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Boson Gas Mean Field Model Trapped by Weak Harmonic Potentials in Mesoscopic Scaling

By

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Abstract

A model of a mean-field interacting boson gas trapped by a weak harmonic potential is considered by means of the theory of Random Point Field [RPF]. In the previous work, the weak potential limit of the RPF which describes the position distribution of the constituent particles of the gas has been obtained. The limiting RPF concerns the distribution of the gas near the bottom of the harmonic potentials.

In this note, we deal with the same model in the different scale to consider the large scale distribution of the system.

§ 1. Introduction and Main Results

We consider the quantum statistical mechanical models of boson gas in $\mathbb{R}^d$ ($d > 2$) equipped with a κ-parameterized family of one-particle Hamiltonians of harmonic oscillators:

$$h_\kappa = \frac{1}{2}\sum_{j=1}^{d}\Big(-\frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{\kappa^2} - \frac{1}{\kappa}\Big), \tag{1.1}$$

which are self-adjoint operators in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}^d)$.

In this model, the limit $\kappa \to \infty$ (weak harmonic limit [WHL]) can be considered as a kind of "thermodynamic limit". In the previous paper [15], the position distribution of the constituent bosons of the model and its WHL is studied by means of random point

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fields [RPF]. For the RPF and its applications to the statistical mechanical models, see e.g., [8, 9, 4, 5, 7, 11, 12, 13, 14, 3]. As anticipating from the superficial similarity between the usual TDL for free boson gas and WHL of the present model, the position distributions of the both cases converge to the same RPF weakly for low density gases. However, for high density (i.e., Bose-Einstein condensation [BEC] phase), they behave completely different way. In fact, the distribution converges weakly in the usual TDL, while it does not in WHL. The difference may also be understood heuristically in terms of the difference between the ground state wave functions of the both models. However, only the distribution around the bottom of the parabolic potential are concerned there since the behavior of WHL was studied by means of the weak limit. For the general aspects on models of Boson gases and BEC, see e.g. [2, 18, 10, 6] and the references cited therein.

The purpose of the paper is to study the WHL of the model in the different scaling in oder to clarify the global feature of the distribution of the constituent bosons. The heuristic physical picture of this scaling is as follows: In the previous paper, the harmonic potential is supposed to be so weak that the gas trapped in a large region even in the macroscopic sense. That is, the gas distributes in the volume much greater than the laboratory scale. In the present case, the potential is supposed to be weak but that the boson gas is trapped in a region of the laboratory scale. That is to say, the gas contained in a container of laboratory size. In other words, suppose that the gas in the container consists of N bosons. Then we consider the limit in which $N \to \infty$ and the mass of the individual bosons tends to 0 proportional to $1/N$ so that the total mass of the system remains unchanged.

Substantial part of the argument for the present subject are common with those in the previous paper [15]. Hence we refer those parts with [15], in the way that Lemma I.3.2 for Lemma 3.2 in [15] and (I.2.3) for the (in)equality (2.3) in [15] and so on. Moreover, we organize the presentation of this note in a completely parallel way to the previous paper. For each proposition or lemma, we give the same number with the corresponding proposition or lemma which plays a similar role in [15]. Especially for those which are almost the same to the corresponding ones, their proofs are omitted or replaced by brief notes which indicate the different points.

We consider the mean-field interacting bosons trapped in the harmonic potential (1.1). Its grand-canonical partition function is given by

$$\Xi_{\lambda,\kappa}(\beta,\mu) = \sum_{n=0}^{\infty} e^{\beta(\mu n - \lambda n^2/2\kappa^d)} \operatorname{Tr}_{\mathfrak{H}^n_{sym}}[\otimes^n G_\kappa(\beta)]. \tag{1.2}$$

Here, $\mathfrak{H}^n_{sym} = \otimes^n_{sym} L^2(\mathbb{R}^d)$ is the n-fold symmetric Hilbert space tensor product of $\mathfrak{H} = L^2(\mathbb{R}^d)$ and $G_\kappa(\beta) = e^{-\beta h_\kappa}$ the one particle Gibbs semigroup. The zeroth term in (1.2) equals to 1 by definition. We deal with the case of $d > 2, \beta > 0, \lambda > 0$ and arbitrary $\mu \in \mathbb{R}$. Hereafter, we suppress the symbol λ from the left-hand side of (1.2).

The spectrum of the operator (1.1) is discrete and has the form:

$$\text{Spec}\, h_\kappa = \{\, \epsilon_\kappa(s) := |s|_1/\kappa \,|\, s = (s_1, \cdots, s_d) \in \mathbb{Z}_+^d \,\} \tag{1.3}$$

where $|s|_1 := \sum_{j=1}^d |s_j|$ and $\mathbb{Z}_+$ is the set of all non-negative integers. The ground state is denoted by

$$\Omega_\kappa(x) = \frac{1}{(\pi\kappa)^{d/4}} e^{-x^2/2\kappa} \tag{1.4}$$

in this paper, where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $x^2 := \sum_{j=1}^d x_j^2$.

Integral kernel of $G_\kappa(\beta) = e^{-\beta h_\kappa}$ has the explicit form (the Mehler's formula):

$$G_\kappa(\beta; x, y) = \frac{\exp\{-(2\kappa)^{-1}(|x|^2 + |y|^2)\tanh(\beta/2\kappa) - |x-y|^2/(2\kappa \sinh(\beta/\kappa))\}}{\{\pi\kappa(1 - e^{-2\beta/\kappa})\}^{d/2}}. \tag{1.5}$$

The operator $G_\kappa(\beta)$ belongs to the trace-class $\mathfrak{C}_1(\mathfrak{H})$, with the trace-norm equals to $\text{Tr}\, G_\kappa(\beta) = 1/(1 - e^{-\beta/\kappa})^d = O(\kappa^d)$ for large κ. (For the trace-class and other related topics, see e.g., [17].) The largest eigenvalue of $G_\kappa(\beta)$ coincides with the operator norm $\|G_\kappa(\beta)\| = 1$. We write all the eigenvalues of operator $G_\kappa(\beta)$ in decreasing order:

$$g_0^{(\kappa)} = 1 > g_1^{(\kappa)} = e^{-\beta/\kappa} \geqslant g_2^{(\kappa)} \geqslant \cdots .$$

We use the RPFs to describe the position distribution of the constituent particles of the system, see [11, 12, 13, 14, 15]. Here, RPFs are probability measures on $Q(\mathbb{R}^d)$, the space of all non-negative integer valued Radon measures on $\mathbb{R}^d$. The RPF $\nu_{\kappa,\beta,\mu}$ for the present model is characterized by the following generating functional:

$$\int_{Q(\mathbb{R}^d)} e^{-\langle f, \xi\rangle}\, \nu_{\kappa,\beta,\mu}(d\xi) \tag{1.6}$$

$$= \frac{1}{\Xi_\kappa(\beta,\mu)} \sum_{n=0}^\infty e^{\beta\mu n - \beta\lambda n^2/2\kappa^d} \text{Tr}\,_{\mathfrak{H}_{sym}^n}[\otimes^n (G_\kappa(\beta) e^{-f})] \,,$$

where $f \in C_0(\mathbb{R}^d)$, $f \geqslant 0$. The measure $\nu_{\kappa,\beta,\mu}$ is a finite RPF whose *Janossy measure* can be given explicitly, see Remark 2.1. In the RPF formalism, the positions of the constituent bosons are expressed by a point measure

$$\xi = \sum_{j=1}^N \delta_{x_j},$$

if there are N bosons in the system and they locate at $\{\, x_1, x_2, \cdots, x_n \,\} \in \mathbb{R}^d$. We discussed the distribution of ξ directly in [15]. In the present paper, we consider the distribution of the scaled object

$$\xi_\kappa = \frac{1}{\kappa^d}\sum_{j=1}^N \delta_{x_j/\kappa},$$

i.e.,

$$\langle f, \xi_\kappa\rangle = \kappa^{-d}\langle f(\,\cdot\,/\kappa), \xi\rangle.$$

Note that ξ_κ is an element of $\mathcal{M}(\mathbb{R}^d)$, the space of all non-negative Radon measures on $\mathbb{R}^d$, rather than $Q(\mathbb{R}^d)$. Let T_κ be this transformation, i.e.,

$$T_\kappa : \mathcal{Q}(\mathbb{R}^d) \ni \xi \mapsto \xi_\kappa \in \mathcal{M}(\mathbb{R}^d).$$

Since $\mathcal{M}(\mathbb{R}^d) \supset Q(\mathbb{R}^d)$, we may regard T_κ the transformation in $\mathcal{M}(\mathbb{R}^d)$.

Let $\mathcal{N}_{\kappa,\beta,\mu}$ be the probability measure on $\mathcal{M}(\mathbb{R}^d)$ induced by the map T_κ from $\nu_{\kappa,\beta,\mu}$, i.e., $\mathcal{N}_{\kappa,\beta,\mu} = \nu_{\kappa,\beta,\mu}T_\kappa^{-1}$. Here, $\nu_{\kappa,\beta,\mu}$ may be regarded as a probability measure on $\mathcal{M}(\mathbb{R}^d)$ concentrated on $Q(\mathbb{R}^d)$.

We introduce two more symbols to indicate certain elements in $\mathcal{M}(\mathbb{R}^d)$. Let η_r [$\eta_{0,\rho}$, respectively] $\in \mathcal{M}(\mathbb{R}^d)$ be defined by

$$\langle \eta_r, f\rangle = \int_{(\mathbb{R}^d)^2} \frac{rf(x)}{e^{\beta(p^2+x^2)/2} - r}\frac{dp\,dx}{(2\pi)^d}$$

$$\left[\ \langle \eta_{0,\rho}, f\rangle = \rho f(0) + \int_{(\mathbb{R}^d)^2} \frac{f(x)}{e^{\beta(p^2+x^2)/2} - 1}\frac{dp\,dx}{(2\pi)^d}, \qquad \text{respectively}\ \right].$$

Let us recall the critical parameter

$$\mu_c := \lambda\int_{[0,\infty)^d}\frac{dp}{e^{\beta|p|_1}-1} = \frac{\lambda\zeta(d)}{\beta^d} = \lambda\int_{(\mathbb{R}^d)^2}\frac{1}{e^{\beta(p^2+x^2)/2}-1}\frac{dp\,dx}{(2\pi)^d}, \tag{1.7}$$

which divides the low density (normal phase) and the high density (BEC phase) regions of the system. Let r_* be the unique solution of

$$\frac{\mu}{\lambda} = \frac{\log r_*}{\beta\lambda} + \frac{1}{\beta^d}\int_{[0,\infty)^d}\frac{r_* dp}{e^{|p|_1} - r_*} = \frac{\log r_*}{\beta\lambda} + \int_{(\mathbb{R}^d)^2}\frac{r_*}{e^{\beta(p^2+x^2)/2} - r_*}\frac{dp\,dx}{(2\pi)^d} \tag{1.8}$$

for $\mu < \mu_c$.

Theorem 1.1. *(i) For $\mu < \mu_c$ (normal phase), $\mathcal{N}_{\kappa,\beta,\mu}$ converges weakly to $\delta_{\eta_{r_*}}$, the probability measure concentrated on the singleton $\eta_{r_*} \in \mathcal{M}(\mathbb{R}^d)$, in the limit $\kappa \to \infty$.*

(ii) For $\mu > \mu_c$ (BEC phase), $\mathcal{N}_{\kappa,\beta,\mu}$ converges weakly to $\delta_{\eta_{0,(\mu-\mu_c)/\lambda}}$, in the limit $\kappa \to \infty$.

Remark 1.2. *To prove the theorem, it is enough to show the following limit of generating functionals:*

$$\lim_{\kappa\to\infty}\int_{\mathcal{M}(\mathbb{R}^d)} e^{-\langle f,\xi_\kappa\rangle}\mathcal{N}_{\kappa,\beta,\mu}(d\xi_\kappa) = \lim_{\kappa\to\infty}\int_{Q(\mathbb{R}^d)} e^{-\langle f(\,\cdot/\kappa),\xi\rangle/\kappa^d}\nu_{\kappa,\beta,\mu}(d\xi) \tag{1.9}$$

$$= \begin{cases} \exp\Big[-\displaystyle\int_{(\mathbb{R}^d)^2}\frac{r_* f(x)}{e^{\beta(p^2+x^2)/2}-r_*}\frac{dp\,dx}{(2\pi)^d}\Big] & \text{for } \mu<\mu_c \\ \exp\Big[-\dfrac{\mu-\mu_c}{\lambda}f(0)-\displaystyle\int_{(\mathbb{R}^d)^2}\frac{f(x)}{e^{\beta(p^2+x^2)/2}-1}\frac{dp\,dx}{(2\pi)^d}\Big] & \text{for } \mu>\mu_c. \end{cases}$$

§ 2. Preliminary Arguments and Estimates

We will prove Theorem 1.1 in the direction indicated in Remark 1.2. First, let us write the left-hand side (1.9) as the ratio $\tilde{\Xi}_\kappa(\beta,\mu)/\Xi_\kappa(\beta,\mu)$. The representations of $\tilde{\Xi}_\kappa(\beta,\mu)$ and $\Xi_\kappa(\beta,\mu)$ are given in the form of integration of Fredholm determinants. We also give the miscellaneous estimates needed for the evaluation of these integrals.

§ 2.1. $\Xi_\kappa(\beta,\mu)$ and $\tilde{\Xi}_\kappa(\beta,\mu)$

The expression of $\Xi_\kappa(\beta,\mu)$ has been obtained in [15] as follows. The operator

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}U(\sigma)$$

is the projection operator on $\mathfrak{H}^n=\otimes^n L^2(\mathbb{R}^d)$ onto its subspace $\mathfrak{H}^n_{sym}$, where $\mathfrak{S}_n$ is the n-th symmetric group and

$$U(\sigma)\varphi_1\otimes\cdots\otimes\varphi_n=\varphi_{\sigma^{-1}(1)}\otimes\cdots\otimes\varphi_{\sigma^{-1}(n)} \qquad \text{for } \sigma\in\mathfrak{S}_n,\ \varphi_1,\cdots,\varphi_n\in L^2(\mathbb{R}^d).$$

The grand-canonical partition function can be written as

$$\begin{aligned}\Xi_\kappa(\beta,\mu) &= \sum_{n=0}^\infty e^{\beta\mu n-\beta\lambda n^2/2\kappa^d}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{Tr}_{\otimes^n L^2(\mathbb{R}^d)}\big[\big(\otimes^n G_\kappa(\beta)\big)U(\sigma)\big] \\ &= \sum_{n=0}^\infty\frac{e^{\beta\mu n-\beta\lambda n^2/2\kappa^d}}{n!}\sum_{\sigma\in\mathfrak{S}_n}\int_{(\mathbb{R}^d)^n}\Big(\prod_{j=1}^n G_\kappa(\beta)(x_j,x_{\sigma^{-1}(j)})\Big)dx_1\cdots dx_n \\ &= \sum_{n=0}^\infty\frac{e^{\beta\mu n-\beta\lambda n^2/2\kappa^d}}{n!}\int_{(\mathbb{R}^d)^n}\mathrm{per}\,\big\{G_\kappa(\beta)(x_i,x_j)\big\}_{1\leqslant i,j\leqslant n}dx_1\cdots dx_n\ ,\end{aligned}$$

where per stands for the permanent of matrix.

Remark 2.1. *The point field $\nu_{\kappa,\beta,\mu}$ of (1.6) can also be defined in terms of Janossy measures or exclusion probability* [1]. *That is to say, $\nu_{\kappa,\beta,\mu}$ is a finite point field which assigns the probability*

$$\Pr\{dX_n\} := \frac{e^{\beta\mu n-\beta\lambda n^2/2\kappa^d}}{\Xi_\kappa(\beta,\mu)} \operatorname{per}\big\{G_\kappa(\beta)(x_i,x_j)\big\}_{1\leqslant i,j\leqslant n} dx_1\cdots dx_n$$

to the event $\{dX_n\}$: there are exactly n points, one in each infinitesimal region $\prod_{i=1}^d [x_j^{(i)}, x_j^{(i)}+dx_j^{(i)}),\quad \big(x_j=(x_j^{(1)},\cdots,x_j^{(d)}),\ j=1,\cdots,n\big)$.

We use the generalized *Vere-Jones' formula* [11, 16] in the form

$$\frac{1}{n!}\int \operatorname{per}\{G_\kappa(\beta)(x_i,x_j)\}_{1\leqslant i,j\leqslant n}\ dx_1\cdots dx_n = \oint_{S_r(0)} \frac{dz}{2\pi i z^{n+1}\operatorname{Det}[1-zG_\kappa(\beta)]},$$

where $r>0$ satisfies $r=||rG_\kappa(\beta)||<1$. $S_r(\zeta)$ denotes the integration contour defined by the map $\theta\mapsto\zeta+r\exp(i\theta)$, where θ ranges from $-\pi$ to π, $r>0$ and $\zeta\in\mathbb{C}$.

We also use

$$e^{-\beta\lambda n^2/2\kappa^d} = \sqrt{\frac{\beta\lambda}{2\pi\kappa^d}}\int_{\mathbb{R}} dx\ \exp\Big(-\frac{\beta\lambda}{2\kappa^d}((x+is)^2-2in(x+is))\Big). \tag{2.1}$$

If $s>0$ satisfies

$$e^{\beta\mu-\beta\lambda s/\kappa^d} < r, \tag{2.2}$$

we can take the summation over n together with a scaling of x and the complex integration to get

$$\begin{aligned}
\Xi_\kappa(\beta,\mu) &= \sqrt{\frac{\beta\lambda}{2\pi\kappa^d}}e^{\beta\lambda s^2/2\kappa^d}\int_{\mathbb{R}} dx\oint\frac{dz}{2\pi i}\frac{e^{-\beta\lambda(x^2+2isx)/2\kappa^d}}{\big(z-e^{\beta\mu+\beta\lambda(ix-s)/\kappa^d}\big)\operatorname{Det}[1-zG_\kappa(\beta)]}\\
&= \sqrt{\frac{\kappa^d}{2\pi\beta\lambda}}e^{\beta\lambda s^2/2\kappa^d}\int_{\mathbb{R}} dx\oint\frac{dz}{2\pi i}\frac{e^{-\kappa^d x^2/2\beta\lambda-isx}}{\big(z-e^{\beta\mu+ix-s\beta\lambda/\kappa^d}\big)\operatorname{Det}[1-zG_\kappa(\beta)]}
\end{aligned}$$

$$= \sqrt{\frac{\kappa^d}{2\pi\beta\lambda}}e^{\beta\lambda s^2/2\kappa^d}\int_{\mathbb{R}} dx\,\frac{e^{-isx-\kappa^d x^2/2\beta\lambda}}{\operatorname{Det}[1-e^{\beta\mu+ix-\beta\lambda s/\kappa^d}G_\kappa(\beta)]}. \tag{2.3}$$

$$= \sqrt{\frac{\kappa^d}{2\pi\beta\lambda}}\frac{e^{\beta\lambda s_\kappa^2/2\kappa^d}}{\operatorname{Det}[1-r_\kappa G_\kappa(\beta)]}\int_{\mathbb{R}} dx\,\frac{e^{-is_\kappa x-\kappa^d x^2/2\beta\lambda}}{\operatorname{Det}[1-(e^{ix}-1)r_\kappa G_\kappa(\beta)(1-r_\kappa G_\kappa(\beta))^{-1}]}. \tag{2.4}$$

Here the product property of the Fredholm determinant was used in the last equality. We set $(s,r)=(s_\kappa,r_\kappa)$ which is the unique solution of

$$\begin{cases} r=\exp\big(\beta\mu-\beta\lambda s/\kappa^d\big)\\ s=\operatorname{Tr}\,[rG_\kappa(\beta)(1-rG_\kappa(\beta))^{-1}]. \end{cases} \tag{2.5}$$

Note that $r_\kappa < 1$ which enables the choice of $r < 1$ for $s = s_\kappa$ in (2.2).

Let us express the generating functional for $\mathcal{N}_{\kappa,\beta,\mu}$ (1.9) as

$$\int_{\mathcal{M}(\mathbb{R}^d)} e^{-\langle f,\xi_\kappa\rangle}\mathcal{N}_{\kappa,\beta,\mu}(d\xi_\kappa) = \tilde{\Xi}_\kappa(\beta,\mu)/\Xi_\kappa(\beta,\mu),$$

where

$$\tilde{\Xi}_\kappa(\beta,\mu) = \sum_{n=0}^{\infty} e^{\beta\mu n - \beta\lambda n^2/2\kappa^d}\mathrm{Tr}_{\,\mathfrak{H}^n_{sym}}[\otimes^n \tilde{G}_\kappa(\beta)]$$

and

$$\tilde{G}_\kappa(\beta)(f) := G_\kappa(\beta)^{1/2}\exp[-\kappa^{-d} f(\,\cdot\,/\kappa)]G_\kappa(\beta)^{1/2}\ , \tag{2.6}$$

with the function $f \in C_0(\mathbb{R}^d)$, $f \geqslant 0$. We suppress the f-dependence from $\tilde{G}_\kappa(\beta)(f)$ below for simplicity. Then, as for $\tilde{\Xi}_\kappa(\beta,\mu)$, we get

$$\tilde{\Xi}_\kappa(\beta,\mu) = \sqrt{\frac{\kappa^d}{2\pi\beta\lambda}}\frac{e^{\beta\lambda\tilde{s}_\kappa^2/2\kappa^d}}{\mathrm{Det}[1-\tilde{r}_\kappa\tilde{G}_\kappa(\beta)]} \tag{2.7}$$

$$\times\int_{\mathbb{R}} dx\,\frac{e^{-i\tilde{s}_\kappa x-\kappa^d x^2/2\beta\lambda}}{\mathrm{Det}[1-(e^{ix}-1)\tilde{r}_\kappa\tilde{G}_\kappa(\beta)(1-\tilde{r}_\kappa\tilde{G}_\kappa(\beta))^{-1}]},$$

where $(\tilde{s}_\kappa, \tilde{r}_\kappa)$ is the unique solution of

$$\begin{cases} \tilde{r} = \exp\left(\beta\mu - \beta\lambda\tilde{s}/\kappa^d\right) \\ \tilde{s} = \mathrm{Tr}\,[\tilde{r}\tilde{G}_\kappa(\beta)(1-\tilde{r}\tilde{G}_\kappa(\beta))^{-1}]. \end{cases} \tag{2.8}$$

Obviously, $\tilde{r}_\kappa \in (0, ||\tilde{G}_\kappa(\beta)||^{-1})$. Note also that r_κ and $\tilde{r}_\kappa$ satisfy the following conditions respectively:

$$\frac{1}{\kappa^d}\mathrm{Tr}\,[r_\kappa G_\kappa(\beta)(1-r_\kappa G_\kappa(\beta))^{-1}] = \frac{\beta\mu - \log r_\kappa}{\beta\lambda}, \tag{2.9}$$

$$\frac{1}{\kappa^d}\mathrm{Tr}\,[\tilde{r}_\kappa\tilde{G}_\kappa(\beta)(1-\tilde{r}_\kappa\tilde{G}_\kappa(\beta))^{-1}] = \frac{\beta\mu - \log \tilde{r}_\kappa}{\beta\lambda}. \tag{2.10}$$

By (2.6), it follows that $\tilde{G}_\kappa(\beta) \leqslant G_\kappa(\beta)$ and the operator $\tilde{G}_\kappa(\beta)$ also belongs to the trace-class $\mathfrak{C}_1(\mathfrak{H})$. We put the eigenvalues of $\tilde{G}_\kappa(\beta)$ in the decreasing order

$$\tilde{g}_0^{(\kappa)} = ||\tilde{G}_\kappa(\beta)|| \geqslant \tilde{g}_1^{(\kappa)} \geqslant \cdots .$$

Then, we have $g_j^{(\kappa)} \geqslant \tilde{g}_j^{(\kappa)}$ $\quad (j = 0, 1, 2, \cdots)$ by the *min-max principle*.

§ 2.2. Approximations of One-Particle Gibbs Semigroups

In this subsection, we consider the limit of the Fredholm determinants to calculate $\tilde{\Xi}_\kappa(\beta,\mu)/\Xi_\kappa(\beta,\mu)$. Recall that the denominator $\Xi_\kappa(\beta,\mu)$ is the same with that in the previous paper [15]. The numerator $\tilde{\Xi}_\kappa(\beta,\mu)$ is different from that in [15], since the definition of $\tilde{G}_\kappa(\beta)$ has been modified. (2.6) Let P_κ be the orthogonal projection from $\mathfrak{H}$ to its one-dimensional subspace spanned by the vector Ω_κ. We put $Q_\kappa = I - P_\kappa$ and

$$f^{(\kappa)}(x) = 1 - e^{-\kappa^{-d} f(x/\kappa)}, \quad f_\kappa(x) = \kappa^d \Big(1 - e^{-\kappa^{-d} f(x)}\Big).$$

Obviously, $f^{(\kappa)}(\kappa x) = \kappa^{-d} f_\kappa(x)$ and $0 \leqslant f_\kappa(x) \uparrow f(x)$ hold.

Lemma 2.2. *Suppose that the sequence* $\{\hat{r}_\kappa\} \subset (0,1)$ *converges to* $\hat{r} \in (0,1]$.
(i) If $\hat{r} < 1$,

$$\lim_{\kappa\to\infty} \mathrm{Det}[1 + \sqrt{f^{(\kappa)}}\hat{r}_\kappa G_\kappa(\beta)(1 - \hat{r}_\kappa G_\kappa(\beta))^{-1}\sqrt{f^{(\kappa)}}] \tag{2.11}$$

$$= \exp\Big[\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\hat{r} f(x)}{e^{\beta(p^2+x^2)/2} - \hat{r}} \frac{dp\,dx}{(2\pi)^d}\Big].$$

holds.
(ii) If $\hat{r} = 1$,

$$\lim_{\kappa\to\infty} \mathrm{Det}[1 + \sqrt{f^{(\kappa)}}\hat{r}_\kappa Q_\kappa G_\kappa(\beta) Q_\kappa (1 - \hat{r}_\kappa Q_\kappa G_\kappa(\beta) Q_\kappa)^{-1}\sqrt{f^{(\kappa)}}] \tag{2.12}$$

$$= \exp\Big[\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(x)}{e^{\beta(p^2+x^2)/2} - 1} \frac{dp\,dx}{(2\pi)^d}\Big].$$

holds.

Proof : (i) From the Mehler's formula (1.5) and the semi-group property $G_\kappa(\beta)^n = G_\kappa(n\beta)$, it follows that

$$\hat{r}_\kappa^n \mathrm{Tr}\,[G_\kappa(\beta)^n f^{(\kappa)}] = \hat{r}_\kappa^n \int_{\mathbb{R}^d} \frac{e^{-\kappa^{-1}x^2\tanh(\beta n/2\kappa)}}{[\pi\kappa(1 - e^{-2\beta n/\kappa})]^{d/2}} f^{(\kappa)}(x)\,dx$$

$$= \hat{r}_\kappa^n \int_{\mathbb{R}^d} \frac{e^{-\kappa x^2\tanh(\beta n/2\kappa)}}{[\pi\kappa(1 - e^{-2\beta n/\kappa})]^{d/2}} f_\kappa(x)\,dx \tag{2.13}$$

$$\longrightarrow \hat{r}^n \int_{\mathbb{R}^d} \frac{e^{-\beta n x^2/2}}{(2\pi\beta n)^{d/2}} f(x)\,dx \quad \text{as} \quad \kappa\to\infty,$$

where we have changed the integral variable x to κx at the second equality. Thanks to (I. A1), we have

$$\frac{1}{\pi\kappa(1 - e^{-2\beta n/\kappa})} \leqslant c_1$$

for $\kappa \geqslant 1$. Since

$$\hat{r}_0 := \sup_{\kappa\geqslant 1} r_\kappa < 1,$$

we obtain

$$\hat{r}_\kappa^n \mathrm{Tr}\,[G_\kappa^n(\beta) f^{(\kappa)}] \leqslant c_1^{d/2} \hat{r}_0^n \int_{\mathbb{R}^d} f(x)\,dx$$

from the expression in (2.13). Note that the right hand-side of the above inequality is independent of κ and summable with respect to n. Hence by the dominated convergence theorem, we get

$$\begin{aligned} &\lim_{\kappa\to\infty} \mathrm{Tr}\,[\hat{r}_\kappa G_\kappa(\beta)(1-\hat{r}_\kappa G_\kappa(\beta))^{-1} f^{(\kappa)}] \\ &= \lim_{\kappa\to\infty} \sum_{n=1}^\infty \hat{r}_\kappa^n \mathrm{Tr}\,[G_\kappa^n f^{(\kappa)}] = \sum_{n=1}^\infty \hat{r}^n \int_{\mathbb{R}^d} \frac{e^{-n\beta x^2/2}}{(2\pi\beta n)^{d/2}} f(x)\,dx \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\hat{r} e^{-\beta(p^2+x^2)/2} f(x)}{1-\hat{r} e^{-\beta(p^2+x^2)/2}} \frac{dp\,dx}{(2\pi)^d}. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} &\|\sqrt{f^{(\kappa)}}\hat{r}_\kappa G_\kappa(\beta)(1-\hat{r}_\kappa G_\kappa(\beta))^{-1}\sqrt{f^{(\kappa)}}\|_{HS}^2 \\ &\leqslant \mathrm{Tr}\,[\hat{r}_\kappa G_\kappa(\beta)(1-\hat{r}_\kappa G_\kappa(\beta))^{-1} f^{(\kappa)}]\|f^{(\kappa)}\|_\infty \\ &\times \|\hat{r}_\kappa G_\kappa(\beta)(1-\hat{r}_\kappa G_\kappa(\beta))^{-1}\| = O(1)\frac{\|f\|_\infty}{\kappa^d}\frac{\hat{r}_0}{1-\hat{r}_0}, \end{aligned}$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm. Hence by the formula $\mathrm{Det}[1+A] = e^{\mathrm{Tr}\,A}\mathrm{Det}_2[1+A]$ and the continuity of the regularized determinant $\mathrm{Det}_2[1+A]$ with respect to the Hilbert-Schmidt norm $\|A\|_{HS}$ followed by the cyclicity of the Fredholm determinant, we get (2.11).

(ii) For $n \leqslant \kappa/\beta$, we have

$$\frac{1}{\pi\kappa(1-e^{-2\beta n/\kappa})} \leqslant \frac{3}{2\pi\beta n},$$

thanks to (I.A1). Using the elementary bound

$$e^{-\kappa x^2 \tanh(\beta n/2\kappa)} \leqslant e^{-c_2\beta n x^2},$$

we obtain

$$\hat{r}_\kappa^n \mathrm{Tr}\,[G_\kappa^n(\beta) f^{(\kappa)}] \leqslant \int_{\mathbb{R}^d} \frac{3^{d/2} e^{-c_2\beta n x^2}}{(2\pi\beta n)^{d/2}} f(x)\,dx$$

as in (i), where $c_2 = (e-1)/(e+1)$. Note that the right-hand side of the above inequality is independent of κ and summable with respect to n. Hence by the dominated convergence theorem, we get

$$\lim_{\kappa\to\infty} \sum_{n=1}^{\lceil\kappa/\beta\rceil} \hat{r}_\kappa^n \mathrm{Tr}\,[G_\kappa^n(\beta) f^{(\kappa)}] = \sum_{n=1}^\infty \int_{\mathbb{R}^d} \frac{e^{-\beta n x^2/2}}{(2\pi\beta n)^{d/2}} f(x)\,dx$$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{e^{-\beta(p^2+x^2)/2}f(x)}{1-e^{-\beta(p^2+x^2)/2}}\frac{dp\,dx}{(2\pi)^d},$$

as above. Here $\lceil y\rceil$ denotes the largest integer which does not exceed y. We need the formula including $Q_\kappa G_\kappa(\beta)Q_\kappa$ instead of $G_\kappa(\beta)$. The difference behaves like

$$\sum_{n=1}^{\lceil\kappa/\beta\rceil} \hat{r}_\kappa^n(\Omega_\kappa, G_\kappa^n(\beta)f^{(\kappa)}\Omega_\kappa) \leqslant \frac{\kappa}{\beta}\int_{\mathbb{R}^d}\frac{e^{-x^2/\kappa}}{(\pi\kappa)^{d/2}}f^{(\kappa)}(x)\,dx = O(\kappa^{1-d}),$$

as $\kappa\to\infty$. Thus, one gets

$$\lim_{\kappa\to\infty}\sum_{n=1}^{\lceil\kappa/\beta\rceil}\hat{r}_\kappa^n\mathrm{Tr}\,[(Q_\kappa G_\kappa(\beta)Q_\kappa)^n f^{(\kappa)}] = \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{e^{-\beta(p^2+x^2)/2}f(x)}{1-e^{-\beta(p^2+x^2)/2}}\frac{dp\,dx}{(2\pi)^d}.$$

For $n>\kappa/\beta$, we have

$$\begin{aligned}&\hat{r}_\kappa^n\mathrm{Tr}\,[(Q_\kappa G_\kappa(\beta)Q_\kappa)^n f^{(\kappa)}]\\ &= \hat{r}_\kappa^n\int_{\mathbb{R}^d}\Big[\frac{e^{-\kappa^{-1}x^2\tanh(n\beta/2\kappa)}}{[\pi\kappa(1-e^{-2\beta n/\kappa})]^{d/2}} - \frac{e^{-x^2/\kappa}}{(\pi\kappa)^{d/2}}\Big]f^{(\kappa)}(x)\,dx\\ &\leqslant \frac{B'}{\kappa^{d/2}}e^{-n\beta/2\kappa}\int f^{(\kappa)}(x)\,dx,\end{aligned}$$

where we used (I, 2.13). Thus we get

$$\begin{aligned}&\sum_{n=\lceil\kappa/\beta\rceil+1}^{\infty}\hat{r}_\kappa^n\mathrm{Tr}\,[(Q_\kappa G_\kappa(\beta)Q_\kappa)^n f^{(\kappa)}]\\ &\leqslant \frac{B''}{\kappa^{d/2}}\frac{1}{1-e^{-\beta/2\kappa}} = O(\kappa^{1-d/2}),\end{aligned}$$

for large κ. Now, the equality (2.12) follows from a similar argument as in (i). Note that $\|Q_\kappa G_\kappa(\beta)Q_\kappa(1-Q_\kappa G_\kappa(\beta)Q_\kappa)^{-1}\| = g_1^{(\kappa)}/(1-g_1^{(\kappa)}) = O(\kappa)$. □

§ 2.3. Estimates for the Scaled Mean-Field Interaction

In the followings, we use the notation $B_\kappa = \hat{O}(\kappa^\alpha)$, which means that there exist two numbers $c_1\geqslant c_2>0$ such that

$$c_1\kappa^\alpha \geqslant B_\kappa \geqslant c_2\kappa^\alpha.$$

We put $W_\kappa := (G_\kappa(\beta))^{1/2}\sqrt{f^{(\kappa)}}$ and define $D_\kappa := G_\kappa(\beta)-\tilde{G}_\kappa(\beta) = W_\kappa W_\kappa^*$.

Lemma 2.3. *For large $\kappa>0$, the following asymptotics hold:*

$$(\Omega_\kappa, D_\kappa\Omega_\kappa) = ||W_\kappa^*\Omega_\kappa||_2^2 = \frac{f(0)+o(1)}{\kappa^d},$$

$$\|W_\kappa\| \leqslant \frac{\|f\|_\infty^{1/2}}{\kappa^{d/2}}, \quad \|D_\kappa\| \leqslant \frac{\|f\|_\infty}{\kappa^d},$$

$$\lim_{\kappa\to\infty} \operatorname{Tr} D_\kappa = \int_{\mathbb{R}^d} \frac{e^{-\beta x^2/2}}{(2\pi\beta)^{d/2}} f(x)\,dx \tag{2.14}$$

and

$$g_0^{(\kappa)} - \tilde{g}_0^{(\kappa)} = (\Omega_\kappa, D_\kappa\Omega_\kappa)(1 + O(\kappa^{1-d})) = \frac{f(0)+o(1)}{\kappa^d}.$$

Proof: The first equality is a straightforward consequence of (1.4). The estimate for $\|W_\kappa\|$ and $\|D_\kappa\|$ come from the above definitions and $\|G_\kappa(\beta)\| = 1$. The next limit is the $n = 1$ case of (2.13).

Now by the *min-max* principle, for $d > 2$ and κ large enough, we obtain the following estimates from the value $g_1^{(\kappa)} = \exp(-\beta/\kappa)$:

$$\begin{aligned} g_0^{(\kappa)} &= 1 \geqslant \tilde{g}_0^{(\kappa)} \geqslant (\Omega_\kappa, \tilde{G}_\kappa(\beta)\Omega_\kappa) = 1 - (\Omega_\kappa, D_\kappa\Omega_\kappa) \\ &= 1 - \frac{f(0)+o(1)}{\kappa^d} > g_1^{(\kappa)} = e^{-\beta/\kappa} \geqslant \tilde{g}_1^{(\kappa)}. \end{aligned} \tag{2.15}$$

It follows from $0 \leqslant \tilde{G}_\kappa(\beta) \leqslant G_\kappa(\beta)$ that $0 \leqslant Q_\kappa\tilde{G}_\kappa(\beta)Q_\kappa \leqslant Q_\kappa G_\kappa(\beta)Q_\kappa$ and therefore

$$\operatorname{Spec} Q_\kappa\tilde{G}_\kappa(\beta)Q_\kappa \subset [0, e^{-\beta/\kappa}]$$

hold. Then, we get

$$\|(\tilde{g}_0^{(\kappa)} - Q_\kappa\tilde{G}_\kappa(\beta)Q_\kappa)^{-1}\| = O(\kappa)$$

and

$$\|W_\kappa^* Q_\kappa[\tilde{g}_0^{(\kappa)} - Q_\kappa\tilde{G}_\kappa(\beta)Q_\kappa]^{-1}Q_\kappa W_\kappa\| = O(\kappa^{1-d}).$$

As in (I.2.18), we get the equality

$$g_0^{(\kappa)} - \tilde{g}_0^{(\kappa)} = (W_\kappa^*\Omega_\kappa, (1 - W_\kappa^* Q_\kappa[\tilde{g}_0^{(\kappa)} - Q_\kappa\tilde{G}_\kappa(\beta)Q_\kappa]^{-1}Q_\kappa W_\kappa)W_\kappa^*\Omega_\kappa), \tag{2.16}$$

which yields $g_0^{(\kappa)} - \tilde{g}_0^{(\kappa)} = (\Omega_\kappa, D_\kappa\Omega_\kappa)(1 + O(\kappa^{1-d}))$. □

§ 2.4. Evidence of Two Thermodynamic Regimes

Now we return to the conditions (2.9). We recall here the behavior of r_κ. Following proposition is the same with Proposition I.2.4.

Proposition 2.4. (a) $\{r_\kappa\}$ *converges to* $r_* \in (0,1)$ *as* $\kappa \to \infty$, *where* r_* *is the unique solution of*

$$\frac{\mu}{\lambda} = \frac{\log r_*}{\beta\lambda} + \frac{1}{\beta^d}\int_{[0,\infty)^d} \frac{r_* dp}{e^{|p|_1} - r_*} = \frac{\log r_*}{\beta\lambda} + \int_{(\mathbb{R}^d)^2} \frac{r_*}{e^{\beta(p^2+x^2)/2} - r_*}\frac{dp\,dx}{(2\pi)^d}, \tag{2.17}$$

if and only if $\mu < \mu_c$.
(b) $\kappa^d(1-r_\kappa) \longrightarrow \lambda/(\mu-\mu_c)$, *and hence* $\lim_{\kappa\to\infty} r_\kappa = 1$, *if and only if* $\mu > \mu_c$.
(c) $\lim_{\kappa\to\infty} r_\kappa = 1$ *and* $\kappa^d(1-r_\kappa) \longrightarrow +\infty$, *if and only if* $\mu = \mu_c$.

§ 3. Proof of Theorem 1.1

§ 3.1. The case $\mu < \mu_c$ (normal phase).

Since it is enough to prove the convergence of the generating functionals (1.9), we evaluate $\Xi_\kappa(\beta,\mu)$ and $\tilde{\Xi}_\kappa(\beta,\mu)$. For the denominator $\Xi_\kappa(\beta,\mu)$, we refer the result of the preceding paper [15].

Lemma 3.1.

$$\Xi_\kappa(\beta,\mu) = \frac{e^{\kappa^d(\beta\mu-\log r_\kappa)^2/2\beta\lambda}\big(1+O(\kappa^{d-3\alpha})\big)}{\sqrt{1+\beta\lambda\kappa^{-d}\mathrm{Tr}\,[r_\kappa G_\kappa(\beta)(1-r_\kappa G_\kappa(\beta))^{-2}]\mathrm{Det}(1-r_\kappa G_\kappa(\beta))}}$$

The corresponding expression for $\tilde{\Xi}_\kappa(\beta,\mu)$ is the following:

Lemma 3.2.

$$\tilde{\Xi}_\kappa(\beta,\mu) = \frac{e^{\kappa^d(\beta\mu-\log \tilde{r}_\kappa)^2/2\beta\lambda}\big(1+O(\kappa^{d-3\alpha})\big)}{\sqrt{1+\beta\lambda\kappa^{-d}\mathrm{Tr}\,[\tilde{r}_\kappa \tilde{G}_\kappa(\beta)(1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta))^{-2}]}\mathrm{Det}(1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta))}.$$

Recall that $\tilde{G}_\kappa(\beta)$ in this paper and the previous paper [15] are different; (2.6) and (I.2.5). Thereby, we need to prove the lemma. However, it may be proved in the same way as Lemma I.3.2, since we have the following lemma which is similar to Lemma I.3.3.

Lemma 3.3. *For large* κ *one gets:*

(*i*) $\tilde{r}_\kappa \geqslant r_\kappa$,

(*ii*) $\tilde{r}_\kappa - r_\kappa = O(\kappa^{-d})$,

(*iii*) $\mathrm{Tr}\Big[\dfrac{r_\kappa G_\kappa(\beta)}{1-r_\kappa G_\kappa(\beta)}\Big] = \hat{O}(\kappa^d), \qquad \mathrm{Tr}\Big[\dfrac{\tilde{r}_\kappa \tilde{G}_\kappa(\beta)}{1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta)}\Big] = \hat{O}(\kappa^d)$,

(*iv*) $\mathrm{Tr}\Big[\dfrac{r_\kappa G_\kappa(\beta)}{(1-r_\kappa G_\kappa(\beta))^2}\Big] - \mathrm{Tr}\Big[\dfrac{\tilde{r}_\kappa \tilde{G}_\kappa(\beta)}{(1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta))^2}\Big] = O(1)$.

Proof : Except for obvious changes, the proof for Lemma I.3.3 also works here.

Now, let us consider the limit of the ratio $\tilde{\Xi}_\kappa(\beta,\mu)/\Xi_\kappa(\beta,\mu)$ to derive (1.9). From Lemma 3.1 and Lemma 3.2, we have

$$\lim_{\kappa\to\infty}\frac{\tilde{\Xi}_\kappa(\beta,\mu)}{\Xi_\kappa(\beta,\mu)}=\lim_{\kappa\to\infty}\sqrt{\frac{1+\beta\lambda\kappa^{-d}\mathrm{Tr}\left[r_\kappa G_\kappa(\beta)(1-r_\kappa G_\kappa(\beta))^{-2}\right]}{1+\beta\lambda\kappa^{-d}\mathrm{Tr}\left[\tilde{r}_\kappa \tilde{G}_\kappa(\beta)(1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta))^{-2}\right]}}$$

$$\times\frac{\mathrm{Det}[1-\tilde{r}_\kappa G_\kappa(\beta)]}{\mathrm{Det}[1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta)]}\frac{\mathrm{Det}[1-r_\kappa G_\kappa(\beta)]}{\mathrm{Det}[1-\tilde{r}_\kappa G_\kappa(\beta)]}e^{\kappa^d(2\beta\lambda)^{-1}[(\beta\mu-\log\tilde{r}_\kappa)^2-(\beta\mu-\log r_\kappa)^2]}.$$

Lemma 3.3 yields that the first factor is equal to $1+O(\kappa^{-d})$. For the second factor, we get

$$\frac{\mathrm{Det}[1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta)]}{\mathrm{Det}[1-\tilde{r}_\kappa G_\kappa(\beta)]}=\mathrm{Det}[1+\tilde{r}_\kappa(G_\kappa(\beta)-\tilde{G}_\kappa(\beta))(1-\tilde{r}_\kappa G_\kappa(\beta))^{-1}]$$

$$=\mathrm{Det}[1+\sqrt{f^{(\kappa)}}\frac{\tilde{r}_\kappa G_\kappa(\beta)}{1-\tilde{r}_\kappa G_\kappa(\beta)}\sqrt{f^{(\kappa)}}]\to\exp\Big[\int_{(\mathbb{R}^d)^2}\frac{r_* f(x)}{e^{\beta(p^2+x^2)/2}-r_*}\frac{dp\,dx}{(2\pi)^d}\Big]$$

from Proposition 2.4(a) and Lemma 2.2(i). For the third factor, we have

$$\frac{\mathrm{Det}[1-\tilde{r}_\kappa G_\kappa(\beta)]}{\mathrm{Det}[1-r_\kappa G_\kappa(\beta)]}=\mathrm{Det}\Big[1-\frac{\tilde{r}_\kappa-r_\kappa}{r_\kappa}\frac{r_\kappa G_\kappa(\beta)}{1-r_\kappa G_\kappa(\beta)}\Big]$$

$$=\exp\Big(-\frac{\tilde{r}_\kappa-r_\kappa}{r_\kappa}\mathrm{Tr}\Big[\frac{r_\kappa G_\kappa(\beta)}{1-r_\kappa G_\kappa(\beta)}\Big]+o(1)\Big)=\exp\Big(-\frac{\tilde{r}_\kappa-r_\kappa}{r_\kappa}\frac{\kappa^d}{\beta\lambda}(\beta\mu-\log r_\kappa)+o(1)\Big),$$

where we have used Lemma 3.3(ii) and (2.9). Note that

$$\Big\|\frac{r_\kappa G_\kappa(\beta)}{1-r_\kappa G_\kappa(\beta)}\Big\|_{HS}^2\leqslant\Big\|\frac{r_\kappa G_\kappa(\beta)}{1-r_\kappa G_\kappa(\beta)}\Big\|\mathrm{Tr}\Big[\frac{r_\kappa G_\kappa(\beta)}{1-r_\kappa G_\kappa(\beta)}\Big]=O(\kappa^d).$$

It also follows from Lemma 3.3 that the fourth factor is equal to

$$\exp\Big[\frac{\kappa^d}{2\beta\lambda}(2\beta\mu-\log r_\kappa\tilde{r}_\kappa)\log\Big(1+\frac{r_\kappa-\tilde{r}_\kappa}{\tilde{r}_\kappa}\Big)\Big]=\exp\Big[-\frac{\tilde{r}_\kappa-r_\kappa}{r_\kappa}\frac{\kappa^d}{\beta\lambda}(\beta\mu-\log r_\kappa)+O(\kappa^{-d})\Big].$$

Because of Remark 1.2, Theorem 1.1(i) follows. □

§ 3.2. The case $\mu>\mu_c(\beta)$ (condensed phase).

For

$$p_j^{(\kappa)}:=\frac{r_\kappa g_j^{(\kappa)}}{1-r_\kappa g_j^{(\kappa)}}\quad(j=0,1,\cdots),$$

we have gotten

$$s_\kappa=\sum_{j=0}^\infty p_j^{(\kappa)}=\hat{O}(\kappa^d),\quad 1+p_0^{(\kappa)}=\frac{1}{1-r_\kappa}=\frac{\kappa^d(\beta^d\mu-\zeta(d)\lambda)}{\beta^d\lambda}(1+o(1))=\hat{O}(\kappa^d),$$

$$p_1^{(\kappa)} = \hat{O}(\kappa), \quad \sum_{j=1}^{\infty} p_j^{(\kappa)\,2} = O(\kappa^d). \tag{3.1}$$

in the previous paper [15]. ¿From these facts, the following lemma has been proven there.

Lemma 3.4. *For large κ, the asymptotics*

$$\Xi_\kappa(\beta,\mu) = \sqrt{\frac{2\pi\beta\lambda}{e^2\kappa^d}} \frac{\beta^{d-1} e^{\kappa^d(\beta\mu - \log r_\kappa)^2/2\beta\lambda}}{(\beta^d\mu - \zeta(d)\lambda)\mathrm{Det}(1 - r_\kappa G_\kappa(\beta))}(1 + o(1)) \tag{3.2}$$

holds.

For $\tilde{p}_j^{(\kappa)} := \tilde{r}_\kappa \tilde{g}_j^{(\kappa)}/(1 - \tilde{r}_\kappa \tilde{g}_j^{(\kappa)})$ $(j = 0, 1, \cdots)$, we get similar estimates on $\tilde{p}_j$, which will be used to obtain the asymptotics for $\tilde{\Xi}_\kappa(\beta,\mu)$.

Lemma 3.5.

$$1 - \tilde{r}_\kappa \tilde{g}_0^{(\kappa)} = \frac{\beta^d\lambda(1 + o(1))}{\kappa^d(\beta^d\mu - \zeta(d)\lambda)}, \quad |1 - \tilde{r}_\kappa| = O(\kappa^{-d}),$$

$$\tilde{p}_0^{(\kappa)} = \hat{O}(\kappa^d), \quad \tilde{p}_1^{(\kappa)} = \hat{O}(\kappa), \quad \sum_{j=1}^{\infty} \tilde{p}_j^{(\kappa)} = \hat{O}(\kappa^d), \quad \sum_{j=1}^{\infty} \tilde{p}_j^{(\kappa)\,2} = O(\kappa^d).$$

Proof : Proposition 2.4(b) , Lemma 3.3(i), Lemma 2.3 and $\tilde{r}_\kappa < g_0^{(\kappa)-1}$ (see just below (2.8)) yield

$$1 - \hat{O}(\kappa^{-d}) = r_\kappa \leqslant \tilde{r}_\kappa < \tilde{g}_0^{(\kappa)\,-1} = 1 + O(\kappa^{-d}), \tag{3.3}$$

which implies $|1 - \tilde{r}_\kappa| = O(\kappa^{-d})$. In the variational formula

$$\tilde{g}_1^{(\kappa)} = \sup_{\psi \perp \tilde{\Omega}} \frac{(\psi, \tilde{G}_\kappa(\beta)\psi)}{(\psi,\psi)} = \sup_{\psi \perp \tilde{\Omega}} \frac{[(\psi, G_\kappa(\beta)\psi) - (\psi, D_\kappa\psi)]}{(\psi,\psi)},$$

we can use a linear combination of two excited states of the one particle Hamiltonian h_κ as a trial function ψ which perpendicular to $\tilde{\Omega}$. Here $\tilde{\Omega}$ is the eigenfunction corresponding to the largest eigenvalue $\tilde{g}_0^{(\kappa)}$ of $\tilde{G}_\kappa(\beta)$. Then we get $\tilde{g}_1^{(\kappa)} \geqslant 1 - \hat{O}(\kappa^{-1})$. Together with $\tilde{g}_1^{(\kappa)} \leqslant g_1^{(\kappa)} = 1 - \hat{O}(\kappa^{-1})$, $\tilde{g}_1^{(\kappa)} = 1 - \hat{O}(\kappa^{-1})$ follows. Thus $\hat{O}(\kappa) = \tilde{p}_1^{(\kappa)} \geqslant \tilde{p}_2^{(\kappa)} \geqslant \cdots$ holds.

Now we get

$$|\sum_{j=1}^{\infty} \tilde{p}_j^{(\kappa)} - \sum_{j=1}^{\infty} \tilde{g}_j^{(\kappa)}/(1 - \tilde{g}_j^{(\kappa)})| = |1 - \tilde{r}_\kappa| \sum_{j=1}^{\infty} \tilde{g}_j^{(\kappa)}/\big((1 - \tilde{r}_\kappa \tilde{g}_j^{(\kappa)})(1 - \tilde{g}_j^{(\kappa)})\big)$$

$$= O(\kappa^{1-d})\sum_{j=1}^{\infty}\tilde{g}_j^{(\kappa)}/(1-\tilde{g}_j^{(\kappa)}),$$

which implies

$$\sum_{j=1}^{\infty}\tilde{p}_j^{(\kappa)} = (1+O(\kappa^{1-d}))\sum_{j=1}^{\infty}\tilde{g}_j^{(\kappa)}/(1-\tilde{g}_j^{(\kappa)}).$$

On the other hand, because

$$|\sum_{j=1}^{\infty}\frac{\tilde{g}_j^{(\kappa)}}{1-\tilde{g}_j^{(\kappa)}} - \sum_{j=1}^{\infty}\frac{g_j^{(\kappa)}}{1-g_j^{(\kappa)}}| \leqslant \sum_{j=1}^{\infty}\frac{g_j^{(\kappa)}-\tilde{g}_j^{(\kappa)}}{(1-g_1^{(\kappa)})(1-\tilde{g}_1^{(\kappa)})} \leqslant \frac{\mathrm{Tr}\, D_\kappa}{(1-g_1^{(\kappa)})(1-\tilde{g}_1^{(\kappa)})} = O(\kappa^2),$$

we have

$$\frac{1}{\kappa^d}\sum_{j=1}^{\infty}\frac{\tilde{g}_j^{(\kappa)}}{1-\tilde{g}_j^{(\kappa)}} = \frac{1}{\kappa^d}\sum_{j=1}^{\infty}\frac{g_j^{(\kappa)}}{1-g_j^{(\kappa)}} + O(\kappa^{2-d})$$

$$= \int_{[0,\infty)}\frac{e^{-\beta|p|_1}}{1-e^{-\beta|p|_1}}\,dp + o(1) = \frac{\zeta(d)}{\beta^d} + o(1).$$

Thus we have $\kappa^{-d}\sum_{j=1}^{\infty}\tilde{p}_j^{(\kappa)} = \beta^{-d}\zeta(d) + o(1)$. This yields the fifth equality. Using (2.10), we get

$$\frac{\tilde{r}_\kappa\tilde{g}_0^{(\kappa)}}{\kappa^d(1-\tilde{r}_\kappa\tilde{g}_0^{(\kappa)})} = -\frac{\log\tilde{r}_\kappa}{\beta\lambda} + \frac{\mu}{\lambda} - \frac{1}{\kappa^d}\sum_{j=1}^{\infty}\tilde{p}_j^{(\kappa)} = \frac{\mu}{\lambda} - \frac{\zeta(d)}{\beta^d} + o(1) = \frac{\mu-\mu_c}{\lambda} + o(1) > 0,$$

which yields the first and the third equality.

To prove the remaining last bound, it is enough to show that

$$\tilde{p}_j^{(\kappa)} \leqslant 2p_j^{(\kappa)} \quad (j = 1, 2, \cdots) \tag{3.4}$$

hold for large enough κ, because of $\sum_{j=1}^{\infty}p_j^{(\kappa)\,2} = O(\kappa^d)$. In fact, in the expression

$$\tilde{p}_j^{(\kappa)} = \frac{r_\kappa\tilde{g}_j^{(\kappa)}}{1-r_\kappa\tilde{g}_j^{(\kappa)}}\frac{1+(\tilde{r}_\kappa-r_\kappa)/r_\kappa}{1-(\tilde{r}_\kappa-r_\kappa)\tilde{g}_j^{(\kappa)}/(1-r_\kappa\tilde{g}_j^{(\kappa)})},$$

$(\tilde{r}_\kappa-r_\kappa)/r_\kappa = O(\kappa^{-d})$ and $|(\tilde{r}_\kappa-r_\kappa)\tilde{g}_j^{(\kappa)}/(1-r_\kappa\tilde{g}_j^{(\kappa)})| \leqslant (\tilde{r}_\kappa-r_\kappa)/(1-r_\kappa g_1^{(\kappa)}) = O(\kappa^{1-d})$ hold. Because of $\tilde{g}_j^{(\kappa)} \leqslant g_j^{(\kappa)}$, we also have $r_\kappa\tilde{g}_j^{(\kappa)}/(1-r_\kappa\tilde{g}_j^{(\kappa)}) \leqslant p_j^{(\kappa)}$. Thus we get (3.4). □

It is obvious now that the next Lemma can be derived along the same line of reasoning as the proof of Lemma 3.4.

Lemma 3.6. *For large κ, the asymptotics*

$$\tilde{\Xi}_\kappa(\beta,\mu) = \sqrt{\frac{2\pi\beta\lambda}{e^2\kappa^d}}\frac{\beta^{d-1}e^{\kappa^d(\beta\mu-\log\tilde{r}_\kappa)^2/2\beta\lambda}}{(\beta^d\mu-\zeta(d)\lambda)\mathrm{Det}(1-\tilde{r}_\kappa\tilde{G}_\kappa(\beta))}(1+o(1)) \tag{3.5}$$

holds.

In order to calculate the limit of $\tilde{\Xi}_\kappa(\beta,\mu)/\Xi_\kappa(\beta,\mu)$, we use the following lemma, where we put

$$\hat{g}_0^{(\kappa)} := (\Omega_\kappa, \tilde{G}_\kappa(\beta)\Omega_\kappa) + \tilde{r}_\kappa(\Omega_\kappa, \tilde{G}_\kappa(\beta)Q_\kappa(1-\tilde{r}_\kappa Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1}Q_\kappa \tilde{G}_\kappa(\beta)\Omega_\kappa).$$

Lemma 3.7. *For large* κ,

$$\text{(i)} \qquad \tilde{r}_\kappa - r_\kappa = (1-\tilde{g}_0^{(\kappa)})(1+o(1)) = (f(0)+o(1))\kappa^{-d},$$
$$\text{(ii)} \qquad 1-\tilde{r}_\kappa \hat{g}_0^{(\kappa)} = (1-\tilde{r}_\kappa \tilde{g}_0^{(\kappa)})(1+o(1))$$

hold.

Proof: From Lemma 3.5 and Proposition 2.4(b), we have $\tilde{r}_\kappa \tilde{g}_0^{(\kappa)} - r_\kappa = o(\kappa^{-d})$. Hence, (i) follows from $\tilde{r}_\kappa - r_\kappa = \tilde{r}_\kappa(1-\tilde{g}_0^{(\kappa)}) + \tilde{r}_\kappa \tilde{g}_0^{(\kappa)} - r_\kappa, \tilde{r}_\kappa = 1 + O(\kappa^{-d})$ and Lemma 2.3.

By virtue of (2.16), we get

$$\begin{aligned}&\tilde{g}_0^{(\kappa)} - \hat{g}_0^{(\kappa)}\\ &= (W_\kappa^*\Omega_\kappa, W_\kappa^* Q_\kappa[(\tilde{g}_0^{(\kappa)} - Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1} - (\tilde{r}_\kappa^{-1} - Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1}]Q_\kappa W_\kappa W_\kappa^*\Omega_\kappa).\end{aligned}$$

From

$$\begin{aligned}&\|(\tilde{g}_0^{(\kappa)} - Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1} - (\tilde{r}_\kappa^{-1} - Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1}\|\\ &= \|(\tilde{g}_0^{(\kappa)} - Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1}(\tilde{r}_\kappa^{-1} - \tilde{g}_0^{(\kappa)})(\tilde{r}_\kappa^{-1} - Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1}\|\\ &\leqslant \frac{1-\tilde{r}_\kappa \tilde{g}_0^{(\kappa)}}{(1-\tilde{r}_\kappa g_1^{(\kappa)})(\tilde{g}_0^{(\kappa)} - g_1^{(\kappa)})}\end{aligned}$$

and

$$(W_\kappa^*\Omega_\kappa, W_\kappa^* Q_\kappa^2 W_\kappa W_\kappa^*\Omega_\kappa) \leqslant \|D_\kappa\|(\Omega_\kappa, D_\kappa\Omega_\kappa) = O(\kappa^{-2d}),$$

it follows that

$$\tilde{g}_0^{(\kappa)} - \hat{g}_0^{(\kappa)} \leqslant (1-\tilde{r}_\kappa \tilde{g}_0^{(\kappa)})O(\kappa^{2-2d}). \tag{3.6}$$

Therefore, we obtain the asymptotics (ii):

$$1-\tilde{r}_\kappa \hat{g}_0^{(\kappa)} = 1-\tilde{r}_\kappa \tilde{g}_0^{(\kappa)} + \tilde{r}_\kappa(\tilde{g}_0^{(\kappa)} - \hat{g}_0^{(\kappa)}) = (1-\tilde{r}_\kappa \tilde{g}_0^{(\kappa)})(1+o(1)). \qquad \square$$

Now, taking into account (3.2) and (3.5), we can find the asymptotics of the generating functional (1.6):

$$\frac{\tilde{\Xi}_\kappa(\beta,\mu)}{\Xi_\kappa(\beta,\mu)} = \exp\Big(\frac{\kappa^d}{2\beta\lambda}(2\beta\mu - \log r_\kappa \tilde{r}_\kappa)\log\frac{r_\kappa}{\tilde{r}_\kappa}\Big)\frac{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa]}{\mathrm{Det}[1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta)]}$$

$$
\times \frac{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]}{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa]} \frac{\mathrm{Det}[1-r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]}{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]} \tag{3.7}
$$

$$
\times \frac{\mathrm{Det}[1-r_\kappa G_\kappa(\beta)]}{\mathrm{Det}[1-r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]}(1+o(1)).
$$

For the exponent of the first factor, we get

$$
\frac{\kappa^d}{2\beta\lambda}(2\beta\mu - \log r_\kappa \tilde{r}_\kappa)\log\frac{r_\kappa}{\tilde{r}_\kappa} = -\frac{\mu\kappa^d}{\lambda}(1-\tilde{g}_0^{(\kappa)})(1+o(1)) = -\frac{\mu f(0)}{\lambda}(1+o(1)) \tag{3.8}
$$

from Lemma 3.7(i). For the second factor, we use the Feshbach formula, which claims

$$
\mathrm{Det}A = \mathrm{Det}B\,\mathrm{Det}(C - K^T B^{-1}K),
$$

where

$$
A = \begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -K^T B^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & C - K^T B^{-1} K \end{pmatrix} \begin{pmatrix} 1 & -B^{-1}K \\ 0 & 1 \end{pmatrix}.
$$

This formula and Lemma 3.7(ii) yield

$$
\frac{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa]}{\mathrm{Det}[1-\tilde{r}_\kappa \tilde{G}_\kappa(\beta)]} =
$$

$$
\frac{1}{1-\tilde{r}_\kappa(\Omega_\kappa, \tilde{G}_\kappa(\beta)\Omega_\kappa) - (\Omega_\kappa, \tilde{r}_\kappa \tilde{G}_\kappa(\beta)Q_\kappa(1-\tilde{r}_\kappa Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa)^{-1}Q_\kappa \tilde{r}_\kappa \tilde{G}_\kappa(\beta)\Omega_\kappa)}
$$

$$
= 1/(1-\tilde{r}_\kappa \hat{g}_0^{(\kappa)}) = (1+o(1))/(1-\tilde{r}_\kappa \tilde{g}_0^{(\kappa)}).
$$

Since

$$
\frac{\mathrm{Det}[1-r_\kappa G_\kappa(\beta)]}{\mathrm{Det}[1-r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]} = 1 - r_\kappa,
$$

then Proposition 2.4(b) and Lemma 3.5 yield

$$
\text{(the 2nd factor)} \times \text{(the fifth factor)} \to 1
$$

in the limit $\kappa \to \infty$.

Now, (2.12) gives the limit:

$$
\frac{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]}{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa \tilde{G}_\kappa(\beta)Q_\kappa]} = \frac{1}{\mathrm{Det}[1+\tilde{r}_\kappa Q_\kappa(G_\kappa(\beta)-\tilde{G}_\kappa(\beta))Q_\kappa(1-\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa)^{-1}]}
$$

$$
= \mathrm{Det}[1+\sqrt{f^{(\kappa)}}\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa(1-\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa)^{-1}\sqrt{f^{(\kappa)}}]^{-1}
$$

$$
\to \exp\Big[-\int_{(\mathbb{R}^d)^2} \frac{f(x)}{e^{\beta(p^2+x^2)/2}-1}\frac{dp\,dx}{(2\pi)^d}\Big]
$$

for the third factor in (3.7). Here, we have used the cyclicity of the Fredholm determinant.

Lemma 3.7(i), (3.1), (2.9), Proposition 2.4(b) and Lemma 2.3 yield

$$\begin{aligned}
&\frac{\mathrm{Det}[1-r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]}{\mathrm{Det}[1-\tilde{r}_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa]} \\
&= \frac{1}{\mathrm{Det}[1-(\tilde{r}_\kappa - r_\kappa)Q_\kappa G_\kappa(\beta)Q_\kappa(1-r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa)^{-1}]} \\
&= \exp\Big(\frac{\tilde{r}_\kappa - r_\kappa}{r_\kappa}\mathrm{Tr}\,\frac{r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa}{1-r_\kappa Q_\kappa G_\kappa(\beta)Q_\kappa} + o(1)\Big) \\
&= \exp\Big((1-\tilde{g}_0^{(\kappa)})\Big[\kappa^d\frac{\beta\mu - \log r_\kappa}{\beta\lambda} - \frac{r_\kappa}{1-r_\kappa}\Big] + o(1)\Big) \\
&= \exp\{(1-\tilde{g}_0^{(\kappa)})\kappa^d\zeta(d)(1+o(1))/\beta^d\} = \exp\{\beta^{-d}f(0)\zeta(d)+o(1)\}.
\end{aligned}$$

Thus by (3.8), we get

$$\begin{aligned}
&(\text{the 1st factor}) \times (\text{the 4th factor}) \\
&\longrightarrow \exp\Big(-\frac{\beta^d\mu - \zeta(d)\lambda}{\beta^d\lambda}f(0)\Big) = \exp\Big(-\frac{\mu-\mu_c}{\lambda}f(0)\Big).
\end{aligned}$$

Now Theorem 1.1(ii) is derived by collecting the asymptotics of factors that we find above. □

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