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Five-dimensional AGT Relation, q-$\mathcal{W}$ Algebra and Deformed β-ensemble

By

Hidetoshi Awata* and Yasuhiko Yamada**

Abstract

We define a q-deformation of the β-ensemble which satisfies q-$\mathcal{W}_N$ constraint. We also show a relation with the Nekrasov partition function of 5D $SU(N)$ gauge theory with $N_f = 2N$.

§ 1. Introduction

In Ref. [1], Alday Gaiotto and Tachikawa discovered remarkable relations between the 4D $\mathcal{N} = 2$ super conformal gauge theories and the 2D Liouville conformal field theories. Some explanations have been addressed from β-ensemble (generalized matrix model) [2, 3] in Ref. [4]–[7].

In the pure $SU(2)$ case, the AGT relation [8] between the instanton part of the partition functions of the gauge theory and correlation functions of the Virasoro algebra is extended naturally to 5D in Ref. [9] (see also [10]). The instanton counting [11]–[14] of the 5D gauge theory [15] can be viewed as a q-analog of 4D cases, [16]–[18] and there also exists a natural q-deformation of the Virasoro/$\mathcal{W}_N$ algebra. [19]–[22]

In this talk, we will study a 5D extension of the AGT relation with $N_f = 2N$ in terms of β-ensemble. The A_{N-1} type quiver matrix model (the ITEP model) [23] was generalized as a β-ensemble [2] satisfying the $\mathcal{W}_N$ constraint by Ref. [3]. The partition function of the A_{N-1} type β-ensemble is defined as the singular vector of the $\mathcal{W}_N$

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*Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.
e-mail: `awata@math.nagoya-u.ac.jp`
**Department of Mathematics, Faculty of Science, Kobe University, Kobe 657-8501, Japan.
e-mail: `yamaday@math.kobe-u.ac.jp`

algebra as follows[3]

$$Z_N^{c\ell} := \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \prod_{a=1}^{N-1} \Delta^{c\ell}(z^a) e^{W^{c\ell}(z^a, z^{a+1})} \tag{1.1}$$

with $z_i^N := 0$. Here $\Delta^{c\ell}(z) := \prod_{i<j} (1 - z_j/z_i)^\beta (z_i/z_j - 1)^\beta$ is the β-deformed kernel and

$$W^{c\ell}(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \beta \sum_{n>0} \frac{1}{n} (z_i^a)^n p_n^{(a)} - \beta \sum_{j=1}^{r_{a+1}} \log\left(1 - z_j^{a+1}/z_i^a\right) - (s_a + 1) \log z_i^a \right\} \tag{1.2}$$

is the Penner type potential. The partition function $Z_N^{c\ell}$ is a function in coupling constants $p_n^{(a)}$ and is specified by a set of integers r_a and s_a ($n \in \mathbb{N}$ and $a = 1, 2, \cdots, N$). Since $Z_N^{c\ell}$ is the singular vector, it satisfies the $\mathcal{W}_N$ constraint $\mathcal{W}_{c\ell n}^a Z_N^{c\ell} = 0$ $(n > 0)$ with the $\mathcal{W}_N$ generators $\mathcal{W}_{c\ell n}^a$, which Virasoro central charge is $c = N - 1 - N(N^2 - 1)\left(\sqrt{\beta} - 1/\sqrt{\beta}\right)^2$. Under the strategy of Ref. [3], we will introduce a q-deformed β-ensemble which automatically satisfies q-$\mathcal{W}_N$ constraint. The partition function Z_N of the A_{N-1} type q-deformed β-ensemble will be defined as the singular vector of the q-$\mathcal{W}_N$ algebra and is given by replacing $\Delta^{c\ell}(z)$ and $W^{c\ell}(z^a, z^{a+1})$ in (1.1) with

$$\Delta(z) := \prod_{i<j} (1 - z_j/z_i) \prod_{\ell \geq 0} \frac{1 - q^\ell p z_j/z_i}{1 - q^\ell t z_j/z_i} \cdot \prod_{i=1}^{r} z_i^{(r+1-2i)\beta}, \tag{1.3}$$

$$W(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} \left((z_i^a)^n p_n^{(a)} + \sum_{j=1}^{r_{a+1}} \left(p^{\frac{1}{2}} z_j^{a+1}/z_i^a\right)^n \right) - (s_a + 1) \log z_i^a \right\}.$$

Here $[\beta]_q = (q^{\frac{\beta}{2}} - q^{-\frac{\beta}{2}})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$, $t = q^\beta$ and $p = q/t$. Then this satisfies the q-$\mathcal{W}_N$ constraint with the q-$\mathcal{W}_N$ generators defined in (3.12). If we specialize the mass parameters appropriately, the 5D Nekrasov partition function of $SU(N)$ gauge theory with $N_f = 2N$ reduces to the q-hypergeometric function. For $N = 2$, we will show that if we specialize the coupling constants appropriately, Z_2 also reduces to the q-hypergeometric function and coincides with the corresponding Nekrasov partition function.

This paper is organized as follows: In section 2, we start with recapitulating the result of the q-$\mathcal{W}_N$ algebra and also define primary fields. In section 3, we introduce q-deformed β-ensemble which automatically satisfies q-$\mathcal{W}_N$ constraint. Section 4 deals with the $N = 2$ case. Finally in section 5, we explain a reduction of the 5D Nekrasov partition function to the q-hypergeometric function and show a coincidence with the partition function of our q-deformed β-ensemble. Appendix A contains a definition of the Macdonald polynomial and several useful formulas.

Notation. Let $[n]_p := (p^{\frac{n}{2}} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$. Parameters are $q := e^{\hbar/\sqrt{\beta}} = e^{g_s R}$, $t := q^\beta = e^{\hbar\sqrt{\beta}} = e^{g_s\beta R}$, $p := q/t = e^{-\hbar(\sqrt{\beta}-1/\sqrt{\beta})}$, $u := t^\gamma$ and $v := (q/t)^{\frac{1}{2}}$. We will use the same letter p also for the set of power sums $p := (p_1, p_2, \cdots)$, but this appears only at $P_\lambda(x[p])$ or $Z_2(p)$. The integral $\oint \frac{dz}{2\pi i z} f(z)$ denotes the constant term in f.

§ 2. Quantum deformation of $\mathcal{W}_N$ algebra

We start with recapitulating the results of the q-$\mathcal{W}_N$ algebra [21, 22] and define primary fields.

§ 2.1. q-$\mathcal{W}_N$ algebra

We use three kinds of basis for bosons. First we define fundamental bosons h^i_n and Q^i_h for $i = 1, 2, \cdots, N$ and $n \in \mathbb{Z}$ such that[1]

$$(2.1)\quad [h^i_n, h^j_m] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})\frac{[\delta_{ij}N - 1]_{p^n}}{[N]_{p^n}} p^{\frac{n}{2}N\mathrm{sgn}(j-i)}\delta_{n+m,0},$$

$$(2.2)\quad [h^i_n, Q^j_h] = \left(\delta_{ij} - \frac{1}{N}\right)\delta_{n,0}, \quad [Q^i_h, Q^j_h] = 0, \quad \sum_{i=1}^N p^{in} h^i_n = 0, \quad \sum_{i=1}^N Q^i_h = 0$$

with q, $t := q^\beta \in \mathbb{C}$, $p := q/t$, $[n]_p := (p^{\frac{n}{2}} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$ and $\mathrm{sgn}(i) := 1$, 0 or -1 for $i > 0$, $i = 0$ or $i < 0$, respectively. Here $[A, B] := AB - BA$. This bosons correspond to the weights $\vec{h}_i$ of the vector representation whose inner product is $(\vec{h}_i \cdot \vec{h}_j) = \delta_{ij} - 1/N$. This algebra is invariant under the following involutions: $\omega_\pm^2 = 1$,

$$(2.3)\quad \omega_+ : \quad \sqrt{\beta} \mapsto 1/\sqrt{\beta}, \quad (q, t) \mapsto (t, q), \quad h^i_n \mapsto h^{N-i+1}_n, \quad Q^i_h \mapsto Q^{N-i+1}_h,$$

$$(2.4)\quad \omega_- : \quad \sqrt{\beta} \mapsto -\sqrt{\beta}, \quad (q, t) \mapsto (q^{-1}, t^{-1}), \quad h^i_n \mapsto h^{N-i+1}_n, \quad Q^i_h \mapsto Q^{N-i+1}_h.$$

We also use root type bosons $\alpha^a_n := h^a_n - h^{a+1}_n$ and $Q^a_\alpha := Q^a_h - Q^{a+1}_h$ and weight type bosons $\Lambda^a_n := \sum_{b=1}^a h^b_n p^{(b-a-\frac{1}{2})n}$ and $Q^a_\Lambda := \sum_{b=1}^a Q^b_h$ for $a = 1, 2, \cdots, N-1$.

Let us define fundamental vertices $\Lambda_i(z)$ and q-$\mathcal{W}_N$ generators $W^i(z)$ for $i = 1, 2, \cdots, N$ as follows:

$$(2.5)\quad \Lambda_i(z) := :\exp\left\{\sum_{n\neq 0} h^i_n z^{-n}\right\}: q^{\sqrt{\beta}h^i_0} p^{\frac{N+1}{2}-i},$$

$$(2.6)\quad W^i(zp^{\frac{1-i}{2}}) := \sum_{1\le j_1<\cdots<j_i\le N} :\Lambda_{j_1}(z)\Lambda_{j_2}(zp^{-1})\cdots\Lambda_{j_i}(zp^{1-i}):$$

[1] To obtain the $q = 1$ limit, we need to change the normalization of bosons by $h^{i\,\mathrm{old}}_n = h^{i\,\mathrm{new}}_n \sqrt{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})/n^2} = h^{i\,\mathrm{new}}_n \sqrt{[\beta]_{q^n}}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})/n$ $(n \neq 0)$ with $h^{i\,\mathrm{old}}_0 = h^{i\,\mathrm{new}}_0$ and $Q^{i\,\mathrm{old}}_h = Q^{i\,\mathrm{new}}_h$ unchanged. Letting $q \to 1$ yields the four-dimensional case.[3]

and $W^0(z) := 1$. Here $:*:$ stands for the usual bosonic normal ordering such that the bosons h^i_n with non-negative mode $n \geq 0$ are in the right. These generators are obtained from the following quantum Miura transformation:

$$\begin{aligned}(2.7)\quad &\sum_{i=0}^{N}(-1)^i W^i(zp^{\frac{1-i}{2}})p^{(N-i)D_z} \\ &= :\left(p^{D_z}-\Lambda_1(z)\right)\left(p^{D_z}-\Lambda_2(zp^{-1})\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})\right):\end{aligned}$$

with $D_z := z\frac{\partial}{\partial z}$. Remark that p^{D_z} is the p-shift operator such that $p^{D_z}f(z) = f(pz)$. The mode n generator W^i_n is defined by $\sum_{n\in\mathbb{Z}} W^i_n z^{-n} := W^i(z)$.

By using root type bosons we define screening currents $S^a_\pm(z)$ as follows:
(2.8)

$$S^a_\pm(z) := :\exp\left\{\mp\sum_{n\neq 0}\frac{\alpha^a_n}{\xi_\pm^{\frac{n}{2}}-\xi_\pm^{-\frac{n}{2}}}z^{-n}\right\}: e^{\pm\sqrt{\beta}^{\pm 1}Q^a_\alpha} z^{\pm\sqrt{\beta}^{\pm 1}\alpha^a_0}, \qquad \xi_+ = q, \quad \xi_- = t,$$

with $\alpha^a_n := h^a_n - h^{a+1}_n$ and $Q^a_\alpha := Q^a_h - Q^{a+1}_h$. Note that the Langlands duality $\omega_-\omega_+ S^a_+(z) = S^a_-(z)$. We denote the negative mode part of $S^a_\pm(z)$ by $(S^a_\pm(z))_- := \exp\left\{\mp\sum_{n<0}\frac{\alpha^a_n}{\xi_\pm^{\frac{n}{2}}-\xi_\pm^{-\frac{n}{2}}}z^{-n}\right\}$. Then the screening charges defined by $\oint dz S^a_\pm(z)$ commute with any q-$\mathcal{W}_N$ generators

$$(2.9)\qquad [\oint dz S^a_\pm(z), W^b(w)] = 0, \qquad a,b = 1,2,\cdots,N-1.$$

For parameters u and γ with $u := t^\gamma$, let us define the following vertex operators

$$(2.10)\qquad V^a_u(z) := :\exp\left\{\sum_{n\neq 0}\frac{(u^{\frac{n}{2}}-u^{-\frac{n}{2}})\Lambda^a_n}{(q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}})}z^{-n}\right\}: e^{-\gamma\sqrt{\beta}Q^a_\Lambda} z^{-\gamma\sqrt{\beta}\Lambda^a_0},$$

with $\Lambda^a_n := \sum_{b=1}^{a} h^b_n p^{(b-a-\frac{1}{2})n}$ and $Q^a_\Lambda := \sum_{b=1}^{a} Q^b_h$. They satisfy

$$\begin{aligned}(2.11)\quad & g^{a,L}_{u,p}\left(\frac{w}{z}\right)\Lambda_i(z)V^a_u(w) - V^a_u(w)\Lambda_i(z)g^{a,R}_{u,p}\left(\frac{z}{w}\right) \\ &= (u^{-1}-1)\sum_{b=1}^{a}\delta_{i,b}\delta\left(\frac{w}{zu^{\frac{1}{2}}}\right):\Lambda_i(z)V^a_u(w):,\end{aligned}$$

where $g^{a,L}_{u,p}(x)$ and $g^{a,R}_{u,p}(x)$ are inverse of the OPE factors,

$$(2.12)\quad g^{a,L}_{u,p}(x) := \frac{:\Lambda_j(z)V^a_u(w):}{\Lambda_j(z)V^a_u(w)} = \exp\left\{\sum_{n>0}\frac{u^{-\frac{n}{2}}-u^{\frac{n}{2}}}{n}\frac{[\,a\,]_{p^n}}{[\,N\,]_{p^n}}p^{\frac{n}{2}(a-N)}x^n\right\}u^{-\frac{a}{N}},$$

$$(2.13)\quad g^{a,R}_{u,p}(x) := \frac{:V^a_u(w)\Lambda_j(z):}{V^a_u(w)\Lambda_j(z)} = \exp\left\{\sum_{n>0}\frac{u^{\frac{n}{2}}-u^{-\frac{n}{2}}}{n}\frac{[\,a\,]_{p^n}}{[\,N\,]_{p^n}}p^{\frac{n}{2}(N-a)}x^n\right\}$$

for any $j > a$.

§ 2.2. Highest weight module of q-$\mathcal{W}_N$ algebra

Next we refer to the representation of the q-$\mathcal{W}_N$ algebra. Let $\mathcal{F}_\alpha$ be the boson Fock space generated by the highest weight state $|\alpha\rangle$ such that $\alpha^a_n|0\rangle = 0$ for $n \geq 0$ and $|\alpha\rangle := \exp\{\sum_{a=1}^{N-1} \alpha^a Q^a_\Lambda\}|0\rangle$. Note that $\alpha^a_0|\alpha\rangle = \alpha^a|\alpha\rangle$. The dual module $\mathcal{F}^*_\alpha$ is generated by $\langle\alpha|$ such that $\langle 0|\alpha^a_{-n} = 0$ for $n \geq 0$ and $\langle\alpha| := \langle 0|\exp\{-\sum_{a=1}^{N-1} \alpha^a Q^a_\Lambda\}$. The bilinear form $\mathcal{F}^*_\alpha \otimes \mathcal{F}_\alpha \to \mathbb{C}$ is uniquely defined by $\langle 0|0\rangle = 1$.

Let $|\lambda\rangle$ be the highest weight vector of the q-$\mathcal{W}_N$ algebra which satisfies $W^a_n|\lambda\rangle = 0$ for $n > 0$ and $a = 1, 2, \cdots, N-1$ and $W^a_0|\lambda\rangle = \lambda^a|\lambda\rangle$ with $\lambda^a \in \mathbb{C}$. Let M_λ be the Verma module over the q-$\mathcal{W}_N$ algebra generated by $|\lambda\rangle$. The dual module M^*_λ is generated by $\langle\lambda|$ such that $\langle\lambda|W^a_n = 0$ for $n < 0$ and $\langle\lambda|W^a_0 = \lambda^a\langle\lambda|$. The bilinear form $M^*_\lambda \otimes M_\lambda \to \mathbb{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle = 1$. A singular vector $|\chi\rangle \in M_\lambda$ is defined by $W^a_n|\chi\rangle = 0$ for $n > 0$ and $W^a_0|\chi\rangle = (\lambda^a + N^a)|\chi\rangle$ with $N^a \in \mathbb{C}$.

The highest weight vector $|\alpha\rangle \in \mathcal{F}_\alpha$ of the boson algebra is also that of the q-$\mathcal{W}_N$ algebra, i.e., $W^a_n|\alpha\rangle = 0$ for $n > 0$ and $a = 1, 2, \cdots, N-1$. Note that $W^a_0|0\rangle = [N]^a_p|0\rangle$ with $[N]_p := (p^{\frac{N}{2}} - p^{-\frac{N}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$.

For a set of non-negative integers s_a and $r_a \geq r_{a+1} \geq 0$ with $a = 1, \cdots, N-1$, let

$$\pm\alpha^{\pm;a}_{r,s} := (1 + r_a - r_{a-1})\sqrt{\beta}^{\pm 1} - (1 + s_a)\sqrt{\beta}^{\mp 1}, \qquad r_0 := 0, \tag{2.14}$$

$$\pm\widetilde{\alpha}^{\pm;a}_{r,s} := (1 - r_a + r_{a+1})\sqrt{\beta}^{\pm 1} - (1 + s_a)\sqrt{\beta}^{\mp 1}, \qquad r_N := 0. \tag{2.15}$$

The singular vectors $|\chi^\pm_{rs}\rangle \in \mathcal{F}_{\alpha^\pm_{rs}}$ are realized by the screening currents as follows:

$$\begin{aligned}|\chi^\pm_{r,s}\rangle &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dz^a_j}{2\pi i} \cdot S^1_\pm(z^1_1)\cdots S^1_\pm(z^1_{r_1})\cdots S^{N-1}_\pm(z^{N-1}_1)\cdots S^{N-1}_\pm(z^{N-1}_{r_{N-1}})|\widetilde{\alpha}^\pm_{r,s}\rangle \\ &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dz^a_j}{2\pi i z^a_j}(z^a_j)^{-s_a}(S^a_\pm(z^a_j))_- \cdot \Delta(z^a;\xi_\pm,\xi_\mp)\Pi\left(\overline{z^a}, pz^{a+1};\xi_\pm,\xi_\mp\right)|\alpha^\pm_{r,s}\rangle\end{aligned} \tag{2.16}$$

with $z^N := 0$, $\overline{z} := 1/z$, $\xi_+ := q$ and $\xi_- := t$. Note that $\omega_-\omega_+|\chi^+_{r,s}\rangle = |\chi^-_{r,s}\rangle$. Here

$$\Pi(z,w) := \Pi(z,w;q,t) := \prod_{i,j}\exp\left\{\sum_{n>0}\frac{[\,\beta\,]_{q^n}}{n}p^{-\frac{n}{2}}z_i^n w_j^n\right\} = \prod_{i,j}\prod_{\ell\geq 0}\frac{1-q^\ell t z_i w_j}{1-q^\ell z_i w_j}, \tag{2.17}$$

$$\begin{aligned}\Delta(z) := \Delta(z;q,t) &:= \prod_{i<j}\exp\left\{-\sum_{n>0}[\,2\,]_{p^n}\frac{[\,\beta\,]_{q^n}}{n}\frac{z_j^n}{z_i^n}\right\}\cdot\prod_{i=1}^{r} z_i^{(r+1-2i)\beta} \\ &= \prod_{i<j}(1-z_j/z_i)\prod_{\ell\geq 0}\frac{1-q^\ell p z_j/z_i}{1-q^\ell t z_j/z_i}\cdot\prod_{i=1}^{r} z_i^{(r+1-2i)\beta}, \qquad |q|<1\end{aligned} \tag{2.18}$$

with $\beta := \log t/\log q$. Note that $\Delta(cz) = \Delta(z)$.

§ 3. Quantum deformation of β-ensemble

Note that the singular vector in (2.16) is naturally mapped to the Macdonald polynomial [24] defined in the appendix A. [22] As a generalization of this map one can define, under the strategy of Ref. [3], a quantum deformation of the generalized matrix model, i.e., q-deformed β-ensemble.

§ 3.1. q-deformed β-ensemble

With a new parameters $p^{(a)} := (p_1^{(a)}, p_2^{(a)}, \cdots)$ let us define the following vertex operator

$$V_N := \prod_{a=1}^{N-1} \exp\left\{ \sum_{n>0} \frac{\Lambda_n^a}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p_n^{(a)} \right\}, \tag{3.1}$$

with $\Lambda_n^a := \sum_{b=1}^a h_n^b p^{(b-a-\frac{1}{2})n}$ and $Q_\Lambda^a := \sum_{b=1}^a Q_h^b$. Note that $[\Lambda_n^a, \Lambda_m^b] = 0$ for $n, m > 0$. Then $\langle\alpha|V_N$ defines the isomorphism between the boson algebras $\langle h_n^a \rangle_{n\in\mathbb{Z}}^{1\le a<N}$ and $\langle p_n^{(a)}, \alpha^a, \frac{\partial}{\partial p_n^{(a)}} \rangle_{n\in\mathbb{N}}^{1\le a<N}$ by

$$\langle\alpha|V_N h_{-n}^i = \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{b=1}^{N-1} A^{i,b}(p^{-n}) p_n^{(b)} \langle\alpha|V_N, \tag{3.2}$$

$$\langle\alpha|V_N h_n^i = (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) \sum_{b=1}^{N-1} B^{i,b}(p^n) \frac{\partial}{\partial p_n^{(b)}} \langle\alpha|V_N \tag{3.3}$$

for $n > 0$ and $\langle\alpha|V_N h_0^i = h^i \langle\alpha|V_N$ with $h^i = \left[\sum_{b=i}^{N-1} - \sum_{b=1}^{N-1} b/N\right] \alpha^b$. Here

$$A^{i,b}(p) := \frac{[\,N\theta(i \le b) - i\,]_p}{[\,N\,]_p} p^{\frac{1}{2}(b - N\theta(i>b))}, \tag{3.4}$$

$$B^{i,b}(p) := p^{\frac{1}{2}} \delta_{i,b} - p^{-\frac{1}{2}} \delta_{i-1,b} \tag{3.5}$$

with $\theta(P) := 1$ or 0 if the proposition P is true or false, respectively.

The vector $|S_{r,s}^+\rangle := \prod_{a=1}^{N-1} \prod_{k=1}^{r_a} (S_+^a(z_k^a))_- \cdot |\alpha_{r,s}^+\rangle$ in (2.16) also defines another linear map from $\langle h_n^a \rangle_{n\in\mathbb{N}}^{1\le a<N}$ to $\langle \sum_{k=1}^{r_a} (z_k^a)^n \rangle_{n\in\mathbb{N}}^{1\le a<N}$ by

$$h_n^i |S_{r,s}^+\rangle = |S_{r,s}^+\rangle \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{b=1}^{N-1} B^{i,b}(p^n) \sum_{k=1}^{r_b} (z_k^b)^n, \qquad n > 0. \tag{3.6}$$

Let us define the following partition function

Definition 3.1. Let $Z_N := Z_N\left(\{p^{(a)}\}_{a=1}^{N-1}\right) := \langle \alpha_{r,s}^+ | V_N | \chi_{r,s}^+ \rangle$.

Then by (2.16), (2.8) and (3.3), we have

$$
\begin{aligned}
Z_N &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \langle \alpha_{r,s}^+ | V_N S_+^1(z_1^1) \cdots S_+^1(z_{r_1}^1) \cdots S_+^{N-1}(z_1^{N-1}) \cdots S_+^{N-1}(z_{r_{N-1}}^{N-1}) | \widetilde{\alpha}_{r,s}^+ \rangle \\
&= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} (z_j^a)^{-s_a} \exp\left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} (z_j^a)^n p_n^{(a)} \right\} \cdot \Delta(z^a) \Pi\left(\overline{z^a}, p z^{a+1}\right) \\
&= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \prod_{a=1}^{N-1} \Delta(z^a) e^{W(z^a, z^{a+1})}
\end{aligned}
$$

with

(3.7)

$$
W(z^a, z^{a+1}) := \sum_{i=1}^{r_a} \left\{ \sum_{n>0} \frac{[\beta]_{q^n}}{n} \left((z_i^a)^n p_n^{(a)} + \sum_{j=1}^{r_{a+1}} \left(\frac{p^{\frac{1}{2}} z_j^{a+1}}{z_i^a} \right)^n \right) - (s_a + 1) \log z_i^a \right\}.
$$

Here $z^N := 0$. This Z_N is regarded as a q-deformation of the partition function of the generalized matrix model,[3] i.e., β-ensemble. One can define other type of partition functions by acting involutions (2.3), (2.4) and (A.8).

We can calculate this integral by using the Macdonald polynomials $P_\lambda(x)$ with the Young diagram λ, their fusion coefficient $f_{\lambda,\mu}^{\nu}$ and the inner products $\langle *, * \rangle$ and $\langle *, * \rangle_r''$ defined in the appendix A.

Proposition 3.2.

$$
Z_N = \prod_{a=1}^{N-1} \sum_{\lambda_a, \mu_a} f_{\mu_{a-1}, \lambda_a}^{\mu_a + (s_a^{r_a})} P_{\mu_a + (s_a^{r_a})}(z^a) \frac{P_{\lambda_a}(x[p^a])}{\langle \lambda_a \rangle} p^{|\mu_a|} \frac{r_a! \langle \mu_a + (s_a^{r_a}) \rangle_{r_a}''}{\langle \mu_a \rangle} \tag{3.8}
$$

with $\langle 0 \rangle := 1$. *Here* λ_a, μ_a *and* ν_a *are Young diagrams such that* $\lambda_{a,i} \geq \lambda_{a,i+1}$, *and so on.* $P_\lambda(x[p])$ *denotes the Macdonald function in power sums* $p := (p_1, p_2, \cdots)$.

One can show that (3.8) is summed over $(N-2) + (N-3)$ Young diagrams for $N \geq 3$.

For any function $\mathcal{O}$ in z_j^a's, the correlation function with respect to $\mathcal{O}$ is defined by

$$
\langle\!\langle \mathcal{O} \rangle\!\rangle := \frac{1}{Z_N} \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \mathcal{O} \prod_{a=1}^{N-1} \Delta(z^a) e^{W(z^a, z^{a+1})}. \tag{3.9}
$$

The effective action S_{eff} defined by $Z_N =: \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot e^{S_{\text{eff}}}$ is now

(3.10)

$$
S_{\text{eff}} = \sum_{a=1}^{N-1} W(z^a, z^{a+1}) - \sum_{n>0} [2]_{p^n} \frac{[\beta]_{q^n}}{n} \sum_{a=1}^{N-1} \sum_{i<j} \left(\frac{z_j^a}{z_i^a} \right)^n + \beta \sum_{a=1}^{N-1} \sum_{i=1}^{r_a} (r_a + 1 - 2i) \log z_i^a.
$$

§ 3.2. q-$\mathcal{W}_N$ constraint, Loop equation and quantum spectral curve

Next let us define $\hat{\Lambda}_i(z)$ and $\mathcal{W}^i(z)$ by the isomorphism (3.3) as follows:

$$\hat{\Lambda}_i(z)\langle\alpha_{r,s}|V_N := \langle\alpha_{r,s}|V_N\Lambda_i(z), \tag{3.11}$$

$$\mathcal{W}^i(z)\langle\alpha_{r,s}|V_N := \langle\alpha_{r,s}|V_N W^i(z) \tag{3.12}$$

and $\sum_{n\in\mathbb{Z}}\mathcal{W}^i_n z^{-n} := \mathcal{W}^i(z)$, which are the power sum realization of fundamental vertices $\Lambda_i(z)$ and q-$\mathcal{W}_N$ generators $W^i(z)$, respectively. Then the highest weight condition for the singular vector $W^a_n|\chi\rangle = 0$ for $n > 0$ is equivalent to the following q-$\mathcal{W}_N$ constraint:

Theorem 3.3.

$$\mathcal{W}^a_n Z_N = 0, \qquad n > 0. \tag{3.13}$$

Let us define $\widetilde{\Lambda}_i(z)$ and $\widetilde{\mathcal{W}}^i(z)$ by linear maps (3.3) and (3.6) as follows:

$$\langle\alpha^+_{r,s}|V_N|S^+_{r,s}\rangle\widetilde{\Lambda}_i(z) := \langle\alpha^+_{r,s}|V_N\Lambda_i(z)|S^+_{r,s}\rangle, \tag{3.14}$$

$$\langle\alpha^+_{r,s}|V_N|S^+_{r,s}\rangle\widetilde{\mathcal{W}}^i(z) := \langle\alpha^+_{r,s}|V_N W^i(z)|S^+_{r,s}\rangle \tag{3.15}$$

and $\sum_{n\in\mathbb{Z}}\widetilde{\mathcal{W}}^i_n z^{-n} := \widetilde{\mathcal{W}}^i(z)$. Hence

$$\left\langle\!\!\left\langle\, \widetilde{\mathcal{W}}^i(z) \,\right\rangle\!\!\right\rangle = \frac{1}{Z_N}\langle\alpha^+_{r,s}|V_N W^i(z)|\chi^+_{r,s}\rangle. \tag{3.16}$$

Therefore the highest weight condition for the singular vector $W^a_n|\chi\rangle = 0$ for $n > 0$ is equivalent to the following loop equation:

Theorem 3.4.

$$\left\langle\!\!\left\langle\, \widetilde{\mathcal{W}}^a_n \,\right\rangle\!\!\right\rangle = 0, \qquad n > 0. \tag{3.17}$$

The quantum spectral curve should be

$$\left\langle\!\!\left\langle\, \left(p^{D_z} - \widetilde{\Lambda}_1(z)\right)\left(p^{D_z} - \widetilde{\Lambda}_2(zp^{-1})\right)\cdots\left(p^{D_z} - \widetilde{\Lambda}_N(zp^{1-N})\right) \,\right\rangle\!\!\right\rangle = 0 \tag{3.18}$$

which regularity in z is guaranteed by the loop equation (3.17).

Let $(q,t) =: (e^{R\epsilon_2}, e^{-R\epsilon_1}) =: (e^{g_s R}, e^{g_s\beta R})$ with the radius $R \in \mathbb{R}$ of the 5th dimensional circle S^1. Let us rescale the variables as $\tilde{p}^{(a)}_n := g_s p^{(a)}_n$, $\tilde{r}_a := g_s r_a$ and $\tilde{s}_a := g_s s_a$. Under the limit $g_s \to 0$ and $r_a\ s_a \to \infty$ with fixed $\tilde{r}_a$ and $\tilde{s}_a$, the sift operator p^{D_z} tends to a commutative variable and the quantum spectral curve reduces to the usual one.

§ 4. $N = 2$ case

Here we give an example when $N = 2$, i.e., the q-deformed Virasoro case. The partition function Z_2 is now

$$Z_2(p) = \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} z_j^{-s} \exp\left\{\sum_{n>0} \frac{[\beta]_{q^n}}{n} z_j^n p_n\right\} \cdot \Delta(z) = p^{\frac{rs}{2}} \frac{r!\langle s^r \rangle''_r}{\langle s^r \rangle} P_{(s^r)}(x[p]). \tag{4.1}$$

Then we have

Proposition 4.1. *The partition function $Z_2(p)$ substituting $p_n = \sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n$ and $\frac{1-t^n}{1-q^n} p_n = (-1)^{n-1}(\sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n)$ are*

$$\frac{Z_2\left(\sum_i x_i + \frac{1-u}{1-t} y\right)}{Z_2\left(\frac{1-u}{1-t} y\right)} = {}_2\varphi_1^{(q,t)} \left[\begin{matrix} q^{-s}, t^r \\ q^{1-s}t^{r-1}/u \end{matrix}; \frac{qx}{uy}\right], \tag{4.2}$$

$$\omega_{q,t} \frac{Z_2\left(\sum_i x_i + \frac{1-u}{1-t} y\right)}{Z_2\left(\frac{1-u}{1-t} y\right)} = {}_2\varphi_1^{(t,q)} \left[\begin{matrix} t^{-r}, q^s \\ t^{1-r}q^{s-1}/u \end{matrix}; \frac{tx}{uy}\right]. \tag{4.3}$$

with $\omega_{q,t}$ in (A.8). *Here ${}_2\varphi_1^{(q,t)}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ is the multivariate q-hypergeometric function* [28]

$$ {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := \sum_{\substack{\lambda \\ \ell(\lambda) \le M}} P_\lambda(x) \prod_{(i,j)\in\lambda} \frac{(t^{i-1} - aq^{j-1})(t^{i-1} - bq^{j-1})}{(t^{i-1} - cq^{j-1})(1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i})}. \tag{4.4}$$

Since $P_\lambda(x; q, t) = P_\lambda(x; q^{-1}, t^{-1})$, ${}_2\varphi_1^{(q,t)}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ satisfies

$$ {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = {}_2\varphi_1^{(q^{-1},t^{-1})} \left[\begin{matrix} a^{-1}, b^{-1} \\ c^{-1} \end{matrix}; \frac{ab}{qc} x\right]. \tag{4.5}$$

When $M = \infty$,

$$\omega_{q,t} {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = {}_2\varphi_1^{(t,q)} \left[\begin{matrix} a, b \\ c \end{matrix}; \frac{ab}{c} x\right], \qquad M = \infty. \tag{4.6}$$

When $M = 1$, ${}_2\varphi_1^{(q,t)}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ reduces to the usual q-hypergeometric function

$$ {}_2\varphi_1^{(q,t)} \left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := {}_2\varphi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, x\right] := \sum_{n\ge0} x^n \prod_{\ell=0}^{n-1} \frac{(1 - aq^\ell)(1 - bq^\ell)}{(1 - cq^\ell)(1 - q^{\ell+1})}, \qquad M = 1. \tag{4.7}$$

In the next section we will show a relation between our $Z_2\left(x + \frac{1-u}{1-t} y\right)$ and the 5-dimensional $SU(2)$ Nekrasov partition function.

§ 5. Five-dimensional Nekrasov partition function

Let $Q = (Q_1, \cdots, Q_N)$ with $\prod_{i=1}^N Q_i = 1$ and $Q^\pm = (Q_1^\pm, \cdots, Q_N^\pm)$ be sets of complex parameters. The instanton part of the five-dimensional $SU(N)$ Nekrasov partition function with $N_f = 2N$ fundamental matters[2] is written by a sum over N Young diagrams λ_i $(i = 1, 2, \cdots, N)$ as follows(double-sign corresponds):[25, 18]

$$Z^{\rm inst}(Q) = \sum_{\{\lambda_i\}} \prod_{i,j} \frac{N_{\lambda_i \bullet}(vQ_i/Q_j^\pm) N_{\bullet \lambda_i}(vQ_j^\mp/Q_i)}{N_{\lambda_i \lambda_j}(Q_i/Q_j)} \cdot \prod_i \left(\frac{\Lambda_\alpha^\pm}{v^N} \right)^{|\lambda_i|} \tag{5.1}$$

with $v := (q/t)^{\frac{1}{2}}$, $N_{\lambda\mu}(Q) := N_{\lambda\mu}(Q; q, t)$, $\Lambda_\alpha^\pm := \Lambda^{2N} \prod_{j=1}^N \left(\frac{Q_j^\pm}{Q_j^\mp} \right)^{\frac{1}{2}}$ and

$$\begin{aligned} N_{\lambda\mu}(Q; q, t) := & \prod_{(i,j)\in\lambda} \left(1 - Q\, q^{\lambda_i - j} t^{\mu'_j - i + 1}\right) \prod_{(i,j)\in\mu} \left(1 - Q\, q^{-\mu_i + j - 1} t^{-\lambda'_j + i}\right) \\ = & \prod_{(i,j)\in\mu} \left(1 - Q\, q^{\lambda_i - j} t^{\mu'_j - i + 1}\right) \prod_{(i,j)\in\lambda} \left(1 - Q\, q^{-\mu_i + j - 1} t^{-\lambda'_j + i}\right). \end{aligned} \tag{5.2}$$

Here $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a Young diagram such that $\lambda_i \ge \lambda_{i+1}$. λ' is its conjugate Young diagram and $|\lambda| = \sum_i \lambda_i$. $Z^{\rm inst}(Q; Q^+; Q^-)$ is symmetric in masses ${Q^\pm}_j$'s. Note that $N_{\lambda\mu}(Q; q, t)$ satisfies

$$N_{\lambda\mu}(vQ; q, t) = N_{\mu\lambda}(Q/v; q^{-1}, t^{-1}) = N_{\mu'\lambda'}(Q/v; t, q), \tag{5.3}$$

$$N_{\lambda\bullet}(vQ) N_{\bullet\lambda}(vQ') = N_{\bullet\lambda}(v/Q) N_{\lambda\bullet}(v/Q') (QQ')^{|\lambda|}. \tag{5.4}$$

There exists Q such that $N_{\lambda\bullet}(Q)$ vanishes except for $\lambda = (0)$, (n) or (1^n). Hence one can adjust N out of $N_f = 2N$ parameters ${Q^\pm}_i$'s so that (5.1) reduces to all $\lambda_i = (0)$ but a $\lambda_j = (n)$ or (1^n) with $n \in \mathbb{Z}_{\ge 0}$ same as Ref. [26]. For example, if $(Q_1, \cdots, Q_{N-1}, Q_N) = (Q_1^\pm, \cdots, Q_{N-1}^\pm, tQ_N^\pm)/v$ with $\prod_{i=1}^N Q^\pm = v^N/t$ then the right hand side of (5.1) is summed over only $(\lambda_1, \cdots, \lambda_{N-1}, \lambda_N) = ((0), \cdots, (0), (n))$ with $n \in \mathbb{Z}_{\ge 0}$. On the other hand,
if $(Q_1, \cdots, Q_{N-1}, Q_N) = (Q_1^\pm, \cdots, Q_{N-1}^\pm, Q_N^\pm/q)/v$ with $\prod_{i=1}^N Q^\pm = qv^N$ then only $(\lambda_1, \cdots, \lambda_{N-1}, \lambda_N) = ((0), \cdots, (0), (1^n))$ contributes. Therefore we obtain

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[2]The parameters (q, t) are related with those (ϵ_1, ϵ_2) of the Ω background through $(q, t) = (e^{R\epsilon_2}, e^{-R\epsilon_1})$ where R is the radius of the 5th dimensional circle. The parameter Q is related with the vacuum expectation value a of the scalar fields in the vector multiplets and the mass m of the fundamental matter as $Q_i = q^{a_i}$, $Q_i^+ = q^{-m_i}$ and $Q_i^- = q^{-m_{N+i}}$.

Proposition 5.1.

$$(5.5)\quad Z^{\text{inst}}(Q_1^{\pm}/v,\cdots,Q_{N-1}^{\pm}/v,tQ_N^{\pm}/v) = {}_N\varphi_{N-1}\left[\begin{matrix}\frac{Q_1^{\mp}}{vQ_N},\cdots,\frac{Q_N^{\mp}}{vQ_N}\\ \frac{tQ_1}{qQ_N},\cdots,\frac{tQ_{N-1}}{qQ_N}\end{matrix};q^{-1},\frac{\Lambda_N^{\pm}}{v^N}\right]$$

$$= {}_N\varphi_{N-1}\left[\begin{matrix}v\frac{Q_N}{Q_1^{\mp}},\cdots,v\frac{Q_N}{Q_N^{\mp}}\\ \frac{qQ_N}{tQ_1},\cdots,\frac{qQ_N}{tQ_{N-1}}\end{matrix};q,v^N\Lambda_N^{\mp}\right],$$

$$(5.6)\quad Z^{\text{inst}}(Q_1^{\pm}/v,\cdots,Q_{N-1}^{\pm}/v,Q_N^{\pm}/qv) = {}_N\varphi_{N-1}\left[\begin{matrix}\frac{Q_1^{\mp}}{vQ_N},\cdots,\frac{Q_N^{\mp}}{vQ_N}\\ \frac{tQ_1}{qQ_N},\cdots,\frac{tQ_{N-1}}{qQ_N}\end{matrix};t,\frac{\Lambda_N^{\pm}}{v^N}\right]$$

with $\prod_{i=1}^N Q_i^{\pm} = v^N/t$ *for* (5.5) *and* $\prod_{i=1}^N Q_i^{\pm} = qv^N$ *for* (5.6) *and*

$$(5.7)\qquad {}_r\varphi_s\left[\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,x\right] := \sum_{n\geq 0} x^n \prod_{\ell=0}^{n-1}\frac{(-q^{\ell})^{s+1-r}\prod_{i=1}^r(1-q^{\ell}a_i)}{(1-q^{\ell+1})\prod_{i=1}^s(1-q^{\ell}b_i)}.$$

Note that

$$(5.8)\quad {}_r\varphi_{r-1}\left[\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_{r-1}\end{matrix};q,x\right] = {}_r\varphi_{r-1}\left[\begin{matrix}a_1^{-1},\cdots,a_r^{-1}\\ b_1^{-1},\cdots,b_{r-1}^{-1}\end{matrix};q^{-1},\widetilde{x}\right],\qquad \widetilde{x} := \frac{x\prod_{i=1}^r a_i}{q\prod_{i=1}^{r-1} b_i}.$$

When $N=2$, Z^{inst} coincides with the $M=1$ case of the partition function Z_2 of the q-deformed β-ensemble (4.2) similar to Ref. [6]

$$(5.9)\quad Z^{\text{inst}}(Q_1^{\pm}/v,tQ_2^{\pm}/v) = {}_2\varphi_1\left[\begin{matrix}v\frac{Q_2}{Q_1^{\mp}},v\frac{Q_2}{Q_2^{\mp}}\\ \frac{qQ_2}{tQ_1}\end{matrix};q,v^2\Lambda_2^{\mp}\right] = \frac{Z_2\left(x+\frac{1-u}{1-t}y\right)}{Z_2\left(\frac{1-u}{1-t}y\right)},$$

$$(5.10)\quad Z^{\text{inst}}(Q_1^{\pm}/v,Q_2^{\pm}/qv) = {}_2\varphi_1\left[\begin{matrix}\frac{Q_1^{\mp}}{vQ_2},\frac{Q_2^{\mp}}{vQ_2}\\ \frac{tQ_1}{qQ_2}\end{matrix};t,\frac{\Lambda_2^{\pm}}{v^2}\right] = \omega_{q,t}\frac{Z_2\left(\sum_i x_i+\frac{1-u}{1-t}y\right)}{Z_2\left(\frac{1-u}{1-t}y\right)}$$

with

$$(5.11)\quad Q_1^{\pm}Q_2^{\pm} = \frac{q}{t^2},\quad q^s = \frac{Q_1Q_1^{\mp}}{v},\quad t^{-r} = \frac{Q_1Q_2^{\mp}}{v},\quad u^{-1} = \frac{tQ_1^{\mp}Q_2^{\mp}}{q},\quad \frac{qx}{y} = Q_1^{\mp}Q_2^{\mp}\Lambda_2^{\mp}$$

for (5.9) and

$$(5.12)\quad Q_1^{\pm}Q_2^{\pm} = \frac{q^2}{t},\quad q^s = \frac{Q_1Q_1^{\mp}}{v},\quad t^{-r} = \frac{Q_1Q_2^{\mp}}{v},\quad u = \frac{tQ_1^{\mp}Q_2^{\mp}}{q},\quad \frac{y}{tx} = \frac{Q_1^{\mp}Q_2^{\mp}}{\Lambda_2^{\pm}}$$

for (5.10). In the $SU(N)$ case, the Nekrasov partition function (5.5) may coincide with our partition function Z_N by using the formulas (A.11) and the Cor. 1.6 in Ref. [27].

§ A. Macdonald polynomial

Here we recapitulate basic properties of the Macdonald polynomial.[24] Let $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_r)$ with $\lambda_i \geq \lambda_{i+1} \geq 0$ be a Young diagram. λ' is its conjugate. For any λ with $\lambda_1 \leq s$, $|\lambda| := \sum_i \lambda_i$. Let $x := (x_1, \cdots, x_r)$ and $p := (p_1, p_2, \cdots)$ with the power sum $p_n := p_n(x) := \sum_{i=1}^r x_i^n$. For any symmetric function f in x with $r = \infty$, $f(x[p])$ stands for the function f expressed in the power sums p.

The Macdonald polynomials $P_\lambda(x) := P_\lambda(x; q, t)$ are degree $|\lambda|$ homogeneous symmetric polynomials in x defined as eigenfunctions of the Macdonald operator H as follows:

$$HP_\lambda(x) = \varepsilon_\lambda P_\lambda(x), \tag{A.1}$$

$$H := \sum_{i=1}^r \prod_{j(\neq i)} \frac{tx_i - x_j}{x_i - x_j} \cdot q^{D_{x_i}}, \qquad \varepsilon_\lambda := \sum_{i=1}^r q^{\lambda_i} t^{r-i} \tag{A.2}$$

with a normalization condition $P_\lambda(x) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r} + \cdots$. Where q^{D_x} with $D_x := x\frac{\partial}{\partial x}$ is the q-shift operator such that $q^{D_x} f(x) = f(qx)$. Note that $P_\bullet(x) := P_{(0)}(x) = 1$.

Two kinds of inner products are known in which the Macdonald polynomials are orthogonal each other. For any symmetric functions f and g in x, let us define inner product $\langle *, * \rangle$ and another one $\langle *, * \rangle''_r$ as follows: [3]

$$\langle f, g \rangle := \oint \prod_{n>0} \frac{dp_n}{2\pi i p_n} \cdot f(x[p^*])\, g(x[p]), \qquad p_n^* := n \frac{1-q^n}{1-t^n} \frac{\partial}{\partial p_n}, \tag{A.3}$$

$$\langle f, g \rangle''_r := \frac{1}{r!} \oint \prod_{j=1}^r \frac{dx_j}{2\pi i x_j} \cdot \Delta(x)\, f(\overline{x})\, g(x), \qquad \overline{x_j} := \frac{1}{x_j} \tag{A.4}$$

with $\Delta(x)$ in (2.18). Here we must treat the power sums p_n as formally independent variables, *i.e.*, $\frac{\partial}{\partial p_n} p_m = \delta_{n,m}$ for all $n, m > 0$. The inner products of Macdonald polynomials are given by

$$\langle P_\lambda, P_\mu \rangle = \delta_{\lambda,\mu} \langle \lambda \rangle, \qquad \langle \lambda \rangle := \prod_{(i,j)\in\lambda} \frac{1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}}, \tag{A.5}$$

$$\langle P_\lambda, P_\mu \rangle''_r = \delta_{\lambda,\mu} \langle \lambda \rangle''_r. \tag{A.6}$$

Let us denote by $f\left(x\left[\frac{1-u}{1-t}\right]\right)$ the function $f(x[p])$ in the specialization $p_n := (1-u^n)/(1-t^n)$ with $u \in \mathbb{C}$, then [24]

$$P_\lambda\left(x\left[\frac{1-u}{1-t}\right]\right) = \prod_{(i,j)\in\lambda} \frac{t^{i-1} - uq^{j-1}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i + 1}}. \tag{A.7}$$

[3] The usual another inner product $\langle *, * \rangle'_r$ is defined with a different kernel $\Delta'(x) := \prod_{i\neq j}^r \exp\left\{-\sum_{n>0}(1-t^n)/(1-q^n)(x_j^n/x_i^n)/n\right\} = \prod_{i\neq j}^r \prod_{\ell\geq 0}(1-q^\ell x_j/x_i)/(1-tq^\ell x_j/x_i)$ ($|q| < 1$). Note that $C(x) := \Delta(x)/\Delta'(x)$ is a pseudo-constant, i.e., $q^{D_{x_i}} C(x) = C(x)$.

With the involution $\omega_{q,t}$,

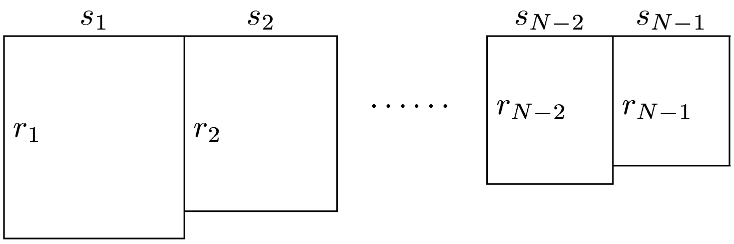
$$\frac{1}{\langle\,\lambda\,\rangle}\omega_{q,t}P_\lambda(x;q,t)=P_{\lambda'}(x;t,q),\qquad \omega_{q,t}(p_n):=(-1)^{n-1}\frac{1-q^n}{1-t^n}p_n. \tag{A.8}$$

Let us denote a function f in the set of variables $(x_1,x_2,\cdots,y_1,y_2,\cdots)$ by $f\,(x,y)$. Let $f^{\nu}_{\lambda,\mu}$ be the fusion coefficient $f^{\nu}_{\lambda,\mu}:=\langle P_\lambda P_\mu,P_\nu\rangle/\langle P_\nu,P_\nu\rangle$, then we have

$$P_\lambda(x)P_\mu(x)=\sum_\nu f^{\nu}_{\lambda,\mu}P_\nu(x), \tag{A.9}$$

$$\frac{P_\nu(x,y)}{\langle\,\nu\,\rangle}=\sum_{\substack{\lambda,\mu\\ \lambda,\mu\subset\nu}}\frac{P_\lambda(x)}{\langle\,\lambda\,\rangle}f^{\nu}_{\lambda,\mu}\frac{P_\mu(y)}{\langle\,\mu\,\rangle}. \tag{A.10}$$

Let us denote the Young diagram decomposing into rectangles as $\lambda=\sum_{i=1}^{N-1}(s_i^{r_i})$, $r_i\geq r_{i+1}$, i.e., $\lambda'=(r_1^{s_1}r_2^{s_2}\cdots r_{N-1}^{s_{N-1}})$,

$\lambda=$ 

.

Then we have the following integral representation of the Macdonald polynomial [22]

$$\begin{aligned}P_\lambda\,(x)&=C^+_\lambda\oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dz^a_j}{2\pi i z^a_j}(z^a_j)^{-s_a}\cdot\Pi\left(x,pz^1\right)\prod_{a=1}^{N-1}\Pi\left(\overline{z^a},pz^{a+1}\right)\Delta(z^a)\\&=C^+_\lambda\langle\alpha^+_{r,s}|\exp\left\{-\sum_{n>0}\frac{h^1_n}{1-q^n}\sum_{i=1}^{M}x_i^n\right\}|\chi^+_{r,s}\rangle,\quad C^+_\lambda:=\prod_{a=1}^{N-1}\frac{p^{-ar_as_a}\langle\,\lambda^{(a)}\,\rangle}{r_a!\langle\,\lambda^{(a)}\,\rangle''_{r_a}}\end{aligned} \tag{A.11}$$

with a singular vector $|\chi^+_{r,s}\rangle$ in (2.16). Here $z_i^N:=0$ and $\lambda^{(1)}:=\lambda$, $\lambda^{(a)}:=\sum_{i=a}^{N-1}(s_i^{r_i})$, i.e., $\lambda^{(a)\prime}=(r_a^{s_a}r_{a+1}^{s_{a+1}}\cdots r_{N-1}^{s_{N-1}})$. Acting $\omega_-\omega_+\omega_{q,t}$ on (A.11) gives

$$P_{\lambda'}\,(x)=C^-_\lambda\langle\alpha^-_{r,s}|\exp\left\{-\sum_{n>0}\frac{h^1_n}{1-q^n}\sum_{i=1}^{M}(-qx_i)^n\right\}|\chi^-_{r,s}\rangle,\qquad C^-_\lambda:=\omega_-\omega_+\frac{C^+_\lambda}{\langle\,\lambda\,\rangle}. \tag{A.12}$$

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