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Quantum dilogarithm identities

By

Kentaro NAGAO*

Abstract

In [Nakb], generalizing the dilogarithm identities in conformal field theories, Nakanishi provided dilogarithm identities for "periodic" quivers. In this note, we suggest a q-deformed version of Nakanishi's identity.

§ 1. Introduction

§ 1.1. Quantum dilogarithm

In this paper we study the following formal power series over $\mathbb{C}(q)$ which is called the *quantum dilogarithm* :

$$\mathbb{E}(y)=\sum_{n=0}^{\infty}\frac{y^n}{(q-q^{-1})(q^2-q^{-2})\cdots(q^n-q^{-n})}.$$

This is a q-deformation of (the exponential of) the classical dilogarithm function

$$\mathrm{Li}_2(y):=-\int_0^y\log(1-t)\,\mathrm{d}\log t=\sum_{n=0}^{\infty}\frac{1}{n^2}\,y^n.$$

The readers may refer [FG09, §1.3] for the history of the quantum dilogarithm.

One of the fundamental property of the quantum dilogarithm is the *quantum pentagon identity* [Sch53, FK94] : we assume $xy=q^2yx$, then we have

$$\mathbb{E}(x)\mathbb{E}(y)=\mathbb{E}(y)\mathbb{E}(q^{-1}xy)\mathbb{E}(x).$$

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*Nagoya University, Nagoya, 464-8602, Japan.
e-mail: `kentaro@math.nagoya-u.ac.jp`

It is natural for us to study quantum dilogarithms as elements in (a completion of) *quantum tori* Given a skew symmetric integer matrix $B = (b_{ij})_{1 \leq i,j \leq n}$ (or, equivalently, a quiver without loops or 2-cycles), the quantum torus QT_B associated to B is the $\mathbb{C}(q)$-algebra generated by $\{x_i^{\pm} \mid 1 \leq i \leq n\}$ with the relation $x_i x_j = q^{2b_{ij}} x_j x_i$. The pentagon identity is an equation in the quantum torus associated to the quiver of type A_1.

§ 1.2. Donaldson-Thomas theory and quantum dilogarithm

Recently, the quantum dilogarithm has appeared in moduli theory of quiver representations [Rei10, KSb, KSa]. It is essential that the denominator of a coefficient in the quantum dilogarithm is the Poincare polynomial of the general linear group (or the polynomial of which computes the order of the general linear group over a finite field with q elements).

Given a 3-dimensional Calabi-Yau category $\mathcal{D}$, *Donaldson-Thomas theory* for $\mathcal{D}$ is the moduli theory of objects in $\mathcal{D}$. Given a quiver with a potential, we can define a 3-dimensional Calabi-Yau category. The Donaldson-Thomas theory associated to such a category is called the non-commutative Donaldson-Thomas theory ([Sze08]).

According to Kontsevich-Soibelman's conjecture, there is an algebra homomorphism from the *motivic Hall algebra* to the quantum torus. A quantum dilogarithm ia the image of the moduli stack which parameterizes direct sums of a spherical object.

On the other hand, we have an interesting generalization of the (classical) pentagon relation which is called the *dilogarithm identity in conformal field theory.* Nakanishi and his collaborators found that *periodicity* of the cluster algebra ([IIKNS10, Kelc]) plays a crucial role for the dilogarithm identity [Naka, IIKKNa, IIKKNb]. In [Nakb], Nakanishi found that the dilogarithm identity is hold for any quiver with "periodicity".

In this note, we provide an analogue of Nakanishi's dilogarithm identity for quantum dilogarithms which generalizes the quantum pentagon identity. This is a consequence of Kontsevich-Soibelman's conjecture on the motivic Donaldson-Thomas theory ([KSb]) combined with the argument in [Nag].

During writing this note, the author was informed that B. Keller proved the identity [Kelb]. G. Kuroki independently found the same identity too [Kur]. After submitting this note, Nakanishi and Kashaev showed how Nakanishi's identity is induced from the quantum one by applying the saddle point method [NK].

§ 2. Statement

§ 2.1. Periodicities in cluster algebras

Let Q be a finite quiver without loops and 2-cycles with vertex set $I = \{1, \dots, n\}$. We put

$$Q(i,j) = \sharp\{\text{arrows from } i \text{ to } j\}, \quad \bar{Q}(i,j) = Q(i,j) - Q(j,i)$$

Since we assume Q has no 2-cycles, $\bar{Q}(i,j)$ determines the quiver Q. For a vertex $k \in I$, the *mutation* $\mu_k Q$ of Q at k is the quiver Q^{new}, where Q^{new} is a finite quiver without loops and 2-cycles and with the same vertex set as Q, which is obtained as follows:

1. reverse all arrows incident with k, and
2. for all vertices $i \neq j$ distinct from k, modify the number of arrows between i and j as follows:

$$\bar{Q}^{\mathrm{new}}(i,j) = \begin{cases} \bar{Q}(i,j) & (Q(i,k) \cdot Q(k,j) \leq 0), \\ \bar{Q}(i,j) + |Q(i,k)| \cdot Q(k,j) & (Q(i,k) \cdot Q(k,j) > 0). \end{cases}$$

For a sequence of vertices $\mathbf{k} = (k_1, \dots, k_l) \in I^l$, we put

$$Q_{\mathbf{k}} := \mu_{k_l}(\cdots(\mu_2(\mu_1 Q))\cdots).$$

Throughout this paper, we use similar notations.

Given a sequence of vertices $\mathbf{k} = (k_1, \dots, k_l) \in I^l$ we have a unique sequence $\varepsilon(1), \dots, \varepsilon(l)$ of signs which satisfied the following ([Nag, Theorem 3.4]).

Let $\mathbb{Z}^I$ be a lattice with basis $(e_i)_{i \in I}$. For a vertex $k \in I$ we define isomorphisms $\phi_{k,\pm} \colon \mathbb{Z}^I \xrightarrow{\sim} \mathbb{Z}^I$ by

$$\phi_{k,+}(e_i) = \begin{cases} e_i + Q(k,i)e_k & i \neq k, \\ -e_k & i = k, \end{cases}$$
$$\phi_{k,-}(e_i) = \begin{cases} e_i + Q(i,k)e_k & i \neq k, \\ -e_k & i = k. \end{cases}$$

We put

$$\phi_{\mathbf{k}} := \phi_{k_l,\varepsilon(l)} \circ \cdots \circ \phi_{k_1,\varepsilon(1)} \colon \mathbb{Z}^I \xrightarrow{\sim} \mathbb{Z}^I.$$

and

$$e(L) := (\phi_{k_{L-1},\varepsilon(L-1)} \circ \cdots \circ \phi_{k_1,\varepsilon(1)})^{-1}(e_{k_L})$$

for $1 \leq L \leq l$. Then we have

$$e(L) \in \begin{cases} (\mathbb{Z}_{\geq 0})^I & (\varepsilon(L) = +), \\ (\mathbb{Z}_{\leq 0})^I & (\varepsilon(L) = -). \end{cases}$$

Definition 2.1. Let $\nu \in \mathfrak{S}_I$ be a permutation. A sequence of vertices $\mathbf{k} = (k_1, \dots, k_l) \in I^l$ is called a ν-period of Q if $\phi_{\mathbf{k}}(e_i) = e_{\nu_{(i)}}$.

Remark. The vector $\phi_{\mathbf{k}}(e_i)$ is nothing but ${}^{\mathrm{t}}g$-vector in the sense of [FZ07]. The condition is equivalent to the criterion of the periodicity using c-vectors, which is given in [Nakb, Theorem 2.6]. (See [NZ].)

§ 2.2. Quantum dilogarithm identity

Let χ_Q be the skew symmetric bilinear form on $\mathbb{Z}^I$ given by $\chi_Q(e_i, e_j) = \bar{Q}_{ij}$[1]. We define the quantum torus QT_Q associated to Q by

$$\mathrm{QT}_Q := \bigoplus_{\mathbf{v}\in\mathbb{Z}^I} \mathbb{C} \cdot y_{\mathbf{v}}$$

where the product is given by

$$y_{\mathbf{v}} y_{\mathbf{v}'} = q^{\chi(\mathbf{v},\mathbf{v}')} y_{\mathbf{v}+\mathbf{v}'}.$$

We also define the completion $\widehat{\mathrm{QT}}_Q$ of QT_Q by

$$\widehat{\mathrm{QT}}_Q := \prod_{\mathbf{v}\in(\mathbb{Z}_{\geq 0})^I} \mathbb{C} \cdot y_{\mathbf{v}} \oplus \bigoplus_{\mathbf{v}\notin(\mathbb{Z}_{\geq 0})^I} \mathbb{C} \cdot y_{\mathbf{v}}.$$

Conjecture. Let $\mathbf{k} = (k_1, \dots, k_l) \in I^l$ be a ν-period of Q. In $\widehat{\mathrm{QT}}_Q$ we have the following equation:

$$\mathbb{E}(y_{\varepsilon(l)e(l)})^{\varepsilon(l)} \cdot \mathbb{E}(y_{\varepsilon(l-1)e(l-1)})^{\varepsilon(l-1)} \cdot \dots \cdot \mathbb{E}(y_{\varepsilon(1)e(1)})^{\varepsilon(1)} = 1.$$

§ 3. Sketch of the "proof"

§ 3.1. Derived equivalences

A potential w of a quiver Q is a linear combination[2] of cyclic paths in the quiver Q. Given a pair (Q, w) of a quiver and its potential, we have the triangulated category $\mathcal{D}_{Q,w}$[3] and the core of its t-structure $\mathcal{A}_{Q,w}$ [4].

For a vertex k, the mutation $\mu_k w$ of the potential w is defined and we have the following derived equivalences ([KY, Kela]):

$$\Phi_{k,+}, \Phi_{k,-} : \mathcal{D}_{Q,w} \xrightarrow{\sim} \mathcal{D}_{\mu_k Q, \mu_k w}.$$

[1]This is nothing but the Euler pairing of the Grothendieck group of $\mathcal{D}_{Q,w}$ (see the next section).
[2]To be precise, we have to consider an infinite linear combination in general.
[3]the derived category of dg modules with finite dimensional cohomologies over Ginzburg's dg algebra
[4]the category of finite dimensional modules over Jacobi algebra

The isomorphisms induced by these equivalences on the Grothendieck groups are nothing but $\phi_{k,+}$ and $\phi_{k,-}$. Given a sequence $\mathbf{k} = (k_1, \ldots, k_l)$ of vertices, we define

$$\Phi_{\mathbf{k}} := \Phi_{k_l,\varepsilon(l)} \circ \cdots \circ \Phi_{k_1,\varepsilon(1)} \colon \mathcal{D}_{Q,w} \xrightarrow{\sim} \mathcal{D}_{\mu_{\mathbf{k}}Q,\mu_{\mathbf{k}}w}.$$

and put

$$\mathcal{T}_{\mathbf{k}} := \mathcal{A}_{Q,w} \cap \Phi_{\mathbf{k}}^{-1}(\mathcal{A}_{\mu_k Q,\mu_k w}[1]).$$

Theorem 3.1 (a special case of [Nag, Theorem 4.2]). *Assume that* $\mathbf{k}$ *is a* ν*-period of* Q*. Then* $\mathcal{A}_{Q,w} = \Phi_{\mathbf{k}}^{-1}(\mathcal{A}_{\mu_k Q,\mu_k w})$*. In particular,* $\mathcal{T}_{\mathbf{k}} = \emptyset$*.*

§ 3.2. Motivic invariants

Each $\varepsilon(i)e(i)$ corresponds to a spherical object $s(i)$ in $\mathcal{A}_{Q,w}$. We define the moduli stack

$$\mathrm{MD}_i := \{s(i)^{\oplus n} \mid n \in \mathbb{Z}_{\geq 0}\},$$

which we regard as an element in the *motivic Hall algebra.* We call it *motivic dilogarithm* in this note. By the definition of the product in the motivic Hall algebra,

$$\mathrm{MD}_l^{\varepsilon(l)} * \cdots * \mathrm{MD}_l^{\varepsilon(l)}$$

coincides with the moduli stack of objects in $\mathcal{T}_{\mathbf{k}}$.

Kontsevich and Soibelman proposed *motivic invariants* of the moduli spaces which would induce a homomorphism from the motivic Hall algebra to the quantum torus such that the image of a motivic dilogarithm is a quantum dilogarithm.

Sending the equation above by this homomorphism, we get a description of the product of the quantum dilogarithms in terms of motivic invariants of the moduli stack of objects in $\mathcal{T}_{\mathbf{k}}$.

In a periodic case, the product is trivial since $\mathcal{T}_{\mathbf{k}}$ is trivial.

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