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Gauge/string duality and thermodynamic Bethe ansatz equations

By

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Abstract

We review recent developments in the study of gluon scattering amplitudes of the four-dimensional maximally supersymmetric Yang-Mills theory at strong coupling based on the gauge/string duality and its underlying integrability. The scattering amplitudes are given by the area of minimal surfaces in five-dimensional anti-de Sitter space with a null polygonal boundary. These minimal surfaces are described by integral equations of the form of the thermodynamic Bethe ansatz equations. Generalizing the result regarding the six-point amplitudes, we observe a general connection between the minimal surfaces and the homogenous sine-Gordon model, which is a class of two-dimensional integrable models associated with certain coset conformal field theories. We also demonstrate that the identification of the underlying integrable models is useful for analyzing the amplitudes by explicitly deriving an expansion of the six-point amplitudes around a special kinematic point.

§ 1. Introduction

§ 1.1. Gauge/string duality and AdS/CFT correspondence

The gauge/string duality emerged as a consequence of a natural development of the study of string solitons such as black holes (p-branes) and D-branes, and has been a central subject in string theory since mid-nineties. The studies of the matrix models for non-perturbative strings and the quantum theory of black holes in string theory are notable examples based on this duality. A basic picture of the duality is that at weak coupling the string solitons are described by open strings/gauge theories in flat space, whereas at strong coupling they are described by closed strings/gravity.

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In this talk, we focus on a particular form of the duality called the AdS/CFT correspondence. This is the duality between the string theory on five-dimensional anti-de Sitter space (AdS_5) times five-dimensional sphere (S^5) and the four-dimensional $SU(N_c)$ super Yang-Mills (SYM) theory which has the maximal $\mathcal{N}=4$ supersymmetry. The $\mathcal{N}=4$ SYM theory is known to be a conformal field theory (CFT), leading to the name of the correspondence. More precisely, the duality states that the two theories are two facets of one entity: for $N_c \gg 1$, when the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ is kept small, the theory is well described by $\mathcal{N}=4$ SYM, whereas the description by the classical strings/gravity on $AdS_5 \times S^5$ is appropriate for $\lambda \gg 1$. On the string side, $\lambda = 4\pi g_s N_c = R^4/\alpha'^2$, where g_s is the string coupling, R is the radius of $AdS_5 \times S^5$ and α' is the inverse string tension. The duality has been studied intensively for large N_c, but is expected to hold also for finite N_c. Schematically,

$$\boxed{\begin{array}{c}\text{String theory on } AdS_5\times S^5 \\ \lambda = R^4/\alpha'^2 \gg 1\end{array}} \quad \underset{\text{strong/weak}}{\overset{\text{dual}}{\Longleftrightarrow}} \quad \boxed{\begin{array}{c}\text{4 dim. } \mathcal{N}=4\ SU(N_c)\ \text{SYM} \\ \lambda = g_{YM}^2 N_c \ll 1\end{array}}$$

This AdS/CFT correspondence has attracted much attention. First, the correspondence embodies interesting long-standing theoretical ideas: the equivalence between large N_c gauge theory and string theory, and the holography which states that quantum gravity is described by lower dimensional non-gravitational theory. Second, because of the strong/weak nature, one can study the gauge theory at strong coupling by classical strings/gravity. In fact, there are many works on the applications of the correspondence, for example, to low energy hadron physics (holographic QCD and AdS/QCD), quark gluon plasma and quantum entanglement. In particular, the application to gluon scattering amplitudes of $\mathcal{N}=4$ SYM is the subject of this talk.

§ 1.2. Integrability underlying AdS/CFT correspondence

Among the works on the AdS/CFT correspondence, the discovery of the underlying integrability in the planar limit ($N_c \gg 1$) opened up new dimensions. Here, the integrability means on the string side that the string sigma model on $AdS_5 \times S^5$ classically admits a flat current with a spectral parameter which generates infinitely many conserved charges. On the gauge side, it means that the dilatation operators representing the anomalous dimension for lower loops are, in the planar limit, identified with Hamiltonians of integrable quantum spin chains. This discovery of the integrability enabled one to compare in detail the gauge and the string side beyond (nearly) supersymmetric sectors which are protected from quantum corrections at strong coupling. Furthermore, assuming that this integrability holds for arbitrary coupling, one can expect to

- solve the four-dimensional SYM theory exactly including the spectrum,
- solve the important string theory on $AdS_5 \times S^5$, in spite that solving string theory on curved space-*time* is generally very difficult,
- prove (or disprove) the AdS/CFT correspondence,
- deeply understand the AdS/CFT correspondence, and gain useful insights into and, if necessary, firm theoretical grounds for applications.

As a state of the art of the study of the AdS/CFT correspondence based on the integrability, there is now a proposal: *the spectrum of the string theory on* $AdS_5 \times S^5$ *and the four-dimensional* $\mathcal{N} = 4$ $SU(N_c)$ *SYM theory for large* N_c *and arbitrary coupling* λ *is obtained by solving a certain set of equations.* (For details, see the article by Prof. Tateo [1].) This set of equations takes the form of the thermodynamic Bethe ansatz (TBA) equations or the Y-system, which appear in the study of finite-size effects of (1+1)-dimensional integrable models. This proposal has been checked up to 4-loop order for a simple single-trace operator called the Konishi operator. The spectrum of this operator at 5-loop order has also been computed by using the Lüscher formula.

Given this impressive progress in understanding the AdS/CFT correspondence, one may also expect that the integrability must shed new light on applications of the correspondence. It turned out that this is indeed the case: Based on their earlier work [2] that gluon scattering amplitudes of $\mathcal{N} = 4$ SYM at strong coupling for large N_c are given by minimal surfaces in AdS_5, Alday and Maldacena initiated a program to compute the amplitudes by using the integrability [3]. In this program, the minimal surfaces in AdS_5 are described by a set of integral/functional equations. Surprisingly, these again take the form of the TBA equations/Y-system [4, 5, 6]. However, the TBA equations/Y-system here are different from those for the spectral problem mentioned above. Thus, the gluon scattering amplitudes at strong coupling/minimal surfaces in AdS_5 provide another example in which one finds unexpected connections between the AdS/CFT correspondence and the TBA equations/Y-systems. Schematically,

$$\boxed{\text{Gluon Scattering Amplitudes at Strong Coupling}}$$
$$\Uparrow \quad \text{ref. [2]}$$
$$\boxed{\text{Minimal Surfaces in } AdS_5}$$
$$\Uparrow \quad \text{refs. [3, 4, 5, 6]}$$
$$\boxed{\text{Thermodynamic Bethe Ansatz Equations}}$$

§ 1.3. Plan of talk

In this talk, we next give a brief summary on the scattering amplitudes of $\mathcal{N}=4$ SYM both at weak and strong coupling in section 2. We then review developments in the study of the scattering amplitudes based on the AdS/CFT correspondence and its underlying integrability in section 3. (See [3, 4, 5, 6, 7, 8] and references therein.) We move on to a discussion on the integrable models and the CFTs associated with the TBA equations/Y-systems for the minimal surfaces in section 4. In particular, we observe [6] that the TBA equations for the minimal surfaces in AdS_3 and AdS_4, corresponding to some kinematic configurations, coincide with those of the homogeneous sine-Gordon (HSG) model [9], which is a class of (1+1)-dimensional integrable models associated with certain coset or generalized parafermion CFTs. This generalizes the connection [4] between the minimal surfaces in AdS_5 for the six-point amplitudes and the $\mathbb{Z}_4$-symmetric integrable model. Finally, we derive an expansion of the six-point amplitudes near the CFT limit corresponding to a special kinematic point [7] in section 5. This demonstrates that the identification of the associated integrable models and CFTs is actually useful for analyzing the amplitudes at strong coupling. We conclude with a summary and discussion on future directions in section 6.

§ 2. Gluon scattering amplitudes of $\mathcal{N}=4$ SYM

§ 2.1. Amplitudes at weak coupling and BDS conjecture

Let us begin with a brief summary of the gluon scattering amplitudes of 4-dimensional $\mathcal{N}=4$ SYM theory at weak coupling $\lambda = g_{YM}^2 N_c \ll 1$. For a review regarding section 2, see for example [10]. This theory contains a gauge field A_μ $(\mu = 0, ..., 3)$, six scalars Φ^i $(i = 1, ..., 6)$ and four fermions ψ^a $(a = 1, ..., 4)$. All the fields take values in the adjoint representation of $SU(N_c)$. This theory is obtained by dimensional reduction from 10-dimensional $\mathcal{N}=1$ SYM theory. The theory also has the superconformal symmetry $psu(2,2\,|\,4)$. The bosonic part $su(2,2) \oplus su(4) \simeq so(2,4) \oplus so(6)$ represents the 4-dimensional conformal symmetry and the R-symmetry. Note that $SO(2,4)$ and $SO(6)$ are the isometries of AdS_5 and S^5, respectively.

In the planar limit $N_c \to \infty$ with the 't Hooft coupling λ kept small, an interesting conjecture is known that the maximally helicity violating (MHV) amplitude has a simple iterative structure to all orders in perturbation. This is called the BDS (Bern-Dixon-Smirnov) conjecture. To state the content of the conjecture, we first note that the n-point amplitudes at L-loop order are decomposed as follows:

$$A_n^{(L)} = N^{(L)} \sum (\text{color factor}) \times \mathcal{A}_n^{(L)} + (\text{multi-trace part})\,.$$

The remainder $\mathcal{A}_n^{(L)}$ after the color factor is factorized is called the color-ordered amplitudes. In the planar limit, the multi-trace part is neglected. From the color-ordered amplitudes, the tree amplitudes are further factorized,

$$\mathcal{A}_n^{(L)} = \mathcal{A}_n^{\text{tree}} \times \mathcal{M}_n^{(L)} .$$

The BDS conjecture states that the scalar part at L-loop order $\mathcal{M}_n^{(L)}$ is given by an iteration of the 1-loop result through the generating function,

$$\mathcal{M}_n = \exp\left[\sum_{k=1}^{\infty} a^k f^{(k)}(\epsilon)\mathcal{M}_n^{(1)}(k\epsilon) + C^{(k)} + \mathcal{O}(\epsilon)\right] ,$$

where $a = \lambda(4\pi e^{-\gamma_E})^\epsilon/8\pi^2$ with γ_E being Euler's constant is the coupling constant customarily used in loop calculations, and $f^{(k)}(\epsilon)$ and $C^{(k)}$ are certain constants independent of external momenta. Note that $\mathcal{N} = 4$ SYM is a massless gauge theory and thus one has to regularize the infrared divergences of the amplitudes by an infrared cut-off and dimensional regularization with $d = 4 - 2\epsilon$. The divergences are canceled in infrared safe quantities, which are obtained by combining the scattering amplitudes. This conjecture has been checked up to higher loops for 4- and 5-point amplitudes.

Probably, it is illuminating to see a concrete example of the 4-point amplitudes,

$$\mathcal{M}_4 = \mathcal{M}_4^{\text{div}} \times \exp\left[\frac{1}{8} f(\lambda)\big(\ln \frac{s}{t}\big)^2 + \text{const.}\right], \tag{2.1}$$

where $\mathcal{M}_4^{\text{div}}$ is the divergent part and s, t are the Mandelstam variables. A remarkable fact is that all the coupling dependence is encoded in the cusp anomalous dimension,

$$f(\lambda) = \frac{\lambda}{2\pi^2}(1 - \frac{\lambda}{48} + \cdots) .$$

§ 2.2. Amplitudes at strong coupling from AdS/CFT correspondence

Now, let us move on to a discussion in the strong coupling region with $\lambda \gg 1$. Based on the AdS/CFT correspondence, Alday and Maldacena argued that the scalar part of the amplitudes at strong coupling is obtained by evaluating the action of the string sigma model on AdS_5 for certain classical string solutions [2]. The saddle-point action gives the area of minimal surfaces, meaning that the amplitudes are given by the area of the minimal surfaces in AdS_5:

$$\mathcal{M}_n \sim e^{-S} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})} .$$

The momentum dependence of the amplitudes come from the boundary condition of the minimal surfaces. This is analyzed by making use of T-dual transformations, and

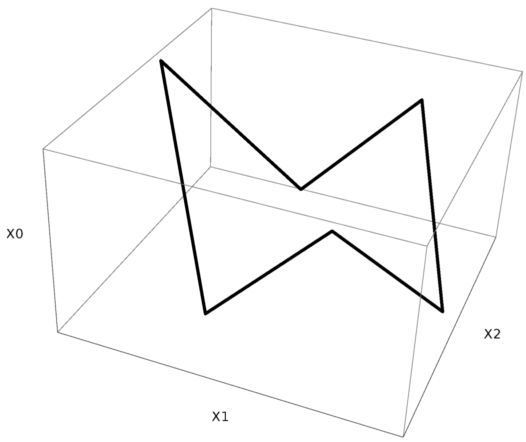


Figure 1. Boundary of a minimal surface on the AdS boundary. The axes in the figure represent the Poincaré coordinates (x^1, x^2, x^0).

it turns out that the surfaces have to end on a polygonal boundary on the boundary of AdS_5. See fig. 1. There, each side of the polygon is null and corresponds to the momentum of an external particle. The momentum conservation $\sum_i p_i^\mu = 0$ implies that the boundary is closed. Thus, denoting the vertices of the polygon by x_i^μ (in terms of the Poincaré coordinates defined below), one has

$$\Delta_i x^\mu := x_i^\mu - x_{i+1}^\mu = p_i^\mu \,. \tag{2.2}$$

The S^5 part is expected to contribute to subleading terms of the amplitudes, but its role is not so clear.

Again, it would be illuminating to see a concrete example of the 4-cusp minimal surfaces describing the 4-point amplitudes. For this purpose, we first parametrize AdS_5 as a hypersurface in $\mathbb{R}^{2,4}$ defined by

$$\vec{Y} \cdot \vec{Y} := -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1 \,. \tag{2.3}$$

The equations of motion for the string coordinates $\vec{Y}(z, \bar{z})$ are

$$\partial\bar{\partial}\vec{Y} - (\partial\vec{Y} \cdot \bar{\partial}\vec{Y})\vec{Y} = 0 \,, \tag{2.4}$$

whereas the Virasoro constraints are

$$(\partial\vec{Y})^2 = (\bar{\partial}\vec{Y})^2 = 0 \,. \tag{2.5}$$

Here, $z, \bar{z}$ are the world-sheet coordinates and $\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$. One can check that these are the equations of the minimal surfaces.

A simple solution to (2.4) and (2.5) is

$$\begin{pmatrix} Y^{-1} + Y^4 & Y^1 + Y^0 \\ Y^1 - Y^0 & Y^{-1} - Y^4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\tau+\sigma} & e^{\tau-\sigma} \\ -e^{-\tau+\sigma} & e^{-\tau-\sigma} \end{pmatrix} , \tag{2.6}$$

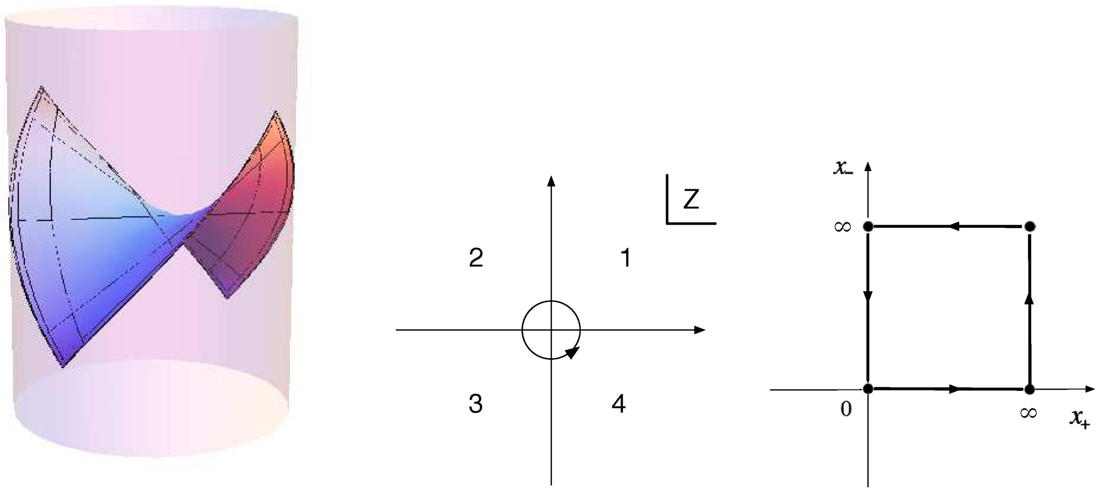


Figure 2. 4-cusp minimal surface in AdS_3 (left), world-sheet z-plane (middle), and surface boundary in the (x^+, x^-)-plane (right).

with $Y^2 = Y^3 = 0$ and $z = \tau + i\sigma$. To see what surface is described by this solution, let us introduce the Poincaré coordinates defined by

$$Y^\mu =: \frac{x^\mu}{r}\,, \quad Y^{-1} + Y^4 =: \frac{1}{r}\,, \quad Y^{-1} - Y^4 = \frac{r^2 + x^\mu x_\mu}{r}\,,$$

where $\mu = 0, 1, 2, 3$. In this coordinate system, the boundary of AdS_5 is located at $r = 0$. Since the surface is embedded in the AdS_3 subspace parametrized by r and $x^\pm := x^0 \pm x^1$, the external momenta given by (2.2) are in $\mathbb{R}^{1,1}$ and correspond to a restricted kinematic configuration. Substituting the solution into these coordinates, one can draw the picture of the surface as in the left figure in fig. 2. In the figure, AdS_3 is represented as a solid cylinder, where the radial direction is parametrized by r and the boundary of the cylinder by x^0 and x^1. The AdS boundary at infinity has been mapped to the boundary of the solid cylinder. To have a closer look, let us go around the world-sheet z-plane far from the origin (middle figure in fig. 2). We then find that the region far from the origin in the first quadrant is mapped to a neighborhood of the origin in the (x^+, x^-)-plane (right figure in fig. 2). Similarly, the regions far from the origin in the second, third, and fourth quadrants are mapped to neighborhoods of $(x^+, x^-) = (\infty, 0), (\infty, \infty), (0, \infty)$, respectively. Thus, as we cross the real or the imaginary axis of the z-plane, the boundary of the surface jumps from one cusp to another and draws null lines. In this way, the solution describes a minimal surface with a polygonal boundary consisting of four cusps and four null sides. The solution (2.6) is thus the solution which we are looking for.

Minimal surfaces corresponding to more general kinematic configurations are obtained by $SO(2,4)$ transformations. According to the prescription by Alday and Maldacena, the area of those minimal surfaces then gives the 4-point amplitudes at strong coupling. Since the surfaces extend to infinity, their area diverges and hence has to be regularized. Either by a dimensional regularization with $d = 4 - 2\epsilon$ or by a cut-off regularization with $r > \epsilon'$, one finds that

$$\mathcal{M}_4 \ \sim \mathcal{M}_4^{\rm div} \times \exp\Big[\frac{1}{8} f(\lambda) \big(\ln \frac{s}{t} \big)^2 + \text{const.} \Big] \, ,$$

where $f(\lambda) = \sqrt{\lambda}/\pi$ and $\mathcal{M}_4^{\rm div}$ is the divergent term. Remarkably, this has the same structure as the BDS formula (2.1) including the divergent term. Furthermore, the value of $f(\lambda)$ here precisely agrees with the cusp anomalous dimension at strong coupling which has been computed in the spectral problem of the AdS/CFT correspondence.

§ 2.3. Insights from strong coupling computation

This agreement of the 4-point amplitudes is very impressive. Moreover, the study of the amplitudes at strong coupling provided very useful insights into the weak coupling side and led to deeper understanding. First, it is known that minimal surfaces in AdS give expectation values at strong coupling of Wilson loops along the boundary of the surfaces. Thus, the above discussion implies that, at strong coupling, the amplitudes are the same as the expectation values of the null polygonal Wilson loops. A natural question here is whether this is also the case on the weak coupling side. It then turned out that the answer is yes, as far as comparison is possible. This correspondence between the amplitudes and the null polygonal Wilson loops are now called the Amplitude/Wilson loop duality.

Second, a detailed analysis on the strong coupling side for n-point amplitudes with $n \to \infty$ revealed that the BDS formula needs to be modified at strong coupling. Subsequent studies confirmed that this is also the case on the weak coupling side for $n \geq 6$. Now, the deviation from the BDS formula is called the remainder function. Given the BDS formula, computing the amplitudes is equivalent to computing the remainder function. The remainder function is thus a central quantity in this subject.

Third, the computation on the strong coupling side manifests the conformal symmetry in a sort of momentum space of the SYM theory, which is (a part of) the T-dualized target space represented by x^μ. This facilitated again the studies on the weak coupling side. Together with earlier observations, the results support the existence of this symmetry also at weak coupling, which is now called the dual conformal symmetry.

The dual conformal symmetry is natural on the strong coupling/string side, since it corresponds to a T-dual symmetry or the Yangian symmetry of the string sigma model. Moreover, once the existence of this symmetry is assumed, that leads to important consequences: The Ward identity associated with this symmetry strongly constrains the form of the amplitudes and, for the n-point amplitudes with $n \leq 5$, the BDS formula turns out to be unique. For $n \geq 6$, the Ward identity allows, in addition to the BDS form, functions of the cross-ratios of the cusp coordinates x_i^μ, which are dual-conformal invariants and related to external momenta by (2.2). The remainder function is thus a function of the cross-ratios.

§ 3. Minimal surfaces in AdS and integrability

In the following, we focus on the strong coupling/string side. Triggered by the computation of the 4-point amplitudes in [3], there were many attempts at constructing the minimal surfaces with more than 4 cusps. For example, cusp solutions are numerically studied in [11], and a special 6-cusp solution is constructed by systematically analyzing finite-gap solutions and their degenerate limits in [12]. However, it turned out that it is very difficult to construct the minimal surfaces with the special null polygonal boundary.

Then, Alday and Maldacena initiated a program of general construction based on integrability [3]. They reduced the analysis of the minimal surfaces to that of the Hitchin system and used related results in the study of the wall-crossing phenomena of $\mathcal{N} = 2$ SYM. Roughly speaking, they showed how to patch the 4-cusp solution (2.6) to form the general n-cusp solution. What is interesting is that the explicit form of the solution still is not available, but it is possible to compute the amplitudes. In this way, they analyzed the 8-point amplitudes corresponding to the 8-cusp solution in AdS_3.

Subsequently, the 6-cusp solution in AdS_5 was discussed in [4] and, together with an argument on general cusp solutions, the 10- and 12-cusp solution in AdS_3 were discussed in [6]. The general construction of the n-cusp solution in AdS_5 was then given in [5]. Below, we would like to explain this general construction. For simplicity, we focus on the case of AdS_3.

§ 3.1. General null polygonal solutions in AdS_3

The first step in this construction is to reduce the analysis of the classical solution of the AdS sigma model to that of the Hitchin system. This step is called the Pohlmeyer reduction. Mathematically, this is equivalent to considering the evolution of a moving frame. Concretely, one first takes a basis in $\mathbb{R}^{2,2} \supset AdS_3$, $q = (\vec{Y}, \partial\vec{Y}, \bar{\partial}\vec{Y}, \vec{N})^t$, where $N_a := \frac{1}{2} e^\alpha \epsilon_{abcd} Y^b \partial Y^c \bar{\partial} Y^d$ and $e^{2\alpha} := \frac{1}{2} \partial\vec{Y} \cdot \bar{\partial}\vec{Y}$. Y_a $(a = -1, 0, 1, 2)$ are the embedding coordinates which parametrize AdS_3 similarly to (2.3). Since q spans a frame at each point of $\mathbb{R}^{2,2}$, derivatives of q are again expressed by linear combinations of the elements of q itself. It is then possible to write the original equations of motion (2.4) and the Virasoro constraints (2.5) in the form of an evolution equation $(d + U)q = q$, where d stands for the world-sheet derivative and U is a certain matrix.

Furthermore, decomposing $SO(2,2)$ vectors by products of $su(2)$ spinors through $so(4) \cong su(2) \oplus su(2)$ and introducing a complex parameter ζ (spectral parameter), the evolution equation is rewritten as

$$0 = \Big[d + B(\zeta)\Big]\psi \,, \tag{3.1}$$

where ψ is a spinor related to q,

$$B_z(\zeta) = \begin{pmatrix} \frac{1}{2}\partial\alpha & -\frac{1}{\zeta}e^{\alpha} \\ -\frac{1}{\zeta}e^{-\alpha}p & -\frac{1}{2}\partial\alpha \end{pmatrix}, \quad B_{\bar z}(\zeta) = \begin{pmatrix} -\frac{1}{2}\bar\partial\alpha & -\zeta e^{-\alpha}\bar p \\ -\zeta e^{\alpha} & \frac{1}{2}\bar\partial\alpha \end{pmatrix},$$

and $p := -2\partial^2\vec{Y}\cdot\vec{N}$. It turns out that p is holomorphic in z. We further decompose the connection $B(\zeta)$ according to the grading with respect to ζ,

$$B_z(\zeta) =: A_z + \frac{1}{\zeta}\Phi_z \,, \quad B_{\bar z}(\zeta) =: A_{\bar z} + \zeta\Phi_{\bar z} \,.$$

The evolution equation of q or (3.1) implies that the original non-linear equations of the string sigma model have been linearized.

The compatibility condition of (3.1), $0 = [\partial + B_z, \bar\partial + B_{\bar z}]$, is expressed as

$$D_{\bar z}\Phi_z = D_z\Phi_{\bar z} = 0 \,, \quad F_{z\bar z} + [\Phi_z, \Phi_{\bar z}] = 0 \,, \tag{3.2}$$

with $D\Phi = d\Phi + [A, \Phi]$. This is nothing but the $su(2)$ Hitchin system, which is obtained by dimensional reduction of the 4-dimensional self-dual (instanton) equations. In the AdS_5 case, one similarly finds the $su(4)$ Hitchin system. Tracing back the argument, a solution to the Hitchin system gives a proper solution to (3.1), and it then gives q and a solution to the original string equations $\vec{Y}$. The formula to reconstruct $\vec{Y}$ is

$$Y_{a\dot a} := \begin{pmatrix} Y^{-1} + Y^2 & Y^1 + Y^0 \\ Y^1 - Y^0 & Y^{-1} - Y^2 \end{pmatrix} = \Psi(\zeta = 1)M\Psi(\zeta = i) \,,$$

where $\Psi(\zeta) = (\psi_1, \psi_2)$ with $\psi_{1,2}(\zeta)$ being properly normalized independent solutions of (3.1), and M is a certain matrix.

Now, we are ready to discuss general cusp solutions. We recall that the number of the cusps is even in AdS_3. For our purpose, we first make changes of variables for the world-sheet coordinates by $dw = \sqrt{p(z)}dz$ and for the potential α by $\hat\alpha = \alpha - \frac{1}{4}\ln p\bar p$. In term of $\hat\alpha$, the compatibility condition of (3.1), or (3.2), reduces to the sinh-Gordon equation $\partial_w\partial_{\bar w}\hat\alpha - 2\sinh\hat\alpha = 0$. Here, we note that the linear problem with $\hat\alpha = 0$ gives the 4-cusp solution (2.6) (with $Y^4 \to Y^2$) in the w-plane. Thus, if we take $p(z)$ to be a polynomial of degree $n-2$, i.e., $p(z) = z^{n-2} + \cdots$, and find a solution where $\hat\alpha \to 0$ as $|w| \to \infty$, that is the $2n$-cusp solution in the original z-plane. The reason is as follows: First, since $w \sim z^{n/2}$ for large $|z|$, if we go around the z-plane once far from

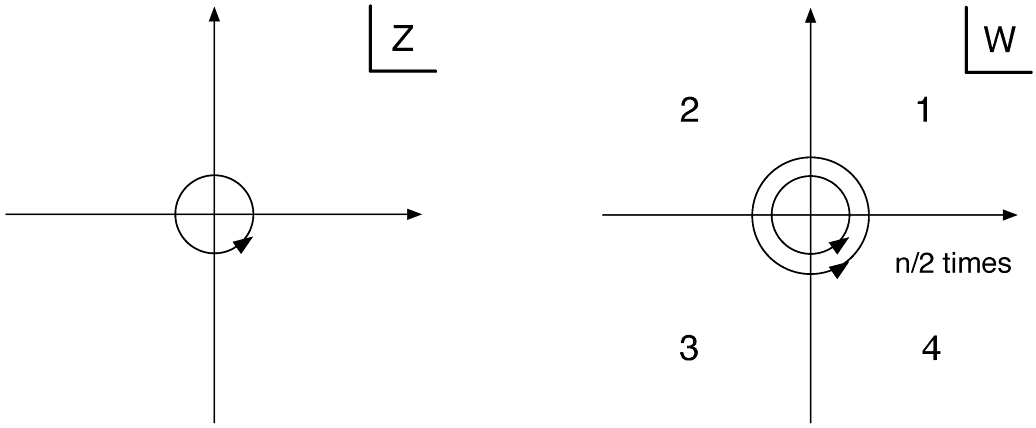


Figure 3. Rotation far from the origin in the z- and w-planes.

the origin, we go around the w-plane $n/2$ times (fig. 3). The solution with $\hat{\alpha} \to 0$ as $|w| \to \infty$ produces one cusp in each quadrant, as explained below (2.6), and thus, from the point of view of the z-plane, the solution has $4 \times n/2 = 2n$ cusps. In a canonical form, the polynomial $p(z)$ has $2(n-3)$ real parameters, which agrees with the number of independent cross-ratios in the $2n$-point scattering for the kinematic configurations corresponding to AdS_3.

§ 3.2. Cross-ratios and area

The above argument does not say anything about the explicit form of the solution, and it is in fact impossible to obtain it. Remarkably, it is however possible to extract physical information without the explicit form of the solution. Let us see how this is possible.

First, we consider the cross-ratios of the cusp coordinates x_i^μ related to external momenta. As we go around the z-plane, we pass through regions in the w-plane with $\mathrm{Re}(w/\zeta + \bar{w}\zeta) > 0$ and $\mathrm{Re}(w/\zeta + \bar{w}\zeta) < 0$ alternatively (Stokes sectors). In each region, the linear problem has a diverging and a decaying solution as $|w| \to \infty$. Let us call them the big and the small solution and denote them by b_i and s_i, respectively. The subscript i labels the region. Explicitly, one has $b_i, s_i \sim (e^{w/\zeta + \bar{w}\zeta}, 0)^t, (0, e^{-(w/\zeta+\bar{w}\zeta)})^t$ for large $|w|$, and the solution of the linear problem is given by

$$\psi(\zeta; z) \sim b_i(\zeta; z) + s_i(\zeta; z)\,.$$

It turns out that the cross-ratios are expressed by these small solutions as

$$\frac{x_{ij}^{\pm} x_{kl}^{\pm}}{x_{ik}^{\pm} x_{jl}^{\pm}} = \frac{(s_i \wedge s_j)(s_k \wedge s_l)}{(s_i \wedge s_k)(s_j \wedge s_l)}(\zeta) =: \chi_{ijkl}(\zeta)\,, \tag{3.3}$$

where $x_{ij}^{\pm} := x_i^{\pm} - x_j^{\pm}$, $s_i \wedge s_j := \det(s_i, s_j)$, and $\zeta = 1$ for $+$ and $\zeta = i$ for $-$. Note that $s_i \wedge s_j$ are independent of z. This formula relates the geometrical data of the minimal surfaces carried by ψ to the physical cross-ratios.

The problem is now how to compute the right-hand side of (3.3). A streamlined solution to this problem is given in [5]. There, one first defines the T- and Y-functions by

$$T_{2k+1} = (s_{-k-1} \wedge s_{k+1})\,, \quad T_{2k} = (s_{-k-1} \wedge s_k)^+\,,$$
$$Y_s = T_{s-1} T_{s+1}\,,$$

where the superscripts $\pm$ stand for the shift of the argument, $f^{\pm}(\zeta) := f(e^{\pm i\pi/2}\zeta)$. Essentially, Y_s are the cross-ratios. For example, $Y_{2k} = -\chi_{-k,k,-k-1,k+1}$. By definition, the products $s_i \wedge s_j$ satisfy the algebraic identity among determinants,

$$(s_i \wedge s_j)(s_k \wedge s_l) = (s_i \wedge s_k)(s_j \wedge s_l) + (s_i \wedge s_l)(s_k \wedge s_j)\,.$$

This gives the functional equations among T_s $(s = 1, ..., n-3)$,

$$T_s^+ T_s^- = T_{s+1} T_{s-1} + 1\,, \tag{3.4}$$

or in terms of Y_s,

$$Y_s^+ Y_s^- = (1 + Y_{s-1})(1 + Y_{s+1})\,. \tag{3.5}$$

These take the well known form of the functional equations which are called the T-system/Hirota equations and the Y-system, respectively. For a review on T- and Y-systems, see for example [13].

Up to here, (3.4) or (3.5) is just algebraic identities. The physical input then comes from the asymptotic behaviors of the Y-functions. A WKB analysis of the linear system (3.1) shows, for example, that

$$\log Y_{2k} \sim \frac{Z_{2k}}{\zeta} + \log \mu_{2k} \quad (\zeta \to 0)\,.$$

Here, Z_s are period integrals $Z_s = \oint_{\gamma_s} \sqrt{p}\, dz$, and $\log \mu_s$ are certain constants, which we call the chemical potentials. In our case of AdS_3, we have $\mu_s = 0$. By using such asymptotic behaviors and assuming certain analyticity of $\log Y_s$, one can convert the Y-system into the following integral equations:

$$\log Y_s(\theta) = -m_s R \cosh\theta + K * \log(1 + Y_{s-1})(1 + Y_{s+1})\,, \tag{3.6}$$

where we have introduced $\theta := \log \zeta$, $m_s R := 2Z_s$, $K(\theta) := 1/\cosh\theta$, and $*$ stands for the convolution, i.e., $f * g = \int \frac{d\theta'}{2\pi} f(\theta - \theta') g(\theta')$. For simplicity, we have displayed the equations when all Z_s are real. The cross-ratios are obtained by solving these equations and setting the spectral parameter to particular values $\zeta = 1, \pm i$.

The set of equations (3.6) are of the form of the thermodynamic Bethe ansatz equations, which are used to analyze finite-size effects of (1+1) dimensional integrable systems with factorizable scattering. In this context, the TBA equations are obtained by minimizing the free energy, m_s are the masses of particles, θ is the rapidity, R is the inverse temperature, and $\log Y_s$ give pseudo energies. It is surprising that the geometrical problem of the minimal surfaces results in the equations of the type of the TBA equations.

The solution to the integral equations (3.6) also gives the area of the minimal surfaces and hence the gluon scattering amplitudes at strong coupling. To see this, we first recall that the area is divergent and needs to be regularized. Here, we adopt the following regularization,

$$\begin{aligned} A(\text{area}) = 4\int d^2z\, e^{2\alpha} &\to 4\int d^2z\,(e^{2\alpha} - \sqrt{p\bar{p}}) + 4\int_{r\geq\epsilon} d^2z\,\sqrt{p\bar{p}} \\ &=: A_{\text{fin}} + A_{w\text{-vol}}\,. \end{aligned}$$

With the help of the analysis of the Hitchin system for $\mathcal{N} = 2$ SYM, one can find that the first finite term of the regularized area, A_{fin}, is nothing but the free energy F associated with the TBA equations (3.6) (up to a sign and a constant):

$$\begin{aligned} A_{\text{fin}} &= \sum_s \int \frac{d\theta}{2\pi} m_s R\cosh\theta \cdot \log(1+Y_s) \ + (\text{const.}) \\ &= -F + (\text{const.})\,. \end{aligned} \tag{3.7}$$

On the other hand, the second term, $A_{w\text{-vol}}$, essentially gives the BDS form,

$$A_{w\text{-vol}} \sim A_{\text{div}} + A_{\text{BDS}} + \cdots\,,$$

where A_{div} is the divergent part and A_{BDS} is the finite part of the BDS formula. Therefore, the most intricate part of the remainder function $\mathcal{R}$ is given by the free energy:

$$\mathcal{R} = -(A - A_{\text{div}} - A_{\text{BDS}}) = F + \cdots\,. \tag{3.8}$$

The ellipses stand for some other terms.

Summarizing, the procedure of computing the strong-coupling amplitudes is as follows:

(1) Solve the integral equations (3.6) and obtain the Y-functions $Y_s(\theta)$.

(2) The area A of the minimal surfaces or the amplitude $\mathcal{M}$ is given by the free energy F associated with the TBA equations and some other terms.

(3) The cross-ratios (3.3) are obtained by evaluating $Y_s(\theta)$ at particular values of the argument $\zeta = e^\theta = 1, \pm i$.

(4) Expressing the amplitudes by the resultant cross-ratios gives the amplitudes as functions of external momenta.

§ 3.3. Minimal surfaces in AdS_5

A similar but more elaborated discussion shows that the minimal surfaces in AdS_5 which have $\hat{n}$-cusps are described by the following Y-system:

$$
\begin{aligned}
\frac{Y_{2,m}^- Y_{2,m}^+}{Y_{1,m}Y_{3,m}} &= \frac{(1+Y_{2,m+1})(1+Y_{2,m-1})}{(1+Y_{1,m})(1+Y_{3,m})}\,, \\
\frac{Y_{3,m}^- Y_{1,m}^+}{Y_{2,m}} &= \frac{(1+Y_{3,m+1})(1+Y_{1,m-1})}{1+Y_{2,m}}\,, \\
\frac{Y_{1,m}^- Y_{3,m}^+}{Y_{2,m}} &= \frac{(1+Y_{1,m+1})(1+Y_{3,m-1})}{1+Y_{2,m}}\,,
\end{aligned}
\tag{3.9}
$$

where $m = 1,..,\hat{n}-5$, and $f^{\pm}(\zeta) = f(e^{\pm i\pi/4}\zeta)$ in this case. This Y-system is non-standard in that $Y_{1,m}$ and $Y_{3,m}$ couple to each other on the left-hand side. A similar Y-system also appears in the study of the spectral problem of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

§ 4. Underlying integrable models and CFTs

We saw that the minimal surfaces in AdS spaces are described by the integral equations of the form of the TBA equations (or the associated Y-systems). A natural question here is: Are these "TBA-like" equations really the TBA equations of any integrable models? In the case of the 6-cusp solution in AdS_5, it has been shown that the integral equations are indeed the TBA equations of the $\mathbb{Z}_4$-symmetric (or A_3-)integrable model, which is obtained by a massive deformation of the $\mathbb{Z}_4$-parafermion CFT [4]. In the following, we would like to show that the answer to the above question is yes for the general cusp solutions in AdS_3 and AdS_4 [6].

Let us first consider the AdS_3 case. To investigate the underlying integrable models, we recall that, if a (1+1)-dimensional integrable model is obtained from a CFT by a relevant perturbation, the free energy described by the TBA equations gives the central charge c of the CFT in the CFT/high-temperature limit $R \to 0$:

$$
F \to -\frac{\pi}{6}c\,.
$$

On the other hand, in the same limit, the period integrals Z_s are vanishing and the minimal surfaces reduce to the regular polygonal surfaces whose boundary forms a regular polygon in a subspace of the AdS boundary after a projection. This class of the solutions in AdS_3 are described by the Painlevé III equation. For the $2n$-cusp solution, the finite part of the regularized area in this limit has been obtained as [3]

$$A_{\rm fin} \to \frac{\pi}{4n}(3n^2 - 8n + 4)\,.$$

To find the free energy, one has to fix the difference between $A_{\rm fin}$ and F in (3.7). This is done by considering another limit where the zeros of the polynomial $p(z)$ become far apart from each other. Since the solution is expected to be a superposition of the $(n-2)$ 6-cusp solutions in this limit, it follows that $F \sim 0$ and $A_{\rm fin} \sim (n-2) \times \frac{7}{12}\pi$. Thus,

$$-F = A_{\rm fin} - \frac{7}{12}(n-2)\pi \to \frac{\pi}{6n}(n-2)(n-3)\,.$$

A candidate of the CFT in the UV/high-temperature limit then has to have central charge $c = (n-2)(n-3)/n$. One can indeed find such a CFT: The coset or the generalized parafermion CFT associated with

$$\frac{\widehat{su}(K)_k}{[\widehat{u}(1)]^{K-1}} \simeq \frac{[\widehat{su}(k)_1]^K}{\widehat{su}(k)_K} \tag{4.1}$$

has the central charge $c = (k-1)K(K-1)/(k+K)$. Thus, the coset CFT with $K = n-2$, $k = 2$ has the correct central charge. In addition, the second representation in (4.1) shows that this is an $su(2)$ coset, and matches the symmetry of the $su(2)$ Hitchin system. Moreover, the degrees of freedom of this coset is $n-3$, which also matches the number of independent cross-ratios $2(n-3)$. We remark that the left and the right sector are described by the same integral equations in the AdS_3 case.

These arguments suggest that the above coset CFT is the right candidate. Proceeding to a consideration away from the CFT point, we note that a massive deformation of this CFT by the adjoint operators is integrable, and gives the homogeneous sine-Gordon model [9]. The model has a factorizable diagonal S-matrix. In the case of the coset $\widehat{su}(n-2)_2/[\widehat{u}(1)]^{n-3}$, the elements of the S-matrix for particles a and b $(a, b = 1, ..., n-3)$ are given up to constant factors by

$$S_{ab}(\theta) \sim \Big[\tanh\frac{1}{2}(\theta + \sigma_{ab} - i\frac{\pi}{2})\Big]^{I_{ab}}\,,$$

where θ is the difference of the rapidities of the particles, I_{ab} is the incidence matrix of $su(n-2)$, and σ_{ab} are certain parameters. By the standard procedure, one can then derive the TBA equations of this HSG model, to find that they coincide with the integral equations for the minimal surfaces in AdS_3. This answers the question at the beginning of this section affirmatively. Precisely speaking, the reality of the parameters σ_{ab} are different and the physical interpretation should be considered further. Keeping this in mind, we have found that the $2n$-cusp solution in AdS_3 is described by the HSG model associated with the coset $\widehat{su}(n-2)_2/[\widehat{u}(1)]^{n-3}$. Schematically,

$$\boxed{\begin{array}{c}2n\text{-cusp minimal surfaces}\\ \text{in } AdS_3\end{array}} \quad \Longleftarrow \quad \boxed{\begin{array}{c}\text{HSG model from}\\ \dfrac{\widehat{su}(n-2)_2}{[\widehat{u}(1)]^{n-3}} \simeq \dfrac{[\widehat{su}(2)_1]^{n-2}}{\widehat{su}(2)_{n-2}}\end{array}}$$

In the case of AdS_5, we already know that the 6-cusp solution is described by the $\mathbb{Z}_4$-symmetric integrable model, which corresponds to the coset (4.1) with $K = 2$ and $k = 4$. Taking into account, again, the symmetry and the degrees of freedom, one may guess that the $\hat{n}$-cusp solution in AdS_5 is described by the HSG model associated with the coset (4.1) with $K = \hat{n} - 4$ and $k = 4$. It turns out, however, that the TBA equations/Y-system of this HSG model are of the standard form and do not agree with (3.9). Instead, they do agree with those for the $\hat{n}$-cusp solution in AdS_4, which are obtained from the AdS_5 case by setting the chemical potentials to zero and hence identifying $Y_{1,m}$ and $Y_{3,m}$ in (3.9) [5]. The reduction from AdS_5 to AdS_4 maintains the $su(4)$ symmetry of the Hitchin system, and the identification among the Y-functions ensures the matching between the degrees of freedom of the coset and the number of independent cross-ratios $2(\hat{n} - 5)$. In addition, the coset CFT has the central charge $c = 3(\hat{n}-4)(\hat{n}-5)/\hat{n}$, which also agrees with the result for the regular polygon solution in AdS_4 [5]. Schematically,

$$\boxed{\begin{array}{c}\hat{n}\text{-cusp minimal surfaces}\\ \text{in } AdS_4\end{array}} \quad \Longleftarrow \quad \boxed{\begin{array}{c}\text{HSG model from}\\ \dfrac{\widehat{su}(\hat{n}-4)_4}{[\widehat{u}(1)]^{\hat{n}-5}} \simeq \dfrac{[\widehat{su}(4)_1]^{\hat{n}-4}}{\widehat{su}(4)_{\hat{n}-4}}\end{array}}$$

The reduction from AdS_5 to AdS_4 seems to suggest a possibility that their Y-systems are related by certain deformations of the underlying CFT/integrable model by the chemical potentials. We see a simple example of such a deformation in the case of the 6-point amplitudes. The identification of the underlying CFT and integrable model in the AdS_5 case is an interesting issue to be discussed further. As a side remark, we note that the $\mathbb{Z}_4$-symmetric integrable model for the 6-cusp solution is a special case of the HSG model.

§ 5. Six-point amplitudes from Z_4-symmetric integrable model

We saw that the HSG model associated with certain cosets describes the minimal surfaces in AdS spaces and hence the scattering amplitudes at strong coupling. This implies an unexpected connection between a four-dimensional SYM theory and (1+1)-dimensional integrable models. Such an identification is not only interesting but also useful in analyzing the amplitudes. We would like to demonstrate this in the case of the 6-point amplitudes corresponding to the 6-cusp solution in AdS_5 by deriving their expansion near the CFT limit [7].

As mentioned before, the 6-cusp minimal surfaces in AdS_5 are described by the TBA equations associated with the $\mathbb{Z}_4$-symmetric integrable model. The model is obtained by an integrable deformation of the $\mathbb{Z}_4$-parafermion CFT by the first energy operator $\varepsilon(x)$ with dimension $D_\varepsilon = \bar{D}_\varepsilon = 1/3$. Its action is given by

$$S = S_{PF} + g \int d^2x\, \varepsilon(x)\,,$$

where S_{PF} is the action of the $\mathbb{Z}_4$-parafermion CFT, which has the central charge $c = 1$. The model contains three particles with mass $m_a = m, \sqrt{2}m$ and m, respectively. The third particle is the anti-particle of the first. The coupling constant g is related to the mass as $g = b_g m^{4/3}$ with b_g being a certain numerical constant. The Y-system of this model is

$$Y_1^+ Y_1^- = 1 + Y_2\,, \quad Y_2^+ Y_2^- = (1 + \mu Y_1)(1 + \mu^{-1} Y_1)\,, \quad Y_1 = Y_3\,, \tag{5.1}$$

where $f^\pm(\theta) = f(\theta \pm \frac{\pi}{4} i)$ and $\log \mu$ is the chemical potential. As in the AdS_3 case, this can be converted to the TBA equations. In the following, we discuss the amplitudes around the CFT limit where all the independent cross-ratios are equal in a certain basis.

First, let us consider the free energy associated with the TBA equations. In general, the free energy of a model on a circle of length $L \gg 1$ with temperature $1/R$ gives the ground state energy $E(R)$ of the model on a circle of length R. This relation is found by evaluating the torus partition function in two different channels. Near the CFT/high-temperature limit $mR := 2|Z| \ll 1$, the CFT perturbation gives an expansion of the free energy. In our case, it reads

$$\begin{aligned} F = E_0 + \frac{1}{4}(mR)^2 - R^2 \sum_{n=1}^{\infty} \frac{(-g)^n}{n!} \Big(\frac{2\pi}{R}\Big)^{2(D_\varepsilon - 1)n + 2} \\ \times \int \Big\langle V(\infty)\, \varepsilon(z_n, \bar{z}_n) \cdots \varepsilon(z_1, \bar{z}_1)\, V(0) \Big\rangle_{\text{CFT}} \prod_{i=2}^{n} (z_i \bar{z}_i)^{D_\varepsilon - 1} dz_2^2 \cdots dz_n^2\,, \end{aligned} \tag{5.2}$$

where E_0 is the CFT ground state energy $-\pi/6$ and V is the vacuum operator. The correlators are connected ones of the CFT on a complex plane, and we have set $z_1 = 1$. $|Z|$ is the absolute value of a period integral $Z =: |Z| e^{i\varphi}$ similar to Z_s in the AdS_3 case.

When the chemical potential vanishes, i.e., $\mu =: e^{i\phi} = 1$, the vacuum operator is the identity, $V = 1$. On the other hand, $\mu \neq 1$ corresponds to a twisted boundary condition of the $\mathbb{Z}_4$-parafermion CFT. The vacuum operator in this case becomes non-trivial. To be explicit, we bosonize the parafermion theory by a free boson Φ. The energy and the vacuum operator are then given by

$$\varepsilon = a_+\, e^{i\sqrt{\frac{2}{3}}\Phi} + a_- e^{-i\sqrt{\frac{2}{3}}\Phi}\,, \quad V = e^{-i\sqrt{\frac{1}{6}}\frac{\phi}{\pi}\Phi}\,,$$

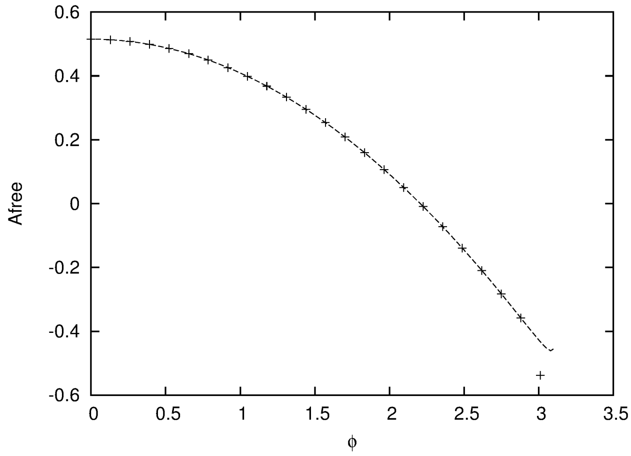


Figure 4. Plot of $A_{\text{free}} := -F$ as a function of ϕ for $|Z| = 0.1$ and $\varphi = -\pi/48$ from numerical results (+) and the high-temperature expansion (5.3) (dashed line).

where $a_\pm$ are certain cocycle factors. Substituting these into the expansion (5.2), one obtains

$$F = E_0 + |Z|^2 - C_{\frac{8}{3}}\, \gamma\Big(\frac{1}{3} + \frac{\phi}{3\pi}\Big)\gamma\Big(\frac{1}{3} - \frac{\phi}{3\pi}\Big)\, |Z|^{\frac{8}{3}} + \mathcal{O}(|Z|^{\frac{16}{3}})\,, \tag{5.3}$$

where $E_0 = -\frac{\pi}{6}(1 - \frac{2\phi^2}{\pi^2})$, $\gamma(z) = \Gamma(x)/\Gamma(1-x)$, and $C_{\frac{8}{3}} = \frac{\pi}{2}\big[\frac{1}{\sqrt{\pi}}\gamma\big(\frac{3}{4}\big)\big]^{\frac{8}{3}}\gamma\big(\frac{1}{6}\big)\gamma\big(\frac{1}{3}\big) \approx 0.18461$. We have also used the explicit value of b_g. This is in good agreement with numerical computations (fig. 4).

Besides the free energy, one can also find an expansion of the Y-functions. From the periodicity and the analyticity, the Y-functions are expanded as

$$Y_a(\theta) = \sum_{n=0}^{\infty} Y_a^{(n)} \cosh\Big(\frac{4}{3}n(\theta - i\varphi)\Big)\,,$$

where $Y_a^{(n)} \sim (mR)^{4n/3}$ as $mR \to 0$. Substituting the expansion into the Y-system (5.1) gives equations to constrain the coefficients $Y_a^{(n)}$. Further using the relation between the Y-functions and the cross-ratios, $U_k = 1 + Y_2\big(\frac{2k+1}{4}\pi i\big)$ $(k = 1, 2, 3)$, one finds the first-order expansion around equal U_k:

$$U_k = 4\cos^2\Big(\frac{\phi}{3}\Big) + y^{(1)}(\phi)\cos\Big(\frac{4\varphi - (2k+1)\pi}{3}\Big) \times |Z|^{\frac{4}{3}} + \mathcal{O}\big(|Z|^{\frac{8}{3}}\big)\,,$$

where $y^{(1)}$ is a function of the chemical potential ϕ. Numerically, this is evaluated as $y^{(1)}(\phi) \approx 5.47669 - 0.484171\phi^2 + 0.0119471\phi^4 + \cdots$. The above relations are inverted to express the data of the minimal surfaces $(|Z|, \varphi, \phi)$ as functions of the cross-ratios,

(5.4)
$$\cos^2\frac{\phi}{3} = \frac{1}{12}\sum_k U_k\,, \quad \tan\frac{4}{3}\varphi = \frac{\sqrt{3}(U_2-U_3)}{2U_1-U_2-U_3}\,, \quad |Z|^{\frac{4}{3}} = \frac{-2U_1+U_2+U_3}{3y^{(1)}(\phi)\cos\frac{4}{3}\varphi}\,.$$

From these, geometrical meaning of $(|Z|, \varphi, \phi)$ in the parameter space (U_1, U_2, U_3) is found.

Collecting all the results in addition to (5.3), the full expression of the remainder function defined in (3.8) is found to be

(5.5)
$$\begin{aligned}\mathcal{R} = &-\left[\frac{\pi}{6}\Big(1-\frac{2\phi^2}{\pi^2}\Big)+\frac{3}{4}\,\mathrm{Li}_2(1-4\beta^2)\right] \\ &-\left[C_{\frac{8}{3}}\gamma\Big(\frac{1}{3}+\frac{\phi}{3\pi}\Big)\gamma\Big(\frac{1}{3}-\frac{\phi}{3\pi}\Big)-\frac{3\big(4\beta^2-1+\log(4\beta^2)\big)}{64\beta^2(4\beta^2-1)^2}y^{(1)}(\phi)^2\right]|Z|^{\frac{8}{3}}+\mathcal{O}\big(|Z|^4\big)\,,\end{aligned}$$

where Li_2 is the dilogarithm and $\beta := \cos(\phi/3)$. By (5.4), this is further expressed in terms of the cross-ratios U_k, which can be directly compared with perturbative computations.

In addition to the above expansion around the CFT limit with $|Z| \ll 1$, it is straightforward to carry out the opposite expansion around the low-temperature/infrared limit with $|Z| \gg 1$, which corresponds to collinear limits in the SYM theory. In fig. 5, we show the remainder function obtained by the first order expansions for $|Z| \ll 1$ and $|Z| \gg 1$. These are again in good agreement with numerical computations. We find that the simple first order expansions well describe the remainder function for all the scale $|Z|$.

§ 6. Summary

The discovery of the integrability opened up new dimensions in the study of the gauge/ string duality or the AdS/CFT correspondence. That has led to a proposal that the full spectrum of $\mathcal{N} = 4$ SYM and the string theory on $AdS_5 \times S^5$ in the planar limit is obtained by solving certain TBA equations/Y-system [1]. Besides this very interesting theoretical development, the integrability has also been applied to the study of the gluon scattering amplitudes: By the AdS/CFT correspondence, the amplitudes at strong coupling are given by the area of the minimal surfaces in AdS_5 with a polygonal boundary which consists of null edges corresponding to external momenta [2]. These minimal surfaces are again described by certain, but different, TBA equations/Y-system

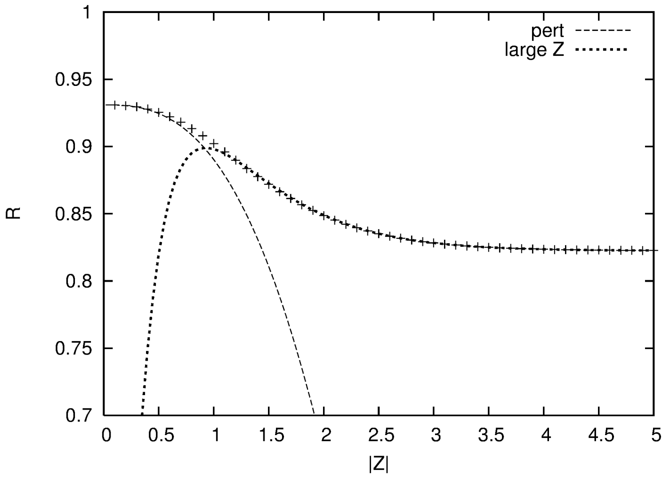
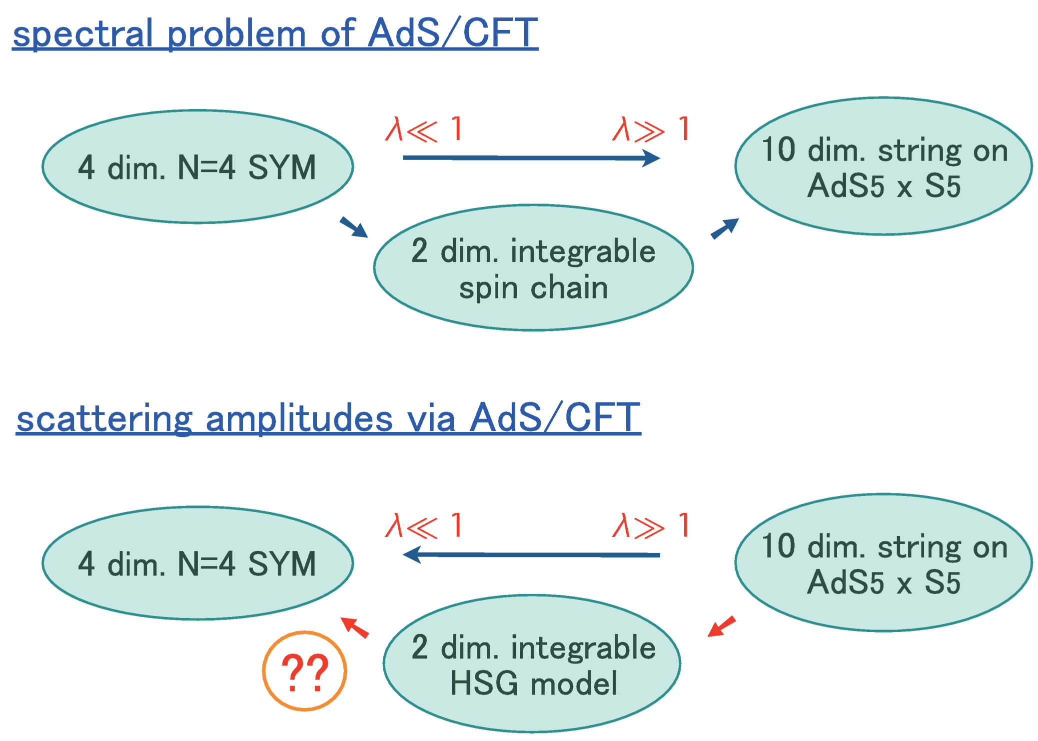


Figure 5. Plot of the remainder function $\mathcal{R}$ as a function of $|Z|$ for $\phi = 0$ and $\varphi = -\pi/48$ from the UV expansion (5.5) (dashed line), the first order IR expansion (dotted line) and numerical results (+).

[3, 4, 5, 6]. The study on the strong coupling side also provided useful insights and facilitated the development on the weak coupling side.

Generalizing the connection between the 6-point amplitudes and the $\mathbb{Z}_4$-symmetric integrable model [4], we observed that the TBA equations/Y-systems for the minimal surfaces in AdS_3 and AdS_4 coincide with those of the HSG model associated with certain coset or generalized parafermion CFTs [6]. Such a connection between the scattering amplitudes of the four-dimensional SYM theory and the (1+1)-dimensional integrable model is not only interesting but also useful for actual computation of the amplitudes. We demonstrated this in the case of the 6-point amplitudes by deriving an expansion near the limit of equal cross-ratios from the $\mathbb{Z}_4$-symmetric integrable model [7].

There may be many future directions. First, although we arrived at the TBA equations /Y-systems in analyzing the amplitudes, the intrinsic reason is not clear. The situation resembles that of the ODE/IM correspondence [14]. Second, the integrable model underlying the AdS_5 case is not yet identified, except for the 6-point case. It would be interesting to clarify whether it is just some deformation of the AdS_4 case by the chemical potentials as in the 6-point case, or corresponds to some new integrable model. Third, it seems that we do not have a formalism to obtain the completely analytic form of the expansion of the Y-functions near the CFT limit. This is important to derive an analytic form of the remainder function, though in some restricted parameter space. Given the recent development on the analytic form of the perturbative remainder function, this is certainly an interesting issue. Finally, recalling the development on the spectral problem where the expansions both from the weak and strong coupling side



finally reached the proposal of the full spectrum, it would be very interesting if one could include the corrections to the strong coupling result. (See the last figure.) On the day before this talk was given, an interesting paper [8] appeared which discusses this issue.

§ 7. Acknowledgements

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References

[1] A. Cavaglia, D. Fioravanti, M. Mattelliano and R. Tateo, article in the proceedings.
[2] L. F. Alday and J. M. Maldacena, JHEP **0706** (2007) 064.
[3] L. F. Alday and J. Maldacena, JHEP **0911** (2009) 082.
[4] L. F. Alday, D. Gaiotto and J. Maldacena, arXiv:0911.4708.

[5] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, J. Phys. A **43** (2010) 485401.
[6] Y. Hatsuda, K. Ito, K. Sakai and Y. Satoh, JHEP **1004** (2010) 108.
[7] Y. Hatsuda, K. Ito, K. Sakai and Y. Satoh, JHEP **1009** (2010) 064.
[8] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, arXiv:1006.2788.
[9] C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, Nucl. Phys. B **484** (1997) 609.
[10] L. F. Alday and R. Roiban, Phys. Rept. **468** (2008) 153.
[11] S. Dobashi, K. Ito and K. Iwasaki, JHEP **0807** (2008) 088.
[12] K. Sakai and Y. Satoh, JHEP **0910** (2009) 001; JHEP **1003** (2010) 077.
[13] A. Kuniba, T. Nakanishi and J. Suzuki, J. Phys. A **44** (2011) 103001.
[14] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **40** (2007) R205.