \,,$$

where $J_\mu(x,v) := \mu^{-3} J(\mu x, v)$. We observe that $\widehat{J}_{\eta^2}$ forces $|k - k' \pmod{2\mathbb{T}^3}| < c\eta^2$, but that $|k + k' \pmod{2\mathbb{T}^3}|$ is essentially unrestricted.

For $n_1, n_2 \leq N$, we again explicitly determine the contractions among all copies of the random potential, and organize the associated expansion in terms of Feynman graphs. For convenience, we introduce the following multi-index notation: Let $n \equiv n_1$, and $\bar{n} \equiv n_1 + n_2$ be fixed. For $j = 1, \ldots, r$, we let

$$\underline{k}^{(j)} := (k_0^{(j)}, \ldots, k_{\bar{n}+1}^{(j)})$$

$$d\underline{k}^{(j)} := \prod_{\ell=0}^{\bar{n}+1} dk_\ell^{(j)}$$

$$d\underline{k}^{(j)}_{\widehat{J}_{\eta^2}} := \prod_{\ell=0}^{\bar{n}+1} dk_\ell^{(j)}\, \widehat{J}_{\eta^2}(k_n^{(j)} - k_{n+1}^{(j)}, \frac{k_n^{(j)} + k_{n+1}^{(j)}}{2})$$

$$K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \epsilon] := \prod_{\ell=0}^{n} \frac{1}{e_\Delta(k_\ell^{(j)}) - \alpha_j - i\epsilon_j} \prod_{\ell'=n+1}^{\bar{n}+1} \frac{1}{e_\Delta(k_{\ell'}^{(j)}) - \beta_j + i\epsilon_j}$$

(3.13)
$$U^{(j)}[\underline{k}^{(j)}] := \prod_{\ell=1}^{n} \widehat{V}_\omega(k_\ell^{(j)} - k_{\ell-1}^{(j)}) \prod_{\ell'=n+2}^{\bar{n}+1} \widehat{V}_\omega(k_{\ell'}^{(j)} - k_{\ell'-1}^{(j)}) \,,$$

where $\epsilon_j := (-1)^j \epsilon$, $\alpha_j, \beta_j \in \mathbb{R}$, and where we note that $\overline{\widehat{V_\omega}(k)} = \widehat{V_\omega}(-k)$.

Moreover, let

$$(3.14) \qquad \underline{\alpha} := (\alpha_1, \ldots, \alpha_r) \quad , \quad d\underline{\alpha} := \prod_{j=1}^{r} d\alpha_j \, ,$$

and likewise for $\underline{\beta}$, $\underline{\xi}$ and $d\underline{\beta}$, $d\underline{\xi}$. Then,

$$\mathbb{E}\left[\left(\langle J_{\eta^2}, W_{\phi_t} \rangle\right)^r\right] = \frac{e^{2r\epsilon t}\eta^{r\bar{n}}}{(2\pi)^{2r}} \int_{(\mathbb{R}\times\mathbb{R})^r} d\underline{\alpha} \, d\underline{\beta} \, e^{-it\sum_{j=1}^{r}(-1)^j(\alpha_j-\beta_j)}$$

$$\int_{(\mathbb{T}^3)^{(\bar{n}+2)r}} \Big[ \prod_{j=1}^{r} d\underline{k}_{\widehat{J_{\eta^2}}}^{(j)} \Big] \mathbb{E}\Big[ \prod_{j=1}^{r} U^{(j)}[\underline{k}^{(j)}] \Big]$$

$$(3.15) \qquad \prod_{j=1}^{r} K^{(j)}[\underline{k}^{(j)}, \alpha_j, \beta_j, \epsilon] \, \widehat{\phi}_0^{(j)}(k_0^{(j)}) \, \overline{\widehat{\phi}_0^{(j)}(k_{\bar{n}+1}^{(j)})} \, ,$$

where

$$(3.16) \qquad \widehat{\phi}_0^{(j)} := \begin{cases} \widehat{\phi}_0 \text{ if } j \text{ is even} \\ \overline{\widehat{\phi}_0} \text{ if } j \text{ is odd} \, . \end{cases}$$

Clearly, the expectation $\mathbb{E}$ produces a sum of $O((\bar{n}r)!)$ singular integrals, which we organize via Feynman graphs. We note that the number of Feynman graphs at fixed $\bar{n}$ is superfactorial in $\bar{n}$.
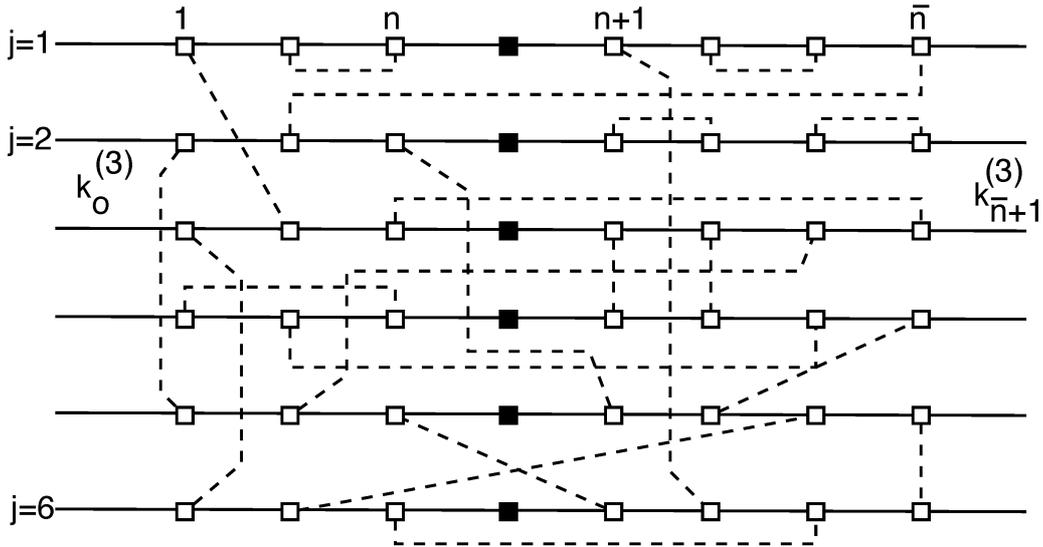


Figure 2. A (completely connected) contraction graph for the case $r = 6$, $n = 3$, $\bar{n} = 7$. The $\widehat{J}_{\eta^2}$-vertices are drawn in black, while the $\widehat{V}_\omega$-vertices are shown in white. The $r$ particle lines are solid, while the lines corresponding to contractions of pairings of

random potentials are dashed. For $j = 3$ in the notation of (3.15), the momenta $k_0^{(3)}$ and $k_{\bar{n}+1}^{(3)}$ are written above the corresponding propagator lines.

**Definition 3.2.** We denote

$$(3.17) \qquad \mathbb{E}_{Disc}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{k}^{(j)}]\Big] := \prod_{j=1}^{r} \mathbb{E}\Big[U^{(j)}[\underline{k}^{(j)}]\Big]$$

as the expectation based on completely disconnected graphs. It includes contractions among random potentials $\widehat{V}_\omega$ only if they lie on the same particle line. We denote

$$(3.18) \quad \mathbb{E}_{N-D}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{k}^{(j)}]\Big] := \mathbb{E}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{k}^{(j)}]\Big] - \mathbb{E}_{Disc}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{k}^{(j)}]\Big],$$

as the expectation based on non-disconnected graphs. It is defined by the condition that there is at least one connectivity component connecting at least two different particle lines. Moreover, we refer to

$$(3.19) \qquad \mathbb{E}_{2-Conn}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{k}^{(j)}]\Big] := \mathbb{E}\Big[\prod_{j=1}^{r} \Big(U^{(j)}[\underline{k}^{(j)}] - \mathbb{E}\Big[U^{(j)}[\underline{k}^{(j)}]\Big]\Big)\Big]$$

as the expectation based on 2-connected graphs.

We make the key observation that for $n_1, n_2 \leq N$, the expectation based on disconnected graphs of the $r$-th power precisely produces the $r$-th power of the expectation,

$$(3.20) \qquad \mathbb{E}_{Disc}\Big[\big|\langle J_{\eta^2}, W_{t;n_1,n_2}\rangle\big|^r\Big] = \Big|\mathbb{E}\langle J_{\eta^2}, W_{t;n_1,n_2}\rangle\Big|^r.$$

Therefore, the difference

$$\Big(\mathbb{E}\Big[\big|\langle J_{\eta^2}, W_{t;n_1,n_2}\rangle - \mathbb{E}\langle J_{\eta^2}, W_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}$$

$$= \Big(\mathbb{E}_{2-Conn}\Big[\big|\langle J_{\eta^2}, W_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}$$

$$(3.21) \qquad = \Big(\mathbb{E}_{2-Conn}\Big[\Big(\big|\int d\xi dv \widehat{J}_{\eta^2}(\xi,v) \overline{\widehat{\phi}_{n_2,t}(v - \tfrac{\xi}{2})} \widehat{\phi}_{n_1,t}(v + \tfrac{\xi}{2})\big|^2\Big)^{\frac{r}{2}}\Big]\Big)^{\frac{1}{r}},$$

involves only the expectation $\mathbb{E}_{2-Conn}$ based on *2-connected graphs*.

The main technical result of [14] is the following proposition which asserts that the expectation based on 2-connected graphs is small (a factor $\eta^{2/5}$ smaller than the corresponding a priori bound).

**Proposition 3.3.** *([14], in part joint with Laszlo Erdös). Let $s \geq 2$, with $s\bar{n} \in 2\mathbb{N}$, and let $T = \eta^2 t > 0$, where $\epsilon = \frac{1}{t}$. Moreover, let $\pi \in \Pi_{s;\bar{n},n}^{(\widehat{J}_{\eta^2})\,2-conn}$ be a 2-connected graph on $s$ particle lines.*

*Then, there exists a finite constant $c = c(T)$ such that*

$$(3.22) \qquad |\operatorname{Amp}_{\widehat{J}_{\eta^2}}(\pi)| \ \leq \ \eta^{\frac{2}{5}} \underbrace{(\log \frac{1}{\eta})^3 \, (c(T) \log \frac{1}{\eta})^{\frac{s\bar{n}}{2}}}_{\text{a priori bound}} \ .$$

*In particular, setting $r = s$, this bound holds for every non-disconnected graph.*

*Proof idea:* The strategy of the proof is based on systematically disconnecting the contraction lines between different particle lines, and to reduce the problem to the $L^4$-case (i.e., the case $r = 2$). Then, we exploit momentum conservation constraints for estimates similar to the ones used for bounding the crossing diagrams in the previous section.                                                                    $\square$

The upper bounds on the contributions of the remainder term in the resolvent expansion are similar. We shall not address them in any detail here because their discussion is more technical. As a result of the above, we obtain the following:

**Corollary 3.4.**   *The sum of disconnected graphs yields, in the kinetic scaling limit $\eta \to 0$ with $(T, X) = \eta^2(t, x)$,*

$$\lim_{\eta \to 0} \mathbb{E}_{Disc} \left[ \left| \left\langle W_T^{(\eta^2)}, J \right\rangle \right|^r \right] \ = \ \left| \left\langle F_T, J \right\rangle \right|^r$$

*for any finite $r \in 2\mathbb{N}$, and any test function $J(X, V)$.*
   *Thus,*

$$\lim_{\eta \to 0} \mathbb{E} \underbrace{\left[ \left| \left\langle W_T^{(\eta^2)}, J \right\rangle - \left\langle F_T, J \right\rangle \right|^r \right]}_{\text{contains only non-disconnected graphs}} \ = \ 0$$

*for any $r \in 2\mathbb{N}$, and thus for all $1 \leq r < \infty$.*

In particular, the variance ($r = 2$) tends to zero as $\eta \to 0$. Moreover, from convergence in higher mean, one immediately obtains convergence in probability.

**Corollary 3.5.**   *The rescaled Wigner transform converges weakly, and in probability, to a solution $F_T(X, V)$ of the linear Boltzmann equation. That is,*

$$\lim_{\eta \to 0} \mathbb{P} \left[ \left| \left\langle W_T^{(\eta^2)}, J \right\rangle - \left\langle F_T, J \right\rangle \right| > \delta \right] \ = \ 0$$

*for any $\delta > 0$, and any test function $J(X, V)$.*

We conclude that in this sense, the average dynamics in the kinetic scaling limit indeed reflects the typical dynamics of the system.

## §4.   Lower bounds on localization lengths

We recall the main conjecture for the 2-dimensional Anderson model:

**Conjecture:** *In dimension 2, Anderson localization holds for all $\eta > 0$. The localization length is of size $O(\exp(\eta^{-c}))$, for some $c > 0$.*

Heuristically, the localization length is the typical diameter of the support of localized eigenfunctions. There are many various precise definitions, adapted to different contexts, which we will not address in detail here.

In the important work [49], C. Shubin, W. Schlag and T. Wolff, prove the following.

**Theorem 4.1.**   *(C. Shubin, W. Schlag, T. Wolff, [49]) Assume that the random potential in the Anderson model is either Gaussian or Bernoulli. Then, with probability one, the localization length of eigenstates (outside a small exceptional energy range) is bounded below by $O(\eta^{-2})$ in $d = 1$, and by $O(\eta^{-2+\delta})$ in $d = 2$.*

In their proof, the authors of [49] use techniques of harmonic analysis to establish for the Anderson model in $d = 1, 2$ and small $\eta > 0$, that with probability one, most eigenstates are in frequency space concentrated on shells of thickness $\leq \eta^2$ in $d = 1$, and $\leq \eta^{2-\delta}$ in $d = 2$. The eigenenergies are required to be bounded away from the edges of the spectrum of the nearest neighbor laplacian $\Delta$, and in $d = 2$, also away from its center. By the uncertainty principle, this implies the asserted lower bounds of order $O(\eta^{-2})$ in $d = 1$, and and $O(\eta^{-2+\delta})$ in $d = 2$, on the localization lengths in position space. Closely related to their work are the papers [43, 44, 45] by Magnen, Poirot, Rivasseau, and [46] by Poirot, who used, amongst others, ideas stemming from the renormalization group analysis of fermionic manybody systems, to study the Greens functions associated to $H_\omega$.

The class of methods employed in [49] and [43, 44, 45, 46] is known to be extremely powerful in $d = 1, 2$, but less suitable for $d = 3$. In the paper [13], an entirely different approach is taken to prove the following result.

**Theorem 4.2.**   *(T. C., [13]) In $d = 3$, the localization length of eigenfunctions is bounded from below by $\eta$ of order $O(\frac{\eta^{-2}}{|\log \eta|})$, for $\eta > 0$ small, and with probability one.*

The link between the localization length and the bounds proven for the Boltzmann limit in [13] is a joint result of the author with L. Erdös and H.-T. Yau. The argument can be outlined as follows. We write the eigenfunctions of $H_\omega$ (for finite $L$ at first) in the basis of Kronecker deltas on $\Lambda_L$. It is impossible that the localization length is smaller than the spreading size of each basis vector $\delta_x$ obtained from $e^{-itH_\omega}\delta_x$.

Thus, the estimate

$$(4.1) \qquad \mathbb{E}[\, \| \, e^{-itH_\omega} \phi_0 \, - \, e^{-it\Delta} \phi_0 \, \|_2^2 \, ] \, \ll \, 1$$

with $\phi_0 = \delta_x$, and some $t > 0$, implies that the localization length is $\geq O(t)$. However, this bound is proven as part of the error estimate for the Boltzmann limit in [13], as addressed in Section 2, for $t = O(\eta^{-2})$.

## § 4.1.   Decaying Random Potentials

The localization conjecture for the weakly disordered Anderson model in dimension 2 is a notoriously hard open problem. It is natural to ask whether there is an interpretation in the context of *renormalization group theory* that elucidates more clearly the nature of some of the obstacles.

In this section, we consider the weakly disordered Anderson model with a decaying random potential. As will be described, this model exhibits a transition parametrized by the decay exponent, from a scattering region, to a region where localization is conjectured.

To be precise, the Hamiltonian has the form

$$(4.2) \qquad H_\omega \; = \; \Delta \, + \, \eta V_\omega(x)$$

where

$$(4.3) \qquad V_\omega(x) \, \sim \, \frac{\omega_x}{\langle x \rangle^\sigma}$$

and $\langle x \rangle := \sqrt{1 + x^2}$.

The following results have been established in the more recent literature: In the supercritical case $\sigma > \frac{1}{2}$, it was proven by Bourgain in [10] that with large probability, $H_\omega$ (with Bernoulli or Gaussian randomness) has, for small $\eta$, pure a.c. spectrum in

$$(4.4) \qquad I_\tau := (-4 + \tau, -\tau) \cup (\tau, 4 - \tau)$$

($\tau > 0$ arbitrary, but fixed), noting that $\mathrm{spec}\Delta = [-4, 4]$ in $d = 2$; moreover, the wave operators were constructed, and asymptotic completeness was established. The (generalized) eigenfunctions are therefore delocalized. Certain other classes of lattice Schrödinger operators with decaying random potentials have been proven to exhibit a.c. spectrum, scattering, and asymptotic completeness by Bourgain in [11], and by Rodnianski and Schlag in [47]. We also note the contextually related work of Denissov in [20].

As noted in the previous section, Schlag, Shubin and Wolff have proven lower bounds on the localization length of eigenfunctions for $d = 2$ and $\sigma = 0$, of the form $\eta^{-2+\delta}$, for any $\delta > 0$, [49].

We shall here address the case $0 < \sigma \leq \frac{1}{2}$ in dimension $d = 2$. Our main results are as follows.

For the critical decay exponent $\sigma = \frac{1}{2}$, the problem is *marginal* in the language of renormalization group theory. Accordingly, we obtain a comparison of the *logarithm* of the localization length to powers of $\eta$, yielding lower bounds on the localization length that are *exponential* in $\frac{1}{\eta}$, of the form $2^{\eta^{-\frac{1}{4}+\delta}}$ ($\delta > 0$ arbitrary).

In the subcritical case $0 < \sigma < \frac{1}{2}$, it is suspected that the model exhibits a significant component of point spectrum. In the language of renormalization group theory, the potential scales like a *relevant* perturbation, whereby we obtain a comparison of the localization length to powers of $\eta$. Consequently, our lower bounds on the localization lengths are *polynomial* in $\frac{1}{\eta}$ for $0 < \sigma < \frac{1}{2}$, of the form $\eta^{-\frac{2-\delta}{1-2\sigma}}$ ($\delta > 0$ arbitrary).

**Theorem 4.3.** *(T. C., [15]) For $\mu > 0$ sufficiently small, $0 < \eta \ll \mu$, any fixed $\tau$ with $\eta \ll \tau < \mu$, and any arbitrary $\delta > 0$, the lower bound on the localization length $\ell_\sigma(\eta)$ of eigenfunctions for eigenvalues in $I_\tau$ satisfies the following estimates:*

- *In the subcritical case $0 < \sigma < \frac{1}{2}$, there exist positive constants $\eta_0(\sigma, \delta) \ll 1$ and $C_\sigma$ for every fixed $0 < \sigma < \frac{1}{2}$ such that*

$$\ell_\sigma(\eta) \geq C_\sigma \eta^{-\frac{2-\delta}{1-2\sigma}} \tag{4.5}$$

  *for all $\eta < \eta_0(\sigma, \delta)$.*

- *In the critical case $\sigma = \frac{1}{2}$, there exists a positive constant $\eta_0(\delta) \ll 1$ such that*

$$\ell_{\sigma=\frac{1}{2}}(\eta) \geq 2^{\eta^{-\frac{1}{4}+\delta}} \tag{4.6}$$

  *for all $\eta < \eta_0(\delta)$.*

These estimates "interpolate" between the lower bound $\eta^{-2+\delta}$ of Schlag-Shubin-Wolff for $\sigma = 0$, and $\infty$ corresponding to pure a.c. spectrum for $\sigma > \frac{1}{2}$ due to Bourgain.

## § 4.2.   Outline of the proof

The proof uses Feynman graph expansions adapted to a dyadic partition of unity on $\mathbb{Z}^2$, combined with the smoothing of resolvent multipliers due to dyadic restriction, inspired by Bourgain in [10].

We introduce a dyadic partition of unity,

$$\sum_{j=0}^{\infty} P_j = 1$$

where we require that $P_j$ and $v_\sigma$ satisfy:

- $P_j \sim \chi(2^j < |x| \le 2^{j+1})$.

- $\|\mathcal{F}(P_j P_{j'})\|_{L^1(\mathbb{T}^2)} \sim 1$ if $|j - j'| \le 1$

- $P_j P_{j'} = 0$ if $|j - j'| > 1$.

- $V_\omega(x) = v_\sigma(x)\omega_x$ where $v_\sigma$ satisfies

$$(4.7) \quad |\mathcal{F}(P_j P_{j'} v_\sigma^2)| \lesssim 2^{-2\sigma j}|\mathcal{F}(P_j P_{j'})| \sim 2^{-2\sigma j}|\mathcal{F}(P_j^2)| \qquad \text{if } |j - j'| \le 1 \,.$$

This implies

$$(4.8) \qquad\qquad |x|^\sigma |v_\sigma(x)| \lesssim 1 \qquad , \qquad 0 < \sigma \le \frac{1}{2}$$

because

$$(4.9) \qquad \|P_j v_\sigma\|_{\ell^\infty(\mathbb{Z}^2)} = \|P_j^2 v_\sigma^2\|_{\ell^\infty(\mathbb{Z}^2)}^{1/2} \le \|\mathcal{F}(P_j^2 v_\sigma^2)\|_{L^1(\mathbb{T}^2)}^{1/2} \sim 2^{-\sigma j}$$

holds.

Next, we partition $V \equiv V_\omega$ into

$$V = \sum_{j=0}^{J+1} V_j$$

where $0 \le j \le J$ account for the dyadic annuli at scales $|x| \sim 2^j$, and where $J + 1$ accounts for the unbounded region $|x| > 2^{J+1}$,

$$(4.10) \qquad\qquad V_j(x) = P_j(x)\, v_\sigma(x)\, \omega_x$$

for $0 \le j \le J$, and

$$(4.11) \qquad\qquad V_{J+1}(x) = \Big( \sum_{j=J+1}^\infty P_j(x) \Big)\, v_\sigma(x)\, \omega_x \,.$$

It is evident that

$$(4.12) \qquad \mathbb{E}[V_j(x)V_{j'}(x')] = \delta_{|j-j'|\le 1} P_j(x)P_{j'}(x)v_\sigma^2(x)\delta_{x,x'} \lesssim 2^{-2\sigma j}\delta_{x,x'}$$

and

$$(4.13) \qquad\qquad \mathbb{E}[V_{J+1}(x)V_{J+1}(x')] \lesssim 2^{-2\sigma J}\delta_{x,x'} \,.$$

The expectations $\mathbb{E}[\prod_i V_{j_i}(x_i)]$ satisfy *Wick's theorem*.

Subsequently, we invoke estimates on the restriction of the resolvent $\frac{1}{e_\Delta - \alpha - i\epsilon}$ to dyadic shells, which we adapt from those proven by Bourgain in [10].

**Lemma 4.4.** *Assume that $\alpha \in I_\tau := (-4 + \tau, -\tau) \cup (\tau, 4 - \tau)$. Then,*

$$(4.14) \quad \left\| \left| \frac{1}{e_\Delta - \alpha - i\epsilon} \right| * |\mathcal{F}(P_j P_{j'} v_\sigma^2)| \right\|_{L^\infty(\mathbb{T}^2)} \lesssim \begin{cases} 2^{j(1-2\sigma)} & \text{if } j \leq J \\ \sigma^{-1} 2^{-2\sigma J} \epsilon^{-1} & \text{if } j, j' = J+1 \end{cases}$$

*and*

$$(4.15) \quad \left\| \left| \frac{1}{e_\Delta - \alpha - i\epsilon} \right| * |\mathcal{F}(P_j P_{j'} v_\sigma^2)| \right\|_{L^1(\mathbb{T}^2)} \lesssim \log \frac{1}{\epsilon}$$

*for $0 \leq j, j' \leq J+1$. The implicit constants only depend on $\tau$, and are finite for $\tau > 0$.*

Again, we evaluate the expectation $\mathbb{E}$ explicitly in the resolvent expansion for $\phi_t$, truncated at $N$. Subsequently, we organize the resulting terms by use of Feynman graphs, and in a similar manner as in section 2, we sum the contributions from all scales, and obtain

$$(4.16) \quad \mathbb{E}[\,\| \phi_t - e^{it\Delta}\phi_0 \|_2^2\,] \lesssim \eta^2 \underbrace{\sum_{j=0}^{J} 2^{j(1-2\sigma)}}_{\sim J 2^{(1-2\sigma)J}} + \underbrace{\eta^2 \, \epsilon^{-1} \, \sigma^{-1} 2^{-2\sigma J}}_{J+1-\text{th term}} + (\text{error terms}),$$

where the first term on the rhs stems from the sum over all annuli with $0 \leq j \leq J$. The second term stems from the remainder term indexed by $J + 1$ which accounts for $\{x \in \mathbb{Z}^2 \,|\, |x| > 2^{J+1}\}$. The last term depends on $N, J, \sigma, \epsilon, \eta$, and accounts for various errors terms which we will not address in any detail.

We again note that based on the arguments in the previous section, $t$ is proportional to lower bound on localization length. Hence, our goal is to maximize $t = \epsilon^{-1}$ while minimizing the rhs of (4.16).

The two dominant terms on the rhs of (4.16) are comparable and small for the following choices of parameters:

- *Subcritical $\sigma < \frac{1}{2}$:* We set $\epsilon = t^{-1} \sim 2^{-(1-2\sigma)J}$ and $\eta^{-2+\delta} \sim J 2^J$, and $N \sim \frac{|\log \epsilon|}{\log |\log \epsilon|}$

- *Critical $\sigma = \frac{1}{2}$:* We set $\epsilon = t^{-1} \sim 2^{-J}$ and $\eta^{-\frac{1}{4}+\delta} = J = N$. We note that the exponent $\frac{1}{4}$ is not optimal, owing to the loss of some powers of $\eta$ caused by the error terms.

From the above results, we conclude the following lower bounds on the localization length:

- *Subcritical $\sigma < \frac{1}{2}$:* The localization length is bounded from below by $\gtrsim \left(\eta^{-2+\delta}\right)^{\frac{1}{1-2\sigma}}$. This result interpolates between the bound of Shubin-Schlag-Wolff in [49], and the scattering result of Bourgain in [10].

- *Critical $\sigma = \frac{1}{2}$:* Due to the critical scaling, we obtain a lower bound on the localization length $\gtrsim 2^{-\left(\frac{1}{\eta}\right)^{1/4-\delta}}$. We note that this result and its proof are reminiscent of Nekhoroshev type estimates.

This concludes our outline of the proof of Theorem 4.3.                    □

## §4.3.   Renormalization group interpretation

The localization conjecture for the weakly disordered Anderson model in $d = 2$ is known as extremely hard. In view of the analysis of the weakly disordered Anderson model with decaying random potential, we arrive at the following interpretation of this problem, from the point of view of renormalization group theory:

- For supercritical $\sigma > \frac{1}{2}$, the random potential is a perturbation is RG *irrelevant* perturbation of the lattice laplacian, $\Delta$. In the scaling limit, $\Delta$ dominates over $V_\omega$. Accordingly, as proven by Bourgain in [10], the perturbed Hamiltonian $H_\omega$ is unitarily equivalent to $\Delta$, via the scattering map, for energies in $I_\tau$. In this sense, $\Delta$ and $H_\omega$ belong to the same universality class of Hamiltonians, and restricted to the spectral intervals $I_\tau$, $H_\omega$ has absolutely continuous spectrum.

- For critical $\sigma = \frac{1}{2}$, the random potential is a RG *marginal* perturbation of $\Delta$. In the scaling limit, $\Delta$ and $V_\omega$ scale in the same manner. The problem is scaling critical, and the type of the spectrum is not known at present. It is not known if this problem is strictly marginal, marginally relevant, or marginally irrelevant.

- For subcritical $\sigma < \frac{1}{2}$, the random potential is a RG *relevant* perturbation of $\Delta$. In the scaling limit, $V_\omega$ dominates over $\Delta$, which makes the conjectural presence of point spectrum and localized states plausible. However, RG relevant problems are notoriously difficult, and at present, there are no known methods to attack this problem. In quantum field theory, RG relevant problems appear in the vicinity of RG unstable fixed points, and are far less investigated than RG irrelevant or marginally irrelevant problems.

In this sense, we conclude: **The localization conjecture for the weakly disordered $(0 < \eta \ll 1)$ Anderson model in $d = 2$ is hard because it constitutes a RG *relevant* problem.**

## §5.   Fermi gas in random medium - I. The ideal Fermi gas

The Anderson model neglects the repulsion between the electrons due to Coulomb interactions, and the Pauli principle. This section addresses some recent results concerning the dynamics of an ideal Fermi gas in a random medium, at positive temperature, based on joint work of the author with I. Sasaki, [16]. In the next section, we address a Fermi gas in a random medium with dynamical Hartree-Fock interactions, based on joint work with I. Rodnianski, [17]. Our goal is to investigate the extent to which manybody effects influence the predictions of the weakly disordered Anderson model.

We also refer to [4, 18, 19] for important recent results on the persistence of localization in fermionic manybody models at strong disorders (a topic which is not addressed here).

We consider a fermion gas in a finite box $\Lambda_L := [-\frac{L}{2}, \frac{L}{2}]^d \cap \mathbb{Z}^d$ of side length $L \gg 1$, with periodic boundary conditions, in dimensions $d \geq 3$. We denote its dual lattice by $\Lambda_L^* := \Lambda_L / L \subset \mathbb{T}^d$.

We denote the fermionic Fock space of scalar electrons by

$$(5.1) \qquad \mathfrak{F} = \bigoplus_{n \geq 0} \mathfrak{F}_n \,,$$

where

$$(5.2) \qquad \mathfrak{F}_0 = \mathbb{C} \quad , \quad \mathfrak{F}_n = \bigwedge_1^n \ell^2(\Lambda_L) \,, \; n \geq 1 \,.$$

We introduce creation- and annihilation operators $a_p^+$, $a_q$, for $p$, $q \in \Lambda_L^*$, satisfying the canonical anticommutation relations

$$(5.3) \qquad a_p^+ \, a_q + a_q \, a_p^+ = \delta(p - q) := \begin{cases} L^d & \text{if } \; p = q \\ 0 & \text{otherwise.} \end{cases}$$

We define the fermionic manybody Hamiltonian

$$(5.4) \qquad H_\omega := T + \eta \, V_\omega$$

where

$$(5.5) \qquad T = \int dp \, E(p) \, a_p^+ \, a_p$$

is the kinetic energy operator; here we recall the notation $\int dp \equiv \frac{1}{|\Lambda_L|} \sum_{p \in \Lambda_L^*}$. Moreover,

$$(5.6) \qquad V_\omega := \sum_{x \in \Lambda_L} \omega_x \, a_x^+ a_x$$

couples the fermions to a static random potential. As previously, in the case of the Anderson model, $\{\omega_x\}_{x \in \Lambda_L}$ is a field of i.i.d. real-valued random variables which we assume to be centered, normalized, and Gaussian for simplicity. Thus,

$$(5.7) \qquad\qquad \mathbb{E}[\,\omega_x\,] \;=\; 0 \;\;, \;\; \mathbb{E}[\,\omega_x^2\,] \;=\; 1$$

for all $x \in \Lambda_L$. Moreover, we assume that

$$(5.8) \qquad\qquad E(p) \;=\; \sum_{j=1}^{d} \cos(2\pi p_j)\,,$$

which defines the Fourier multiplier corresponding to the nearest neighbor Laplacian on $\mathbb{Z}^d$.

Let

$$(5.9) \qquad\qquad N \;:=\; \sum_{x \in \Lambda_L} a_x^{+} a_x$$

denote the particle number operator. Clearly,

$$(5.10) \qquad\qquad [H_\omega, N] \;=\; 0$$

holds.

Let $\mathfrak{A}$ denote the $C^*$-algebra of bounded operators on $\mathfrak{F}$. We consider the dynamics on $\mathfrak{A}$ given by

$$(5.11) \qquad\qquad \alpha_t(A) \;=\; e^{itH_\omega}\, A\, e^{-itH_\omega}$$

generated by the random Hamiltonian $H_\omega$.

## §5.1.  Statement of the main results

We consider a normalized, translation-invariant

$$(5.12) \qquad\qquad \rho_0 : \mathfrak{A} \longrightarrow \mathbb{C}\,,$$

which is deterministic in the sense that it does not depend on $\{\omega_x\}_x$. Accordingly, we define the associated time-evolved state

$$(5.13) \qquad\qquad \rho_t(A) \;:=\; \rho_0(\,e^{itH_\omega}\, A\, e^{-itH_\omega}\,)\,,$$

with $t \in \mathbb{R}$, and initial condition given by $\rho_0$. We particularly focus on the dynamics of the averaged two-point functions

$$(5.14) \qquad\qquad \mathbb{E}[\,\rho_t(\,a_p^{+} a_q\,)\,]\,,$$

where $p, q \in \Lambda_L^*$. Clearly,

$$(5.15) \qquad \mathbb{E}[\,\rho_0(\,a_p^+ a_q\,)\,] \; = \; \rho_0(\,a_p^+ a_q\,) \; = \; \delta(p-q)\,\frac{1}{L^d}\,\rho_0(\,a_p^+ a_p\,)\,,$$

where

$$(5.16) \qquad \delta(k) \; := \; L^d \delta_k\,,$$

and where

$$(5.17) \qquad \delta_k \; = \; \begin{cases} 1 \text{ if } k \equiv 0 \bmod \Lambda_L^* \\ 0 \text{ otherwise} \end{cases}$$

denotes the Kronecker delta on the lattice $\Lambda_L^*$ (mod $\mathbb{T}^d$). We remark that for fermions,

$$(5.18) \qquad 0 \;\leq\; \frac{1}{L^d}\,\rho_0(\,a_p^+ a_p\,) \;\leq\; 1\,,$$

since $\|a_p^{(+)}\| = L^{d/2}$ in operator norm, $\forall p \in \Lambda_L^*$.

## § 5.2. Boltzmann limit of the momentum distribution function

We denote the microscopic time, position, and velocity variables by $(t, x, p)$, and the corresponding macroscopic variables by $(T, X, V) = (\eta^2 t, \eta^2 x, v)$. We prove that the momentum distribution $f_t(q)$ converges to a solution of a linear Boltzmann equation in the limit $\eta \to 0$.

**Theorem 5.1.** *(T. C., I. Sasaki, [16]) We assume that $\rho_0$ is translation invariant. Then, the averaged two-point functions are translation invariant,*

$$(5.19) \qquad \mathbb{E}[\rho_t(\,a^+(f)a(g)\,)] \; = \; \int dp\,\overline{f(p)}\,g(p)\,\mathbb{E}[\rho_t(\,a_p^+ a_p\,)]\,,$$

*(i.e., diagonal in $a_p^+, a_p$) for any $f, g \in \mathcal{S}(\mathbb{T}^d)$ of Schwartz class, and the thermodynamic limit*

$$(5.20) \qquad \Omega_T^{(2;\eta)}(f;g) \; := \; \lim_{L\to\infty} \mathbb{E}[\rho_{T/\eta^2}(\,a^+(f)\,a(g)\,)]$$

*exists for all $f, g \in \mathcal{S}(\mathbb{T}^d)$, and $T > 0$.*

*For any $T > 0$ and all $f, g \in \mathcal{S}(\mathbb{T}^d)$, the limit*

$$(5.21) \qquad \Omega_T^{(2)}(f;g) \; := \; \lim_{\eta\to 0} \Omega_T^{(2;\eta)}(f;g)$$

*exists, and is the inner product of $f, g$ with respect to a Borel measure $F_T(p)dp$,*

$$(5.22) \qquad \Omega_T^{(2)}(f;g) \; = \; \int dp\, F_T(p)\,\overline{f(p)}\,g(p)\,,$$

*where $F_T(V)$ satisfies the linear Boltzmann equation*

$$(5.23) \qquad \partial_T F_T(V) \; = \; 2\,\pi \int_{\mathbb{T}^d} dU \, \delta(\, E(U) - E(V)\,)\,(\, F_T(U) - F_T(V)\,)\,,$$

*with initial condition*

$$(5.24) \qquad\qquad F_0(p) \; = \; \lim_{L \to \infty} \frac{1}{L^d} \, \rho_0(\, a_p^+ \, a_p\,)$$

*for $p \in \mathbb{T}^d$.*

## § 5.3.   Outline of the Proof

The proof of Theorem 5.1 can be sketched as follows. We consider the Heisenberg evolution of the creation- and annihilation operators,

$$(5.25) \qquad\qquad a(f,t) \; := \; e^{itH_\omega} a(f) e^{-itH_\omega}\,,$$

where $f$ is a test function.

We make the key observation that since $H_\omega$ is bilinear in $a^+, a$, it follows that $a(f,t)$ is a linear superposition of annihilation operators. Therefore, there exists a function $f_t$ such that

$$(5.26) \qquad\qquad a(f,t) \; = \; a(f_t) \; = \; \int dp \, \overline{f_t(p)} \, a_p \; = \; (a^+(f_t))^*$$

In particular,

$$i\partial_t a(f_t) = [\, H_\omega \,,\, a(f_t)\,]$$
$$= \int dp \, f_t(p) \, E(p) \, a_p \; + \; \eta \int dp \int du \, f_t(p) \, \widehat{V}_\omega(u - p) \, a_u$$
$$(5.27) \qquad\qquad = a(\, \Delta f_t\,) \; + \; a(\, \eta \, V_\omega^{(1)} f_t\,)\,,$$

and moreover, it is clear that $a(f,0) = a(f_0) = a(f)$. As before, $\Delta$ is the nearest neighbor Laplacian on $\Lambda_L$. Moreover, $H_\omega^{(1)} = H_\omega|_{\mathfrak{F}_1}$ denotes the 1-particle Anderson Hamiltonian, and $V_\omega^{(1)} = V_\omega|_{\mathfrak{F}_1}$ is the 1-particle multiplication operator $(V_\omega^{(1)} f)(x) = \omega_x f(x)$.

Thus, $f_t$ solves the 1-particle random Schrödinger equation

$$(5.28) \qquad\qquad i\partial_t f_t \; = \; H_\omega^{(1)} f_t \; := \; \Delta f_t \; + \; \eta \, V_\omega^{(1)} f_t$$

$$(5.29) \qquad\qquad\qquad\qquad f_0 \; = \; f\,.$$

Accordingly,

$$\rho_t(\, a^+(f)\, a(g)\,) = \rho_0(\, a^+(f_t)\, a(g_t)\,)$$

$$= \int dp\, dq\, \rho_0(\, a_p^+\, a_q)\, \overline{f_t(p)}\, g_t(q)$$

$$(5.30) \qquad = \int dp\, J(p)\, \overline{f_t(p)}\, g_t(p)$$

where the momentum conservation condition

$$(5.31) \qquad \rho_0(\, a_p^+\, a_q)\, =\, \delta(p-q)\, J(p)$$

follows from translation invariance of $\rho_0$, with

$$(5.32) \qquad 0\, \leq\, J(p)\, =\, \frac{1}{L^d}\rho_0(\, a_p^+ a_p\,)\, =\, \frac{1}{1+e^{h(p)}}\, \leq\, 1\,.$$

The fact that $J(p) \leq 1$ is a consequence of the Pauli principle.

For $N \in \mathbb{N}$, we expand $f_t$, $g_t$ into the truncated Duhamel series,

$$(5.33) \qquad f_t\, =\, f_t^{(\leq N)}\, +\, f_t^{(>N)}\,,$$

with

$$(5.34) \qquad f_t^{(\leq N)}\, :=\, \sum_{n=0}^{N} f_t^{(n)}\,.$$

The Duhamel term of $n$-th order (in powers of $\eta$) is given by

$$(5.35) \qquad \widehat{f_t^{(n)}}(p) := \eta^n\, e^{\epsilon t} \int d\alpha\, e^{it\alpha} \int dk_0 \cdots dk_n\, \delta(p-k_0)$$

$$\Big( \prod_{j=0}^{n} \frac{1}{E(k_j) - \alpha - i\epsilon} \Big) \Big( \prod_{j=1}^{n} \widehat{V}_\omega(k_j - k_{j-1}) \Big)\, \widehat{f}(k_n)\,.$$

As in the discussion of the Boltzmann limit for the weakly disordered Anderson model, we choose

$$(5.36) \qquad \epsilon\, =\, \frac{1}{t}$$

so that the factor $e^{\epsilon t}$ remains bounded for all $t$. By

$$(5.37) \qquad f_t^{(>N)}\, =\, i\eta \int_0^t ds\, e^{i(t-s)H_\omega}\, V_\omega^{(1)}\, f_t^{(N)}(s)\,,$$

we account for the Duhamel remainder term.

Accordingly,

$$\rho_t(\,a^+(f)\,a(g)\,) \;=\; \rho_0(\,a^+(f_t)\,a(g_t)\,) \;=\; \sum_{n,\widetilde{n}\in\mathcal{I}_N} \rho_t^{(n,\widetilde{n})}(f;g) \tag{5.38}$$

where

$$\rho_t^{(n,\widetilde{n})}(f;g) \;:=\; \rho_0(\,a^+(f_t^{(n)})\,a(g_t^{(\widetilde{n})})\,) \tag{5.39}$$

for $\mathcal{I}_N := \{1,\ldots,N,>N\}$.

Next, we use the following notation. If $n,\widetilde{n} \le N$, and $n+\widetilde{n}$ is odd, $\mathbb{E}[\rho_t^{(n,\widetilde{n})}(p,q)] = 0$. Thus, let

$$\bar{n} \;:=\; \frac{n+\widetilde{n}}{2} \;\in\; \mathbb{N}, \tag{5.40}$$

and we define $\{u_j\}_{j=0}^{2\bar{n}+1}$ by

$$u_j \;:=\; \begin{cases} k_{n-j} & \text{if } j \le n \\ \widetilde{k}_{j-n-1} & \text{if } j \ge n+1 \,. \end{cases} \tag{5.41}$$

Thus, for $n,\widetilde{n} \le N$ (and $\widehat{V}_\omega(u)^* = \widehat{V}_\omega(-u)$),

$$\mathbb{E}[\rho_t^{(n,\widetilde{n})}(f;g)] \;=\; \eta^{2\bar{n}}\,e^{2\epsilon t} \int d\alpha\,d\widetilde{\alpha}\,e^{it(\alpha-\widetilde{\alpha})}$$

$$\int du_0\cdots du_{2\bar{n}+1}\,\overline{f(u_0)}\,g(u_{2\bar{n}+1})\,J(u_n)\,\delta(u_n - u_{n+1})$$

$$\prod_{j=0}^{n} \frac{1}{E(u_j)-\alpha-i\epsilon} \prod_{\ell=n+1}^{2\bar{n}+1} \frac{1}{E(u_\ell)-\widetilde{\alpha}+i\epsilon}$$

$$\mathbb{E}\Big[\,\prod_{j=1}^{n} \widehat{V}_\omega(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{V}_\omega(u_j - u_{j-1})\,\Big] \tag{5.42}$$

This expression is completely analogous to (2.16) in our previous discussion of the Anderson model, and we organize the expectation with respect to the random potential by use of Feynman diagrams.

Accordingly, to prove the theorem, we show that the Feynman amplitudes of crossing and nesting diagrams yield small error terms, and that the amplitudes of decorated ladder diagrams are dominant.

The sum of Feynman amplitudes associated to decorated ladder diagrams yields the solution of the linear Boltzmann equation, as asserted in the theorem.  □

## §5.4.  Discussion of the Result

An initial condition of particular interest is the Gibbs state (with inverse temperature $\beta$ and chemical potential $\mu$) for a non-interacting fermion gas,

$$(5.43) \qquad \rho_0(A) \;=\; \frac{1}{Z_{\beta,\mu}} \, \mathrm{Tr}\big(\, e^{-\beta(T-\mu N)} A \,\big)$$

where $Z_{\beta,\mu} := \mathrm{Tr}\big(\, e^{-\beta(T-\mu N)} \,\big)$.

The momentum distribution in the free Gibbs state is the *Fermi-Dirac distribution*

$$F_0(p) \;=\; \lim_{L\to\infty} \rho_0\big(\, \frac{1}{L^d}\, a_p^+ \, a_p \,\big). \;=\; \frac{1}{1 + e^{\beta(E(p)-\mu)}}$$

According to it, the probability of having a plane wave with momentum $p$ is $\frac{F_0(p)}{\int dp\, F_0(p)}$.

We make the key observation that for all $0 < \beta \leq \infty$,

$$F_0(p) \;=\; \frac{1}{1 + e^{\beta(E(p)-\mu)}}$$

is a *stationary solution* of the Boltzmann equation. This remains true in zero temperature limit $\beta \to \infty$ where (in the weak sense)

$$\frac{1}{1 + e^{\beta(E(p)-\mu)}} \;\to\; \chi[E(p) < \mu]$$

which is nontrivial whenever $\mu > 0$.

## §5.5.  Persistence of Quasifreeness

A state $\rho_0$ is quasifree (determinantal) if

$$\rho_0\big(\, a^+(f_1) \cdots a^+(f_r) a(g_1) \cdots a(g_s) \,\big)$$

$$= \delta_{r,s} \, \det \big[\, \rho_0\big(\, a^+(f_i) a(g_j) \,\big) \,\big]_{1 \leq i,j \leq r} .$$

In a quasifree state, the particles are uncorrelated, and obey the Pauli principle. In addition to the Boltzmann limit described above, we are interested in the influence of the random potential on the property of quasirandomness.

We observe that since $H_\omega$ is bilinear in the creation- and annihilation operators,

$$(5.44) \qquad K(t) \;:=\; e^{itH_\omega} \, K \, e^{-itH_\omega}$$

is also bilinear in $a^+, a$. Therefore,

$$(5.45) \qquad \rho_t(A) \;=\; \frac{1}{Z_K} \, \mathrm{Tr}\big(\, e^{-K(t)} A \,\big)$$

is quasifree with probability 1 (where $Z_K := \mathrm{Tr}(e^{-K})$).

However, the average $\mathbb{E}[\rho_t(\,\cdot\,)]$ is *not quasifree*, for any $\eta > 0$, and $t > 0$. Notably, quasifreeness is a *nonlinear* condition on determinants.

However, the kinetic scaling limit produces a quasifree limiting state.

**Theorem 5.2.** *(T. C., I. Sasaki, [16]) Assume that $\rho_0$ is number conserving, quasifree, and translation invariant. Then, the following holds. For any normal ordered monomial in creation- and annihilation operators,*

$$(5.46) \qquad a^+(f_1) \cdots a^+(f_r)\, a(g_1) \cdots a(g_r)\,,$$

*with $r, s \in \mathbb{N}$ and Schwartz class test functions $f_j, g_\ell \in \mathcal{S}(\mathbb{T}^d)$, and any $T > 0$, the macroscopic $2r$-point function*

$$(5.47) \qquad \Omega_T^{(2r)}(\, f_1, \ldots, f_r\,;\, g_1, \ldots, g_r\,)$$
$$:= \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}[\rho_{T/\eta^2}(\, a^+(f_1) \cdots a^+(f_r)\, a(g_1) \cdots a(g_r)\,)]$$

*exists and is quasifree,*

$$(5.48) \qquad \Omega_T^{(2r)}(\, f_1, \ldots, f_r\,;\, g_1, \ldots, g_r\,) \;=\; \det\big[\,\Omega_T^{(2)}(\, f_i\,,\, g_j\,)\,\big]_{1 \le i,j \le r}\,.$$

*The macroscopic 2-point function is the same as in Theorem 5.1,*

$$(5.49) \qquad \Omega_T^{(2)}(\, f\,;\, g\,) \;=\; \int dp\, F_T(p)\, \overline{f(p)}\, g(p)\,,$$

*and $F_T(p)$ solves the linear Boltzmann equation (5.23) with initial condition (5.24).*

For the proof, we employ the fact that the main estimate

$$\lim_{\eta \to 0} \lim_{L \to \infty} \Big| \mathbb{E}[\rho_{T/\eta^2}(\, a^+(f_1) \cdots a^+(f_r)\, a(g_1) \cdots a(g_r)\,)]$$

$$- \det\big[\,\Omega_T^{(2)}(\, f_i\,;\, g_j\,)\,\big]_{1 \le i,j \le r} \Big| \;=\; 0$$

can be interpreted as a corollary of (3.11) of Theorem 3.1.                    $\square$

## §6.   Fermi gas in random medium - II. Fermi gas with dynamical Hartree-Fock interaction

In this section, we include particle interactions between the fermions, modeled in dynamical Hartree-Fock theory. As a consequence, it is necessary to control both randomness and the nonlinearities arising from the self-interaction of the field. The results presented here are based on joint work with I. Rodnianski, [17].

We consider the time-dependent Hamiltonian

$$(6.1) \qquad\qquad H(t) \;=\; T \,+\, \eta\, V_\omega \,+\, \lambda\, W(t)$$

where the fermion-fermion interaction is modeled by

$$(6.2) \qquad W(t) \;=\; \sum_{x,y} v(x-y) \left\{ \mathbb{E}[\,\rho_t(a_x^+ a_x)\,]\, a_y^+ a_y \,-\, \mathbb{E}[\,\rho_t(a_y^+ a_x)\,]\, a_x^+ a_y \right\}.$$

The terms on the rhs correspond to the Hartree-Fock direct and exchange term, respectively. The coupling constant $\lambda$ accounts for the strength of interaction between the fermions. The kinetic energy operator $T$ and the operator $V_\omega$ which describes the interaction of each fermion with the static random potential are as in the previous section. For technical reasons that we will not further address here, we assume that $\|\widehat{v}\|_{H^{3/2+\delta}(\mathbb{T}^3)} < C$ for $\delta > 0$ arbitrary but fixed.

We are interested in the dynamics of two-point function, which is determined by

$$
\begin{aligned}
& i\partial_t\, \rho_t(\,a_p^+ a_q\,) \\
&= (\,E(p) - E(q)\,)\, \rho_t(\,a_p^+ a_q\,) \\
&\quad + \lambda \int du\, \mathbb{E}[\,\rho_t(\,\tfrac{1}{L^d} a_u^+ a_u\,)\,]\, (\,\widehat{v}(u-p)\, \rho_t(\,a_u^+ a_q\,) \,-\, \widehat{v}(q-u)\, \rho_t(\,a_p^+ a_u\,)\,) \\
(6.3) &\quad + \eta \int du\, \widehat{\omega}(u-p) \rho_t(\,a_u^+ a_q\,) \,-\, \widehat{\omega}(q-u) \rho_t(\,a_p^+ a_u\,)
\end{aligned}
$$

for any realization of the random potential, where $\widehat{\omega}(u) := \sum_{x \in \Lambda_L} \omega_x e^{-2\pi i u \cdot x}$ is well-defined, almost surely (where we will ultimately let $L \to \infty$).

We make the following key observations:

- For a generic realization of the random potential, the problem is *not* translation invariant.

- The equation (6.3) for the momentum distribution function $\rho_t(\,a_p^+ a_p\,)$ does not close.

However, we can close the equation for the momentum distribution function by taking the expectation, $\mathbb{E}$. This is because the $\mathbb{E}$-average is translation invariant, due

to the homogeneity of the randomness. Then, the average state $\mathbb{E}[\rho_t(\,\cdot\,)] : \mathfrak{A} \to \mathbb{C}$ solves

$$i\partial_t \mathbb{E}[\,\rho_t(\,A\,)\,] = \mathbb{E}[\,\rho_t(\,[\,H(t)\,,\,A\,]\,)\,]$$

(6.4)                          $$\mathbb{E}[\rho_0] = \rho_0\,.$$

This is a self-consistent nonlinear initial value problem determining $\mathbb{E}[\rho_t(\,\cdot\,)]$.

We note that for every realization of $V_\omega$, we have

$$\rho_t(\,A\,) \;=\; \rho_0(\,\mathcal{U}_t^* \, A\, \mathcal{U}_t\,)$$

for $A \in \mathfrak{A}$, with $\mathcal{U}_t$ unitary,

$$i\partial_t \mathcal{U}_t \;=\; H(t)\mathcal{U}_t\,,$$

and $\mathcal{U}_0 = \mathbf{1}$. Notably, the Hamiltonian $H(t)$ itself depends on $\mathbb{E}[\rho_t(\,\cdot\,)]$. In particular, we note that

(6.5)                    $$\rho_t(\,a^+(f)\,a(g)\,) \;=\; \rho_0(\,a^+(f,t)\,a(g,t)\,)\,.$$

The Heisenberg evolution of the creation- and annihilation operators is determined by

(6.6)                          $$a(f,t) \;:=\; \mathcal{U}_t^*\,a(f)\mathcal{U}_t\,.$$

Similarly as in the case discussed for the ideal Fermi gas, there exists a function $f_t$ such that

(6.7)                          $$a(f,t) \;=\; a(f_t)\,,$$

where $f_t$ is the solution of the 1-particle random Schrödinger equation

(6.8)          $$i\partial_t f_t(p) \;=\; E(p)f_t(p) \;+\; \eta\,(\,\widehat{V}_\omega * f_t\,)(p) \;-\; \lambda\,(\,\widehat{v} * \mu_t\,)(p)f_t(p)$$

with initial condition

(6.9)                          $$f_0 \;=\; f\,.$$

Noting that the Hamiltonian $H(t)$ itself depends on the unknown quantity

(6.10)                    $$\mu_t(p) \;:=\; \frac{1}{L^3}\,\mathbb{E}[\,\rho_t(\,a_p^+ a_p\,)\,]\,,$$

we determine $\mu_t$ by writing the solution to (6.4) in integral form, as an expansion in powers of $\eta$.

For arbitrary test functions $f$ and $g$, we consider the pair correlation function

$$\rho_t(\, a^+(f)\, a(g)\,) = \rho_0(\, a^+(f_t)\, a(g_t)\,)$$
$$= \int dp\, dq\, \rho_0(\, a_p^+\, a_q\,)\, \overline{f_t(p)}\, g_t(q)$$
$$(6.11) \qquad\qquad = \int dp\, J(p)\, \overline{f_t(p)}\, g_t(p)\,.$$

Passing to the last line, we have used the momentum conservation condition

$$(6.12) \qquad\qquad \rho_0(\, a_p^+\, a_q\,) \;=\; J(p)\, \delta(p-q)$$

obtained from the translation invariance of the initial state $\rho_0$, where

$$(6.13) \qquad\qquad 0 \;\leq\; J(p) \;=\; \frac{1}{L^3}\rho_0(\, a_p^+ a_p\,) \;\leq\; 1\,,$$

similarly as in the case of the ideal Fermi gas.

The solution $f_t$ of (6.8), (6.9), satisfies the Duhamel formula

$$(6.14) \qquad\qquad f_t(p) \;=\; U_{0,t}(p)\, f(p) \;+\; i\,\eta \int_0^t ds\, U_{s,t}(p)\, (\, \widehat{V}_\omega * f_s\,)(p)$$

where

$$(6.15) \qquad\qquad U_{s,t}(p) \;:=\; e^{i\int_s^t ds'\, (\, E(p) - \lambda\kappa_{s'}(p)\,)}$$

and

$$(6.16) \qquad\qquad \kappa_s(u) \;:=\; (\, \widehat{v} * \mu_s\,)(u)\,.$$

We note that the term $U_{0,t}(p)f(p)$ solves (6.8) for $\eta = 0$ (no random potential) with initial condition (6.9).

Let $N \in \mathbb{N}$, which remains to be optimized. The $N$-fold iterate of (6.14) is given by the truncated Duhamel expansion with remainder term,

$$(6.17) \qquad\qquad f_t \;=\; f_t^{(\leq N)} \;+\; f_t^{(>N)}\,,$$

where

$$(6.18) \qquad\qquad f_t^{(\leq N)} \;:=\; \sum_{n=0}^{N} f_t^{(n)}\,,$$

and $f_t^{(>N)}$ is the Duhamel remainder term of order $N$. We define

$$(6.19) \qquad\qquad t_{-1} \;:=\; 0\;,\;\; t_j \;=\; s_0 + \cdots + s_j\,,$$

for $j = 0, \ldots, n$, and

$$(6.20) \quad \mathcal{R}(k_0, \ldots, k_n; z) := \int_{\mathbb{R}_+^{n+1}} ds_0 \cdots ds_n \left( \prod_{j=0}^n e^{-is_j(E(k_j)-z)} e^{i\lambda \int_{t_{j-1}}^{t_j} ds' \, \kappa_{s'}(k_j)} \right),$$

for $z \in \mathbb{C}$.

The $n$-th order term in the Duhamel expansion is given by

$$(6.21) \qquad f_t^{(n)}(p) := (i\eta)^n \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \int dk_0 \cdots dk_n \, \delta(p - k_0)$$

$$\left[ \prod_{j=0}^n U_{t_{j-1}, t_j}(k_j) \right] \left[ \prod_{\ell=1}^n \widehat{V}_\omega(k_\ell - k_{\ell-1}) \right] f(k_n).$$

Expressed in terms of the time increments $s_j := t_j - t_{j-1}$,

$$f_t^{(n)}(p) = (i\eta)^n \int ds_0 \cdots ds_n \, \delta(t - \sum_{j=0}^n s_j) \int dk_0 \cdots dk_n \, \delta(p - k_0)$$

$$(6.22) \qquad \left[ \prod_{j=0}^n e^{-i \int_{t_{j-1}}^{t_j} ds' (E(k_j) - \lambda \kappa_{s'}(k_j))} \right] \left[ \prod_{\ell=1}^n \widehat{V}_\omega(k_\ell - k_{\ell-1}) \right] f(k_n).$$

Expressing the delta distribution $\delta(t - \sum_{j=0}^n s_j)$ in terms of its Fourier transform, we find

$$f_t^{(n)}(p) = (i\eta)^n \, e^{\epsilon t} \int d\alpha \, e^{-it\alpha} \int dk_0 \cdots dk_n \, \delta(p - k_0)$$

$$(6.23) \qquad \mathcal{R}(k_0, \ldots, k_n; \alpha + i\epsilon) \left[ \prod_{j=1}^n \widehat{V}_\omega(k_j - k_{j-1}) \right] f(k_n).$$

The above three equivalent expressions for $f_t^{(n)}(p)$ have different advantages in different contexts, and will all be used in the sequel.

The Duhamel remainder term of order $N$ is given by

$$(6.24) \qquad\qquad f_t^{(>N)} = i\eta \int_0^t ds \, \mathcal{U}_{s,t} \, V_\omega^{(1)} f_s^{(N)}.$$

We choose

$$(6.25) \qquad\qquad\qquad \epsilon = \frac{1}{t}$$

so that the factor $e^{\epsilon t}$ in (6.23) remains bounded for all $t$.

Substituting the truncated Duhamel expansion for $a^+(f_t)$, $a(g_t)$ in (6.11), one obtains

$$(6.26) \qquad \rho_t(a^+(f) a(g)) = \rho_0(a^+(f_t) a(g_t)) = \sum_{n, \tilde{n}=0}^{N+1} \rho_t^{(n, \tilde{n})}(f, g)$$

where

$$\rho_t^{(n,\widetilde{n})}(f,g) := \rho_0\big(a^+(f_t^{(n)})\, a(g_t^{(\widetilde{n})})\big)\,. \tag{6.27}$$

If $n, \widetilde{n} \le N$, we have

$$\mathbb{E}[\rho_t^{(n,\widetilde{n})}(f,g)] = \eta^{2\bar{n}} \sum_{\pi \in \Gamma_{n,\widetilde{n}}} \int_0^t dt_q \cdots \int_0^{t_2} dt_1 \int_0^t d\theta_q \cdots \int_0^{\theta_2} d\theta_1$$

$$\int du_0 \cdots du_{2\bar{n}+1}\, \overline{f(u_0)}\, g(u_{2\bar{n}+1})\, J(u_n)\, \delta(u_n - u_{n+1})$$

$$\tag{6.28}\Big[\prod_{j=0}^n U_{t_{j-1},t_j}(u_j)\Big]\Big[\prod_{j=n+1}^{2\bar{n}+1} \overline{U_{\theta_{j-1},\theta_j}(u_j)}\Big]$$

$$\mathbb{E}\Big[\prod_{j=1}^n \widehat{V}_\omega(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{V}_\omega(u_j - u_{j-1})\Big]$$

and using (6.23), this is equivalent to

$$\mathbb{E}[\rho_t^{(n,\widetilde{n})}(f,g)] = \eta^{2\bar{n}} e^{2\epsilon t} \sum_{\pi \in \Gamma_{n,\widetilde{n}}} \int d\alpha\, d\widetilde{\alpha}\, e^{it(\alpha - \widetilde{\alpha})}$$

$$\int du_0 \cdots du_{2\bar{n}+1}\, \overline{f(u_0)}\, g(u_{2\bar{n}+1})\, J(u_n)\, \delta(u_n - u_{n+1})$$

$$\tag{6.29}\mathcal{R}(u_0,\ldots,u_n;\alpha+i\epsilon)\,\mathcal{R}(u_{n+1},\ldots,u_{2\bar{n}+1};\widetilde{\alpha}-i\epsilon)$$

$$\mathbb{E}\Big[\prod_{j=1}^n \widehat{V}_\omega(u_j - u_{j-1}) \prod_{j=n+2}^{2\bar{n}+1} \widehat{V}_\omega(u_j - u_{j-1})\Big]$$

where $t_{-1}, \theta_{-1} := 0$ in (6.28).

## §6.1. Statement of main results

We introduce macroscopic variables $(T,X)$, related to the microscopic variables $(t,x)$ by

$$(T,X) = (\zeta t, \zeta x)\,, \tag{6.30}$$

with $\zeta$ a real parameter. We will study kinetic scaling limits associated to different scaling ratios between $\zeta$, $\eta$ and $\lambda$.

The random potential has an average effect on the dynamics of $\mu_t$ by an amount proportional to its variance, $O(\eta^2 t)$, in the time interval $[0,t]$. Since the strength of the fermion pair interactions is $O(\lambda)$, both effects are comparable if $\lambda = O(\eta^2)$. Accordingly, we distinguis the following scaling regimes.

### 6.1.1.   The regime $\lambda \leq C\eta^2$

The interaction between electrons and the effect of the random potential per time unit is comparable if $\lambda = C\eta^2$.

**Theorem 6.1.**      *(T. C., I. Rodnianski, [17]) Assume that $\lambda \leq O(\eta^2)$. Then, for any fixed, finite $T > 0$, and any choice of test functions $f$, $g$,*

$$(6.31) \qquad \lim_{\eta \to 0} \lim_{L \to \infty} \mathbb{E}[\, \rho_{T/\eta^2}(\, a^+(f)a(g)\,)\,] = \int dp\, \overline{f(p)}\, g(p)\, F_T(p)$$

*holds, where $F_T(p)$ satisfies the linear Boltzmann equation*

$$(6.32) \qquad \partial_T F_T(p) = 2\pi \int du\, \delta(\, E(u) - E(p)\,)\,(\, F_T(u) - F_T(p)\,)$$

*with initial condition $F_0 = \mu_0$.*

The Boltzmann equations obtained in the kinetic scaling limit are linear because the Hartree-Fock interactions cancel, due to translation invariance.

*Remarks about the proof.* For the proof, we use the nonlinear evolution

$$(6.33) \qquad\qquad\qquad U_{s,t}(p) := e^{i \int_s^t ds'\,(\, E(p) - \lambda \widehat{v} * \mu_{s'}\,)}$$

as the reference dynamics, instead of free evolution $e^{i(t-s)E(p)}$ as in previous sections, and we invoke the Feynman graph expansion in powers of $\eta$.

Since the free evolution operator depends on the unknown $\mu_t(p)$, and satisfies a *nonlinear* evolution equation, the resolvent calculus used for the problems discussed previously is *unavailable*! Accordingly, the entire analysis in [17] is based on *stationary phase estimates*.

The recombination of contributions associated to decorated ladders is much more difficult for the problem at hand than for the linear problems discussed previously. Our approach involves a very careful analysis of phase cancellations and stationary phase effects.                                                                              □

### 6.1.2.   The regime $\eta = o(\sqrt{\lambda})$

In this regime, the limiting distribution is *stationary*.

**Theorem 6.2.**      *(T. C., I. Rodnianski, [17]) Assume that $\eta^2 = O(\lambda^{1+\delta})$ for $\delta > 0$ arbitrary. Then, for any fixed, finite $T > 0$,*

$$(6.34) \qquad\qquad \lim_{\lambda \to 0} \lim_{L \to \infty} \mathbb{E}[\, \rho_{T/\lambda}(\, a^+(f)\, a(g)\,)\,] = \int dp\, \overline{f(p)}\, g(p)\, F_T(p)\,,$$

*for arbitrary test functions f, g,, and*

$$(6.35) \qquad \partial_T F_T(p) = 0,$$

*for $F_0 = \mu_0$. Accordingly, $F_T = F_0$ is stationary.*

### 6.1.3. The regime $t = T/\eta^2$ and $\lambda = O_\eta(1)$

This regime is very difficult to control, and in [17], we prove a partial result that highlights some interesting aspects about the problem of determining the kinetic scaling limit determined by $T = \eta^2 t$ and $\eta \to 0$, with $\lambda$ small but independent of $\eta$. We are considering, for $\lambda = O(1)$, the rescaled, formal fixed point equation

$$\int dp \, \overline{f(p)} \, g(p) \, \mu_{T/\eta^2}(p) = \mathcal{G}^{(L)}[\, \mu_\bullet(\bullet); \eta; \lambda; T; f, g \,]$$

$$(6.36) \qquad := \mathbb{E}[\, \rho_{T/\eta^2}(\, a^+(f)a(g)\,)\,]$$

for $\mu_\bullet(\bullet)$. The existence and uniqueness of solutions for this fixed point equation is currently an open problem. Below, we will make the assumption that there exist limiting stationary solutions, and determine their form under this hypothesis.

We base our discussion on the following hypotheses for the case $\lambda = O(1)$:

*(H1)* There exist solutions $F^{(\eta)}(T) := \lim_{L\to\infty} \mu_{T/\eta^2}$ of (6.36), such that the limit $w - \lim_{\eta\to 0} F^{(\eta)}(T) =: F(T) = F(0)$ exists and is stationary.

*(H2)* The stationary fixed point solution in (H1) satisfies

$$F(T) = \lim_{\eta\to 0} \lim_{L\to\infty} \mathcal{G}^{(L)}[\, F^{(\eta)}; \eta; \lambda; T; f, g \,]$$

$$(6.37) \qquad = \lim_{\eta\to 0} \lim_{L\to\infty} \mathcal{G}^{(L)}[\, F; \eta; \lambda; T; f, g \,].$$

The first equality sign here is equivalent to *(H1)*, while the second equality sign accounts for the assumption that $F^{(\eta)}$ can be replaced by the limiting fixed point $F$ before letting $\eta \to 0$, to produce the same result.

We note that based on the analysis given in [17], we are able to prove hypothesis *(H2)* if $F^{(\eta)} = F + O(\eta^2)$. Error bounds of order $O(\eta^2)$ require more precise estimates of "crossing" and "nesting" terms in the Feynman graph expansion than considered in this paper, but are available from [27, 28, 29, 30].

**Proposition 6.3.** *Let $\lambda$ be small but independent of $\eta$, and assume that $F \in L^\infty(\mathbb{T}^3)$ independent of t. Then, the thermodynamic limit*

$$(6.38) \qquad \mathcal{G}[\, F; \eta; \lambda; T; f, g \,] := \lim_{L\to\infty} \mathcal{G}^{(L)}[\, F; \eta; \lambda; T; f, g \,]$$

*exists.*

The proof of this proposition follows straightforwardly from results established in [13, 14, 16, 34].

**Theorem 6.4.**    *(T. C., I. Rodnianski, [17]) Assume that $\lambda \leq O_\eta(1)$, and let*

$$(6.39) \qquad\qquad \widetilde{E}_\lambda(u) \ := \ E(u) \ + \ \lambda(\widehat{v} * F)(u) \,.$$

*We assume that $F \in L^\infty(\mathbb{T}^3)$ admits the bounds*

$$(6.40) \qquad \sup_\alpha \int dp \, \frac{1}{|\widetilde{E}_\lambda(q) - \alpha - i\epsilon|} \, , \, \sup_q \int d\alpha \, \frac{1}{|\widetilde{E}_\lambda(q) - \alpha - i\epsilon|} \ \leq \ C \log \frac{1}{\epsilon} \,,$$

*and*

$$\sup_{\alpha_i} \sup_{u \in \mathbb{T}^3} \int dq \, dp \, \frac{1}{|\widetilde{E}_\lambda(q) - \alpha_1 - i\epsilon|} \, \frac{1}{|\widetilde{E}_\lambda(p) - \alpha_2 - i\epsilon|} \, \frac{1}{|\widetilde{E}_\lambda(p \pm q + u) - \alpha_3 - i\epsilon|}$$

$$(6.41) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \ \epsilon^{-b}$$

*for some $0 < b < 1$.*

*Then, $F$ satisfies*

$$(6.42) \qquad\qquad \int dp \, \overline{f(p)} \, g(p) \, F(p) \ = \ \lim_{\eta \to 0} \mathcal{G}[\,F; \eta; \lambda; T; f, g\,] \,,$$

*independent of $T$, if and only if it satisfies*

$$(6.43) \qquad\qquad F(p) \ = \ \mu_0(p) \ = \ \frac{1}{\widetilde{m}_\lambda(p)} \int du \, \delta(\,\widetilde{E}_\lambda(u) - \widetilde{E}_\lambda(p)\,) \, F(u) \,,$$

*where*

$$(6.44) \qquad\qquad \widetilde{m}_\lambda(p) \ := \ 2\pi \int du \, \delta(\,\widetilde{E}_\lambda(u) - \widetilde{E}_\lambda(p)\,)$$

*is the (normalized) measure of the level surface of $\widetilde{E}_\lambda$ for the value $\widetilde{E}_\lambda(p)$.*

We point out the following comments related to Theorem 6.4.

1.  The solution of (6.39) corresponds to a renormalized kinetic energy which is shifted by the average interaction energy for fermion pairs.

2.  The fixed point equation (6.43) for $F$ shows that the stationary kinetic limits of $\mu_t$ are concentrated and equidistributed on level surfaces of the renormalized kinetic energy function $\widetilde{E}_\lambda(\,\cdot\,)$.

3. The bounds (6.40) and (6.41) correspond to the "crossing estimates" in [13, 34, 27, 39]. They ensure sufficient non-degeneracy of the renormalized energy level surfaces so that the Feynman graph expansions introduced below are convergent. However, they do not seem sufficient to prove hypothesis *(H2)* under the assumption that *(H1)* holds.

4. We note that if $\lambda \leq o_\eta(1)$, the stationary solutions found in Theorem 6.4 reduce to those of the linear Boltzmann equation derived in Theorem 6.1.

## References

[1] R. Adami, G. Golse, A. Teta, *Rigorous derivation of the cubic NLS in dimension one*, J. Stat. Phys. **127**, no. 6, 1194–1220 (2007).

[2] M. Aizenman, S. Molchanov, *Localization at large disorder and at extreme energies: an elementary derivation*, Commun. Math. Phys. **157**, 245–278 (1993).

[3] M. Aizenman, R. Sims, S. Warzel, *Absolutely continuous spectra of quantum tree graphs with weak disorder*, Comm. Math. Phys. **264**, no. 2, 371–389 (2006).

[4] M. Aizenman, S. Warzel, *Localization bounds for multiparticle systems*, Comm. Math. Phys. **290**, no. 3, 903–934 (2009).

[5] I. Anapolitanos, I.M. Sigal, *The Hartree-von Neumann limit of many body dynamics*, Preprint http://arxiv.org/abs/0904.4514.

[6] W. Aschbacher, V. Jaksic, Y. Pautrat, C.-A. Pillet, *Transport properties of quasi-free fermions*, J. Math. Phys. **48**, no. 3 (2007)

[7] V. Bach, E.H. Lieb, J.P. Solovej, *Generalized Hartree-Fock theory and the Hubbard model*, J. Stat. Phys. **76** (1-2), 3–89 (1994).

[8] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, *Some considerations on the derivation of the nonlinear quantum Boltzmann equation*, J. Stat. Phys. **116**, no. 1-4, 381–410 (2004).

[9] J. Bellissard, *Random matrix theory and the Anderson model*, J. Statist. Phys. **116**, no. 1-4, 739–754 (2004).

[10] J. Bourgain, *On random Schrödinger operators on* $\mathbb{Z}^2$, Discrete Contin. Dyn. Syst. **8**, no. 1, 1-15, (2002).

[11] J. Bourgain, *Random lattice Schrödinger operators with decaying potential: Some higher dimensional phenomena*, Springer LNM, Vol 1807 (2003), 70-98.

[12] R. Carmona, J. Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser Boston (1990).

[13] T. Chen, *Localization lengths and Boltzmann limit for the Anderson model at small disorders in dimension 3*, J. Stat. Phys., **120** (1-2), 279-337 (2005).

[14] T. Chen, *Convergence in higher mean of a random Schrödinger to a linear Boltzmann evolution*, Comm. Math. Phys., **267**, 355-392 (2006).

[15] T. Chen, *Localization lengths for Schrödinger operators on* $Z^2$ *with decaying random potentials*, Int. Math. Res. Notices, **2005:54**, 3341-3373, 2005.

[16] T. Chen, I. Sasaki, *Boltzmann limit and quasifreeness for a homogenous Fermi gas in a weakly disordered random medium*, J. Stat. Phys., **132** (2), 329-353, 2008.

[17] T. Chen, I. Rodnianski, *Boltzmann limit for a homogenous Fermi gas with Hartree-Fock interactions in a random medium*, http://arxiv.org/abs/0910.2786

[18] V. Chulaevsky, Y. Suhov, *Eigenfunctions in a two-particle Anderson tight binding model*, Comm. Math. Phys. **289**, no. 2, 701–723 (2009).

[19] V. Chulaevsky, Y. Suhov, *Multi-particle Anderson localisation: induction on the number of particles*, Math. Phys. Anal. Geom. **12**, no. 2, 117–139 (2009).

[20] Denissov, S. *Absolutely continuous spectrum of multidimensional Schrödinger operator*, Int. Math. Res. Notices 74, 3963–3982 (2004).

[21] J.-M. Combes, P. Hislop, *Localization for some continuous, random Hamiltonians in d-dimensions*, J. Funct. Anal., **124**, 149-180 (1994).

[22] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger operators*, Springer Verlag (1987).

[23] W. De Roeck, J. Fröhlich, *Diffusion of a massive quantum particle coupled to a quasi-free thermal medium in dimension $d \geq 4$*, http://arxiv.org/abs/0906.5178

[24] S. Denisov, *Absolutely continuous spectrum of multidimensional Schrodinger operator*, Int. Math. Res. Notices **74**, 3963–3982 (2004).

[25] M. Disertori, H. Pinson, T. Spencer, *Density of states for random band matrices*, Comm. Math. Phys. **232**, no. 1, 83 - 124 (2002).

[26] L. Erdös, *Linear Boltzmann equation as the scaling limit of the Schrödinger evolution coupled to a phonon bath*, J. Stat. Phys. **107** (5), 1043-1127 (2002).

[27] L. Erdös, M. Salmhofer, *Decay of the Fourier transform of surfaces with vanishing curvature*, Math. Z. **257** (2), 261–294 (2007).

[28] L. Erdös, M. Salmhofer, H.-T. Yau, *Quantum diffusion for the Anderson model in the scaling limit*, Ann. Henri Poincaré, **8** (4), 621–685 (2007).

[29] L. Erdös, M. Salmhofer, H.-T. Yau, *Quantum diffusion of the random Schrödinger evolution in the scaling limit. II. The recollision diagrams.* Comm. Math. Phys. **271** (1), 1–53 (2007).

[30] L. Erdös, M. Salmhofer, H.-T. Yau, *On the quantum Boltzmann equation*, J. Stat. Phys., **116** (114), 367–380 (2004).

[31] J. Feldman, H. Knörrer, E. Trubowitz, *A Two Dimensional Fermi Liquid, Part 1: Overview*, Comm. Math. Phys, **247**, 1-47 (2004).

[32] A. Figotin, F. Germinet, A. Klein, P. Müller, *Persistence of Anderson localization in Schrödinger operators with decaying random potentials.* Ark. Mat. **45** (1), 15–30 (2007).

[33] J. Fröhlich, T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys. **88**, 151 - 184 (1983).

[34] L. Erdös, H.-T. Yau, *Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation*, Comm. Pure Appl. Math. **53** (6), 667–735 (2000).

[35] N.T. Ho and L.J. Landau, *Fermi gas on a lattice in the van Hove limit*, J. Stat. Phys., **87**, 821–845 (1997).

[36] N.M. Hugenholtz, *Derivation of the Boltzmann equation for a Fermi gas*, J. Stat. Phys., **32**, 231–254 (1983).

[37] A. Klein, *Extended states in the Anderson model on the Bethe lattice*, Adv. Math. **133**, no. 1, 163–184 (1998).

[38] P. Kuchment, *Quantum graphs: an introduction and a brief survey*, Analysis on graphs and its applications, 291–312, Proc. Sympos. Pure Math., **77**, Amer. Math. Soc., Providence, RI, 2008.

[39] J. Lukkarinen, *Asymptotics of resolvent integrals: The suppression of crossings for analytic*

*lattice dispersion relations*, J. Math. Pures Appl. (9) 87, no. 2, 193–225 (2007).

[40] J. Lukkarinen, H. Spohn, *Kinetic Limit for Wave Propagation in a Random Medium*, Arch. Ration. Mech. Anal. **183**, no. 1, 93–162 (2007).

[41] J. Lukkarinen, H. Spohn, *Not to normal order—notes on the kinetic limit for weakly interacting quantum fluids*, J. Stat. Phys. **134**, no. 5-6, 1133–1172 (2009).

[42] J. Lukkarinen, H. Spohn, *Weakly nonlinear Schrödinger equation with random initial data*, http://arxiv.org/abs/0901.3283

[43] J. Magnen, G. Poirot, V. Rivasseau, *Renormalization group methods and applications: First results for the weakly coupled Anderson model*, Phys. A **263**, no. 1-4, 131-140 (1999).

[44] J. Magnen, G. Poirot, V. Rivasseau, *Ward type identities for the 2d Anderson model at weak disorder*, J. Stat. Phys., **93**, 1/2, 331-358 (1998).

[45] J. Magnen, G. Poirot, V. Rivasseau, *The Anderson model as a Matrix model*, Proc. Adv. QFT Conf., 1996.

[46] Poirot, G., *Mean Green's function of the Anderson model at weak disorder with an infra-red cut-off*, Ann. Inst. H. Poincaré Phys. Théor. 70, no. 1, 101-146 (1999).

[47] I. Rodnianski, W. Schlag, *Classical and quantum scattering for a class of long range random potentials*, Int. Math. Res. Notices, **2003:5**, 243-300 (2003).

[48] J. Schenker, *Hölder equicontinuity of the integrated density of states at weak disorder*, Lett. Math. Phys., **70**, no. 3, 195 - 209 (2004).

[49] W. Schlag, C. Shubin, T. Wolff, *Frequency concentration and localization lengths for the Anderson model at small disorders*, J. Anal. Math., **88** (2002).

[50] T. Spencer, M.R. Zirnbauer, *Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions*, Comm. Math. Phys. **252**, no. 1-3, 167–187 (2004).

[51] H. Spohn, *Derivation of the transport equation for electrons moving through random impurities*, J. Statist. Phys., 17, no. 6, 385-412 (1977).

[52] H. Spohn, *The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics*, J. Stat. Phys. **124**, no. 2-4, 1041–1104 (2006).

[53] Stein, E. *Harmonic Analysis*, Princeton University Press (1993).