

# Geometry and Combinatorics of Crystal Melting

By

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## Abstract

We survey geometrical and especially combinatorial aspects of generalized Donaldson-Thomas invariants (also called BPS invariants) for toric Calabi-Yau manifolds, emphasizing the role of plane partitions and their generalizations in the recently proposed crystal melting model. We also comment on equivalence with a vicious walker model and the matrix model representation of the partition function <sup>1</sup>.

## § 1. Introduction

The study of plane partitions, a three-dimensional generalization of partitions, has a long history of more than a century in mathematics [1, 2]. There has recently been a renewed interest in this old topic, both among mathematicians and physicists alike, due to the pioneering discovery that topological A-model [3] on toric Calabi-Yau manifolds [4] can be described by a statistical mechanical model of plane partitions [5, 6]. In more mathematical language, plane partitions count Donaldson-Thomas (DT) invariants [7, 8], whose partition function is equivalent [9, 10] to that of the Gromov-Witten invariants under suitable parameter identifications.

There is an interesting twist to this story, which is the topic of more recent studies in this field. There we study “generalized Donaldson-Thomas invariants” [11, 12], which depend on moduli (mathematically stability conditions, or physically complexified Kähler moduli <sup>2</sup>). These “invariants” are invariant under a generic deformation of

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<sup>2</sup>These two notions are precisely speaking not the same, but the differences will not be important in this paper.

moduli, but can jump when we cross real codimension loci (called walls of marginal stability, which divide the moduli space into chambers) in the moduli space. This jumping is called the wall crossing phenomena, and there are general formulas [11, 12, 13] which govern this jumping phenomena. Generalized DT invariants are indeed generalizations of the original DT invariants, in the sense that the former coincide with the latter only in a specific chamber (hereafter called the topological string chamber) of the moduli space.

Given the richness of these new invariants, the natural question is whether there are combinatorial counterparts to this geometric story. The goal of the present article is to provide an answer to this question. Generalized DT invariants on toric Calabi-Yau manifolds are described by a statistical mechanical model of “crystal melting” [14, 15, 16], formulated here as an enumeration problem of plane partitions and their generalizations. We will also comment on the equivalence with a vicious walker model, following [17]. No prior knowledge in this field is assumed, and this paper is intended to be self-contained, at least as far as the combinatorial aspects are concerned.

This article is organized as follows. In section 2 we define our combinatorial partition function as a sum over suitable evolution of partitions. In section 3 we comment on the representation of the partition function as a unitary matrix integral. The derivation of this matrix integral is given in section 4, based on an equivalence with a vicious walker model. Appendix contains an introduction to generalized Donaldson-Thomas invariants, which readers can consult for geometric side of the story.

## § 2. Definition of the Model

We begin with the following theorem, which states that the partition function for generalized DT invariants  $Z_{\text{gDT}}$  (see Appendix) on a toric Calabi-Yau manifold can be computed exactly by purely combinatorial methods:

**Theorem 2.1** (Szendrői [14], Mozgovoy-Reineke [15], Ooguri-Yamazaki [16]).  
*For a toric Calabi-Yau manifold, the partition function for generalized DT invariants can be written as*

$$(2.1) \quad Z_{\text{gDT}} = Z_{\text{crystal}},$$

*where the statistical mechanical model in the right hand side can equivalently formulated as (1) a crystal melting model (a configuration of molten atoms), (2) dimer model, or (3) a generalization of plane partitions (an evolution of partitions)*<sup>3</sup>.

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<sup>3</sup>The third formulation is available only when the toric Calabi-Yau manifold has no compact 4-cycles. All the examples in this paper fall into this category.

The goal of this section is to define  $Z_{\text{crystal}}$ , using the plane partitions and their generalizations. We use the third formulation, developed in [18, 19, 20, 21]. See [14, 15, 16, 22] for the first and the second description.

*Remark.* As we discussed in the introduction, the LHS of the equation (2.1) depends on the value of the moduli. Correspondingly, RHS also has moduli dependence, and we have different statistical mechanical models for different chambers of the moduli space. The schematic relation (2.1) should be interpreted this way. See (2.16).

Let us begin with standard notations. A partition  $\lambda = (\lambda_i)$  is a non-increasing sequence of integers  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$  such that  $\lambda_n = 0$  for sufficiently large  $n$ . The length  $|\lambda|$  of a partition  $\lambda$  is given by  $|\lambda| := \sum_i \lambda_i$ . We define a transpose  $\lambda^t$  by

$$\lambda^t_i := \#\{j | \lambda_j \leq i\}.$$

This is simply a graphical transpose of a partition, as will be clear from the following example.

**Example 2.2.** For  $\lambda = (4, 2, 1) =$ ,  $|\lambda| = 7$  and  $\lambda^t = (3, 2, 1, 1) =$ .

Given two partitions  $\lambda$  and  $\mu$ , we define  $\lambda \succ^+ \mu$  if and only if

$$\lambda_i = \mu_i + 1 \text{ or } \mu_i \text{ for all } i.$$

We also denote  $\lambda \succ^- \mu$  if and only if  $\lambda^t \succ^+ \mu^t$ , or equivalently

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots .$$

**Example 2.3.** Two partitions  $\lambda = (4, 2, 1) =$  and  $\mu = (3, 2, 1) =$  both satisfy  $\lambda \succ^+ \mu$  and  $\lambda \succ^- \mu$ .

Now we define a plane partition (also called a 3d partition) as a sequence of partitions  $\Pi = \{\lambda(n)\}_{n \in \mathbb{Z}}$  such that

$$(2.2) \quad \dots \prec^+ \lambda(-2) \prec^+ \lambda(-1) \prec^+ \lambda(0) \succ^+ \lambda(1) \succ^+ \lambda(2) \succ^+ \dots,$$

and  $\lambda(n) = \{0\}$  when  $|n|$  sufficiently large. Define the length  $|\Pi|$  of a plane partition  $\Pi = \{\lambda(n)\}$  to be  $|\Pi| = \sum_n |\lambda(n)|$ . Note that this is a finite sum by the assumption above. Our partition function is defined by

$$(2.3) \quad Z_{\text{crystal}}(q) := \sum_{\Pi} q^{|\Pi|}.$$

As is well-known, MacMahon’s formula represents this as an infinite product

$$(2.4) \quad Z_{\text{crystal}}(q) = \prod_{k>0} (1 - q^k)^{-k},$$

which is the same as the generalized DT partition function <sup>4</sup> for  $\mathbb{C}^3$  [3]: <sup>5</sup>

$$(2.5) \quad Z_{\text{crystal}}(q) = Z_{\text{gDT}}^{\mathbb{C}^3}(q),$$

For this reason we hereafter denote the LHS of the above equation (2.5) by  $Z_{\text{crystal}}^{\mathbb{C}^3}$ .

*Remark.* In the definition of a plane partition (2.2) we could have used  $\bar{\succ}, \bar{\prec}$  instead of  $\overset{+}{\succ}, \overset{+}{\prec}$ . This has the effect of replacing  $\{\lambda(n)\}$  by  $\{\lambda(n)^t\}$ , and the partition function is the same either way.

*Remark.* The choice of the weight in (2.3) is the same as in the Schur process of [23].

We have seen that plane partitions correspond to the simplest Calabi-Yau geometry  $\mathbb{C}^3$ . Our next task is to consider a set of partitions corresponding to the (resolved) conifold. We again consider a sequence of partitions  $\Pi = \{\lambda(n)\}$ , but now with a slightly different interlacing conditions, with plus and minus appearing alternately:

$$(2.6) \quad \dots \bar{\prec} \lambda(-2) \overset{+}{\prec} \lambda(-1) \bar{\prec} \lambda(0) \overset{+}{\succ} \lambda(1) \bar{\prec} \lambda(2) \overset{+}{\succ} \dots$$

For such a  $\Pi$ , we define  $|\Pi|_0 := \sum_{n:\text{even}} |\lambda(n)|$  and similarly  $|\Pi|_1 := \sum_{n:\text{odd}} |\lambda(n)|$ . The conifold partition function is then defined by

$$(2.7) \quad Z_{\text{crystal}}^{\text{conifold}}(q_0, q_1) := \sum_{\Pi: \text{satisfying (2.6)}} q_0^{|\Pi|_0} q_1^{|\Pi|_1}.$$

Then (2.1) in this example states that

$$(2.8) \quad Z_{\text{crystal}}^{\text{conifold}}(q_0, q_1) = Z_{\text{gDT}}^{\text{conifold}}(q, Q)$$

under the parameter identification

$$(2.9) \quad q = q_0 q_1, \quad Q = q_1.$$

Infinite product expression for this partition function is known [3] from the study of the topological string: <sup>6</sup>

$$(2.10) \quad Z_{\text{gDT}}^{\text{conifold}}(q, Q) = M(q)^2 \prod_{k>0} (1 + q^k Q)^k \prod_{k>0} (1 + q^k Q^{-1})^k,$$

<sup>4</sup>See Appendix for summary of these invariants.

<sup>5</sup>In the language of topological strings,  $q$  is related to the topological string coupling constant  $g_s$  by  $q = -e^{-g_s}$ .

<sup>6</sup>In the topological strings,  $q = -e^{-g_s}$ ,  $Q = -e^{-t}$ , where  $g_s$  is the topological string coupling constant, and  $t$  is the Kähler moduli of the resolved conifold.

The corresponding statement for  $Z_{\text{crystal}}^{\text{conifold}}$  was shown combinatorially in [22].

The discussion above is a little bit imprecise because we did not specify the moduli dependence of the generalized DT invariants. We therefore consider the following more general partition function which includes the moduli dependence and applies to more general toric geometries.

First, fix an integer  $L$  and a function  $\rho : \{1/2, 3/2, \dots, L - 1/2\} \rightarrow \{\pm 1\}$ . We can periodically extend  $\rho$  to a map  $\sigma : \mathbb{Z}_{1/2} \rightarrow \{\pm 1\}$ , where  $\mathbb{Z}_{1/2}$  is a set of half-integers. This will determine a toric Calabi-Yau geometry. Second, to fix a moduli dependence, we define a bijection  $\theta : \mathbb{Z}_{1/2} \rightarrow \mathbb{Z}_{1/2}$  such that <sup>7</sup>

$$(2.11) \quad \theta(h + L) = \theta(h) + L \quad \text{for all } h \in \mathbb{Z}_{1/2},$$

and

$$(2.12) \quad \sum_{i=1}^L \theta \left( i - \frac{1}{2} \right) = \sum_{i=1}^L \left( i - \frac{1}{2} \right).$$

We then define a generalized plane partition of type  $(L, \rho, \theta)$  (whose totality we are going to denote by  $\mathcal{P}^{(L, \rho, \theta)}$ ) to be a sequence of partitions  $\Pi = \{\lambda(n)\}$  such that

$$(2.13) \quad \begin{aligned} \lambda(i) &\overset{\sigma \circ \theta(i+1/2)}{<} \lambda(i+1) \quad \text{for } \theta \left( i + \frac{1}{2} \right) < 0, \\ \lambda(i) &\overset{\sigma \circ \theta(i+1/2)}{>} \lambda(i+1) \quad \text{for } \theta \left( i + \frac{1}{2} \right) > 0. \end{aligned}$$

We define  $|\Pi|_i := \sum_{n \equiv i \pmod L} |\lambda(n)|$  for  $i = 0, \dots, L - 1$ . We also define

$$q_i^\theta := \begin{cases} q_{\theta^{-1}(i-1/2)+1/2} \cdot q_{\theta^{-1}(i-1/2)+3/2} \cdots q_{\theta^{-1}(i+1/2)-1/2} & (\theta^{-1}(i-1/2) < \theta^{-1}(i+1/2)), \\ q_{\theta^{-1}(i-1/2)-1/2} \cdot q_{\theta^{-1}(i-1/2)-3/2} \cdots q_{\theta^{-1}(i+1/2)+1/2} & (\theta^{-1}(i-1/2) > \theta^{-1}(i+1/2)), \end{cases}$$

where we used  $q_i$  for  $i \in \mathbb{Z}$  by extending  $q_0, \dots, q_{L-1}$  periodically,

$$q_{i+L} = q_i.$$

Note that  $q_i^{\theta=\text{id}} = q_i$ . The partition function is defined by

$$(2.14) \quad Z_{\text{crystal}}^{(L, \rho, \theta)}(q_0, q_1, \dots, q_{L-1}) := \sum_{\Pi \in \mathcal{P}^{(L, \rho, \theta)}} (q_0^\theta)^{|\Pi|_0} (q_1^\theta)^{|\Pi|_1} \cdots (q_{L-1}^\theta)^{|\Pi|_{L-1}}$$

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<sup>7</sup>This notation including half-integers looks cumbersome, but is useful to see the action of the Weyl group of the affine Kac-Moody algebra [28, 19, 20]. The parametrization by  $\theta$  actually covers only half of the chambers, but sufficient for our purposes here. The partition function becomes a finite product in the remaining half.

**Example 2.4.** Let us take  $L = 1$ ,  $\rho = +1$  and then the only choice for  $\theta$  is  $\theta = \text{id}$ , and (2.13) reduces to (2.2). This means  $Z_{\text{crystal}}^{(L=1, \rho=+1, \theta=\text{id})} = Z_{\text{crystal}}^{\mathbb{C}^3}$ .

**Example 2.5.** Take  $L = 2$ ,  $\rho(1/2) = +1$  and  $\rho(3/2) = -1$ .  $\theta$  can in general be written as

$$\theta = \theta_n : \frac{1}{2} \mapsto \frac{1}{2} - n, \quad \frac{3}{2} \mapsto \frac{3}{2} + n.$$

When  $\theta = \theta_0$ , (2.13) is the same as (2.6), and the partition function (2.14) coincides with (2.7). The case of  $n \neq 0$  corresponds to generalized DT invariants in other chambers. The corresponding partition function is given by [24, 25, 26] (again under the identification (2.9))

$$(2.15) \quad Z_{\text{crystal}}^{\text{conifold}}(q_0, q_1; \theta_n) = M(q)^2 \prod_{k>0} (1 + q^k Q)^k \prod_{k>n} (1 + q^k Q^{-1})^k.$$

In particular, in the limit  $n \rightarrow \infty$ , this coincides with the commutative DT partition function for the conifold [3]. In this limit our statistical mechanical model reduces to the gluing of two crystal corners (topological vertices) as in [5, 4].

The general story goes as follows. We can construct from  $(L, \rho)$  a toric Calabi-Yau manifold  $X^{(L, \rho)}$ , which is one of the so-called generalized conifolds. This is a toric Calabi-Yau manifold without compact 4-cycles<sup>8</sup>, and has a connected string of  $L - 1$   $\mathbb{P}^1$ 's. Each  $\mathbb{P}^1$  is either a  $\mathcal{O}(-2, 0)$ -curve or a  $\mathcal{O}(-1, -1)$ -curve, depending on  $\sigma(i - 1/2) = \sigma(i + 1/2)$  or  $\sigma(i - 1/2) = -\sigma(i + 1/2)$ .<sup>9</sup> We can then consider the partition function of generalized DT invariants on  $X^{(L, \rho)}$ .

The remaining task is to specify the moduli dependence, which in this case is given by an element of the Weyl group of the affine Lie algebra  $\hat{A}_{L-1}$  [28]. The corresponding partition function is denoted by  $Z_{\text{gDT}}^{X^\rho}(q, Q; \theta)$ . Now the following theorem states that this partition function is the same as the crystal partition function of type  $(L, \rho, \theta)$ :

**Theorem 2.6** (Nagao [28]).

$$(2.16) \quad Z_{\text{crystal}}^\sigma(q_0, \dots, q_{L-1}; \theta) = Z_{\text{gDT}}^{X^\rho}(q, Q; \theta),$$

where the parameter identifications are given by<sup>10</sup>

$$q = \pm q_0 \dots q_{L-1}, \quad Q_i = \pm q_i \quad (i = 1, \dots, L - 1).$$

<sup>8</sup>See [27] for discussion of Calabi-Yau geometries with a compact 4-cycle.

<sup>9</sup>This means that the overall sign change of  $\rho$  does not change the geometry. In (2.13), this has the effect of replacing  $\begin{smallmatrix} + & + \\ \succ & \succ \end{smallmatrix}$  by  $\begin{smallmatrix} - & - \\ \succ & \succ \end{smallmatrix}$ . As discussed previously in the case of  $\mathbb{C}^3$ , this does not change the partition function, but will change the matrix model representation of the partition function presented in the next section.

<sup>10</sup>The signs are determined from  $\rho$ . See [15] and section 3.5 of [29].

*Remark.* We can consider further generalizations, by changing the boundary conditions at infinity. This generalized model counts “open generalized Donaldson-Thomas invariants”. See [19, 20, 30, 31].

In the following we concentrate on the case of  $\mathbb{C}^3$  and the resolved conifold.

### § 3. Matrix Model

In the following sections we show that the crystal melting partition function defined in the previous section can be written as a unitary matrix integral:

**Theorem 3.1.**

$$(3.1) \quad Z_{\text{crystal}}^{\mathbb{C}^3}(q) = \lim_{N \rightarrow \infty} \int_{U(N)} dU \det \Theta(U|q),$$

where  $dU$  is the Haar measure of the unitary group and

$$(3.2) \quad \Theta(u|q) = \prod_{k=0}^{\infty} (1 + uq^k)(1 + u^{-1}q^{k+1}).$$

**Theorem 3.2** (Ooguri-Sułkowski-Yamazaki [17]).

$$(3.3) \quad Z_{\text{crystal}}^{\text{conifold}}(q, Q; \theta_n) = C_n Z_{\text{matrix}}^{\text{conifold}}(q, Q; n),$$

where

$$(3.4) \quad C_n = \prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^k} \prod_{k=n+1}^{\infty} \left( \frac{1 - Q^{-1}q^k}{1 - q^k} \right)^n,$$

and

$$(3.5) \quad Z_{\text{matrix}}^{\text{conifold}}(q, Q; n) = \lim_{N \rightarrow \infty} \int_{U(N)} dU \det \left( \frac{\Theta(U|q)}{\Theta(QU|q)} \prod_{k=1}^n (1 + Q^{-1}U^{-1}q^k) \right),$$

where the measure  $dU$  and the function  $\Theta(u|q)$  are the same as in Theorem 3.1.

We shall give derivations of these results in the next section, but before going there some comments are in order.

*Remark.* Theorem 3.1 and Theorem 3.2 for  $n = 0$  are not new, although  $n \neq 0$  case has not previously appeared in the literature as far as the author is aware of. Theorem 3.1 seems to be well-known in the literature, and can be considered as a

reduction of a multi-matrix model in [32]. For Theorem 3.2 with  $n = 0$ , see the paper [33], which also proves similar identities for other groups. A vicious walker model similar to the one for the conifold presented in the next section was also discussed in [34]. In their terminology  $\succ^+, \prec^+$  ( $\succ^-, \prec^-$ ) are called “normal” (“super”) time evolution, and the model is analyzed by identifies involving semi-standard Young tableaux and hook Schur functions (also called supersymmetric Schur functions). They also analyze the scaling limit of the model. See also [35].

*Remark.* The prefactor  $C_n$  simplifies in the limit  $n = 0$  and  $n \rightarrow \infty$ ;  $C_0 = 1$  and  $C_\infty = M(q)$ . In particular in these cases  $C_n$  is independent of  $Q$ .

*Remark.*  $Z_{\text{matrix}}$ , being a partition function of a unitary matrix integral, is a reduction of a  $\tau$ -function of a two-dimensional integrable Toda chain [36] (see e.g. [37]). Similar integrable structures appeared in topological strings context in [38]. See also [39, 40] for the appearance of thermodynamic Bethe ansatz equations in the study generalized DT invariants.

Finally, let us discuss the thermodynamic limit of our model, using the matrix integral given above. The thermodynamic limit is the limit  $g_s \rightarrow 0$ , where the string coupling constant  $g_s$  is related to the parameter  $q$  by  $q = e^{-g_s}$ <sup>11</sup>. For small  $g_s$ , the modular transformation of  $\Theta$  with respect to  $g_s$  gives

$$\Theta(e^{i\phi}|e^{-g_s}) = e^{-\frac{\phi^2}{2g_s}} \cdot \left(1 + O(e^{-\frac{1}{g_s}})\right).$$

If we ignore non-perturbative terms in  $g_s$ , this means that the matrix model  $Z_{\text{matrix}}^{\mathbb{C}^3}$  reduces to a Hermitian Gaussian matrix model with unitary measure. This result was originally derived from the Chern-Simons theory on the conifold [41, 42] (see also [43] for crystal melting description).

The spectral curve for this Gaussian matrix model is given by the equation [42, 44]

$$(3.6) \quad e^x + e^y + e^{x-y-T} + 1 = 0,$$

where  $T = Ng_s$  is the 't Hooft coupling. This is the mirror of the resolved conifold. In the limit of  $T \rightarrow \infty$  (which we should take since  $N \rightarrow \infty$ ), the curve reduces to

$$(3.7) \quad e^x + e^y + 1 = 0,$$

which is the mirror of  $\mathbb{C}^3$ .

Next we discuss the spectral curve for the conifold matrix model (3.5)<sup>12</sup>. As before, we take the limit  $g_s \rightarrow 0, N \rightarrow \infty$  with  $T := Ng_s$  fixed, but now we also take  $n \rightarrow \infty$ ,

<sup>11</sup>This is the parameter counting the size  $|\Pi| := \sum_{i=1}^{L-1} |\Pi|_i$  of a generalized plane partition  $\Pi$ .

<sup>12</sup>This is the spectral curve for the matrix model. Our statistical model can equivalently be written as a dimer model, which has its own version of the spectral curve. See [45] and [46].

with  $\tau := ng_s$  fixed. The spectral curve is given by [17]

$$(3.8) \quad e^{x+y} + e^x + e^y + Q_1 e^{2x} + Q_2 e^{2y} + Q_3 = 0,$$

where

$$(3.9) \quad \begin{aligned} Q_1 &= \epsilon^2 \cdot \frac{1 + \mu Q}{(1 + \mu \epsilon^2)(1 + Q \epsilon^2)}, \\ Q_2 &= \mu \cdot \frac{1 + Q \epsilon^2}{(1 + \mu Q)(1 + \mu \epsilon^2)}, \\ Q_3 &= Q \cdot \frac{1 + \mu \epsilon^2}{(1 + \epsilon^2 Q)(1 + \mu Q)}, \end{aligned}$$

and  $\mu = Q^{-1}q^n, \epsilon^2 = e^{-T}$ . This is the mirror [47] of the so-called closed vertex geometry, whose web diagram is shown in Figure 1.

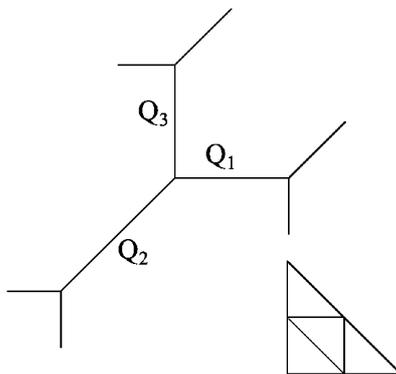


Figure 1. The web diagram for the close vertex geometry. Three  $\mathbb{P}$ 's with size  $Q_1, Q_2, Q_3$  appear symmetrically.

There are two interesting observations on this result. First, (3.9) coincides with the mirror map for this geometry. Second, the curve (3.8) is symmetric under exchanges of  $Q, \mu = Q^{-1}q^n$  and  $\epsilon^2 = e^{-T}$ . Namely, (1) the original Kähler moduli  $Q$  of the resolved conifold, (2) the chamber parameter  $n$  and (3) the 't Hooft parameter  $T$  appear symmetrically in the spectral curve. This is an interesting result, which suggests a possible connection between continuum limit of the wall crossing formulas and the BCOV holomorphic anomaly equation [48].

In the matrix model we are interested in the limit of  $N \rightarrow \infty$ , which means  $T \rightarrow \infty$  or equivalently  $\epsilon \rightarrow 0$ . With appropriate shifts of  $x$  and  $y$ , the equation (3.8) in this limit becomes

$$(3.10) \quad \mu e^{2y} + e^{x+y} + e^x + (1 + Q\mu) e^y + Q = 0.$$

The is the mirror of the so-called Suspended Pinched Point (SPP) geometry, with  $Q$  and  $\mu$  being exponentials of flat coordinates representing sizes of its two  $\mathbb{P}^1$ 's, which encode

two copies of the initial  $\mathcal{O}(-1, -1) \rightarrow \mathbb{P}^1$  geometry, see figure 2. Not only does the spectral curve agree with the mirror curve of the SPP geometry in the limit of  $g_s \rightarrow 0$ , but in fact the matrix integral reproduces the full topological string partition function all orders in  $g_s$  expansion. Indeed, it is known that the SPP topological string partition function<sup>13</sup>, with Kähler parameters  $Q$  and  $\mu$ , is equal to

$$(3.11) \quad Z_{\text{top}}^{\text{SPP}}(q, Q, \mu) = \prod_{k=1}^{\infty} \frac{(1 - Qq^k)^k (1 - \mu q^k)^k}{(1 - q^k)^{3k/2} (1 - \mu Q q^k)^k}.$$

On the other hand, from the explicit structure of the BPS generating function and formulas (2.15), (3.3), (3.4) and (3.5), we find that the value of the matrix integral, in the  $N \rightarrow \infty$  limit, is related to the above topological string partition function as

$$(3.12) \quad Z_{\text{matrix}}(q, Q; n) = Z_{\text{top}}^{\text{SPP}}(q, Q, \mu = Q^{-1}q^n) \cdot \prod_{k=1}^{\infty} (1 - q^k)^{k/2}.$$

This result is consistent with the philosophy of the remodeling conjecture [49], which states that a set of invariants (symplectic invariants) [50] constructed recursively from the spectral curve coincide with topological string partition function on the same geometry. Since symplectic invariants are defined by rewriting loop equations of matrix models purely in the language of spectral curves, the fact that the topological string partition function can be written as a matrix model would prove the remodeling conjecture. Indeed, this type of logic was used in [32, 51] to prove the remodeling conjecture for toric Calabi-Yau manifolds. It would be interesting to know whether similar recursion relations exist in other chambers.

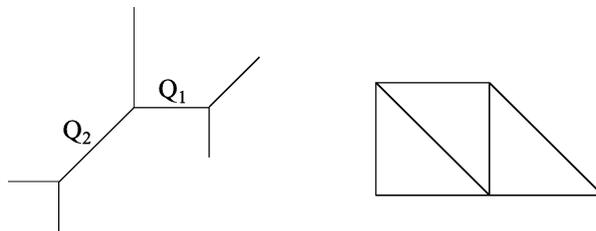


Figure 2. The web diagram for the SPP geometry. This geometry has two  $\mathbb{P}^1$ 's with size  $Q_1$  and  $Q_2$ .

<sup>13</sup>In our context, a topological string partition function is a generalized DT partition function in the topological string chamber, where generalized DT invariants coincide with the original DT invariants of [7, 8]. This chamber is the analogue of  $\theta_{n=\infty}$  in the conifold

§ 4. Derivation of the Matrix Models

There are at least two derivations of the above-mentioned matrix models, one using the vertex operator formalism for free fermions [52] (see [18, 19, 21] for discussion our context) and another using the equivalence with a vicious walker model (non-intersecting paths). Both are presented in [17]. Here we comment on the latter method in the case  $\theta = \text{id}$ . The derivation here is a slightly simplified version of the derivation in [17]. See also [32, 35, 51], which constructs similar matrix models in a particular chamber. In particular [51] treats arbitrary toric Calabi-Yau geometries.

Let us fix a sufficiently large number  $N$ . This is the same  $N$  for the size of the unitary matrix in the previous section, and in the end we take the limit  $N \rightarrow \infty$ . Define

$$(4.1) \quad h_k(t) := \lambda_{N-k+1}(t) + k - 1, \quad (k = 1, \dots, N).$$

Since  $\lambda(t)$  is a partition, we have

$$(4.2) \quad h_k(t) < h_{k+1}(t),$$

for all  $t$ . We also have the boundary condition,

$$(4.3) \quad h_k(t) = k - 1 \text{ when } |t| \text{ large.}$$

Moreover, (2.2) means we have, for each step  $t$ ,

$$(4.4) \quad h_k(t + 1) - h_k(t) = 0 \text{ or } -1,$$

for  $t \geq 0$  and

$$(4.5) \quad h_k(t + 1) - h_k(t) = 0 \text{ or } 1,$$

for  $t < 0$ .

Suppose that we fix a large positive (negative) integer  $t_{\max}$  ( $t_{\min}$ ). We are going to send these numbers of infinity. If we plot the value of  $\{h_k(t)\}_{t=t_{\min}}^{t_{\max}}$  for each  $k$ , we have a set of  $N$  paths. Due to the conditions (4.4), (4.5) the paths move on the graph shown in Figure 3, and (4.3) means we have a fixed boundary condition. Finally, (4.2) means that  $N$  paths are non-intersecting. Summing up, we have a statistical mechanical model of non-intersecting paths (also called a vicious walker model [53]<sup>14</sup>), whose partition function is given by:

$$(4.6) \quad Z \simeq \sum_{\{h_i(t)\}: \text{non-intersecting paths on the graph}} \prod_t q^{\sum_i h_i(t)},$$

where  $\simeq$  shows that we neglected an overall multiplicative constant.

<sup>14</sup>We can also regards this model as a time evolution of  $N$  particles in one dimension. In this language the model is an exclusion process, a variant of the ASEP [54].

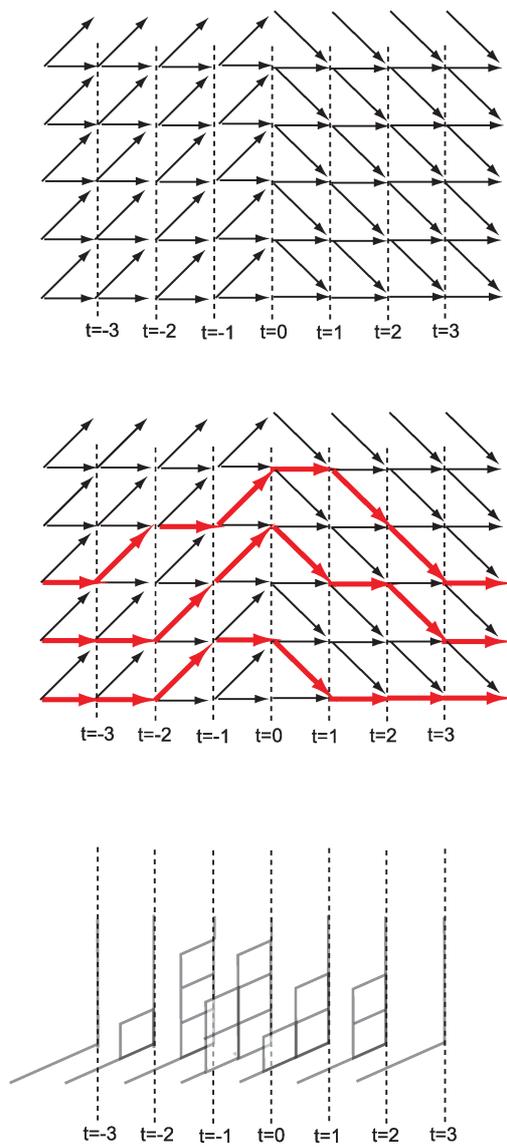


Figure 3. Top: an oriented graph for  $\mathbb{C}^3$ . Middle: an example of 3 non-intersecting paths shown as bold (red) arrows. The location of the  $k$ -th path at time  $t$  gives  $h_k(t)$ . Bottom: The corresponding evolution of Young diagrams.

At first sight it seems difficult in practice to implement the non-intersecting conditions for paths. The following theorem states that we can write the sum over non-intersecting paths as a determinant of a matrix, whose element is defined by a single path:

**Theorem 4.1** (Lindström [55], Gessel-Viennot [56]; Karlin-McGregor [57]). *Suppose we are given an oriented graph without oriented loops. Suppose moreover that each edge  $e$  comes with a weight  $w(e)$ . For a path  $p$  on the graph, we define  $w(p)$  to be the product of the weights for all the edges on the path:  $w(p) := \prod_{e \in p} w(e)$ . We define  $F$  by summing over all non-intersecting paths  $\{p_i\}$  (each  $p_i$  starts from  $a_i$  and ends at  $b_i$ ):*

$$(4.7) \quad F(\{a_i\}, \{b_i\}) = \sum_{\{p_i: a_i \rightarrow b_i\}: \text{non-intersecting}} \prod_i w(p_i),$$

and an  $N \times N$  matrix  $G(a_i, b_j)$  by

$$(4.8) \quad G(a_i, b_j) = \sum_{p: \text{a path from } a_i \text{ to } b_j} w(p).$$

Then

$$(4.9) \quad F(\{a_i\}, \{b_i\}) = \det_{i,j} (G(a_i, b_j)).$$

*Proof.* When we expand the determinant  $\det_{i,j} (G(a_i, b_j))$ , we have contributions from non-intersecting as well as intersecting paths. However, contributions from the latter cancel out because they always come in pairs with an opposite sign (Figure 4)<sup>15</sup>. □

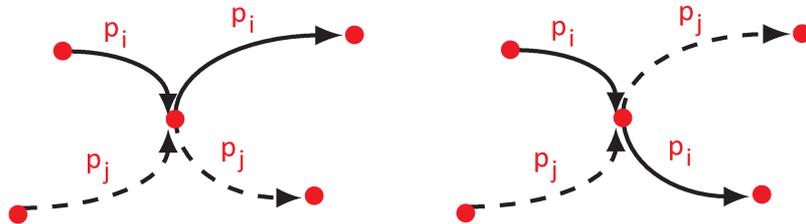


Figure 4. When we expand  $\det G$ , intersecting paths always come in pairs with an opposite sign. The reason is that we can exchange the label for paths after the intersection, without changing the paths themselves.

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<sup>15</sup>When more than two paths intersect at a single point, we need to pick two of them according to a fixed ordering and apply the same argument.

**Example 4.2.** Consider an oriented graph with weights  $w_1, \dots, w_6$  as shown in Figure 5. It is easy to see that

$$(4.10) \quad F(\{a_1, a_2\}, \{b_1, b_2\}) = (w_1 w_4)(w_3 w_5),$$

and

$$(4.11) \quad G(a_i, b_j) = \begin{pmatrix} w_1 w_4 & w_1 w_6 \\ w_2 w_4 & w_2 w_6 + w_3 w_5 \end{pmatrix}.$$

We indeed have  $\det G = w_1 w_4 (w_2 w_6 + w_3 w_5) - (w_2 w_4)(w_1 w_6) = F$ .

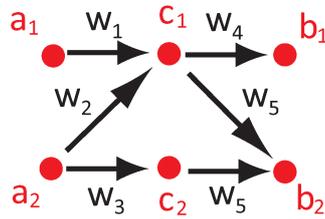


Figure 5. An example of an oriented graph.

*Remark.* The fact that paths are non-intersecting is a manifestation of free fermions, and the determinant is interpreted as a Slater determinant.

Therefore we have

$$(4.12) \quad Z_{\text{crystal}}^{\mathbb{C}^3}(q) = \det_{i,j} (G_{i,j}(q)),$$

where  $G_{i,j}(q)$  is defined as a weighted sum over all possible paths which start at height  $i$  at  $t = t_{\min}$  and end at height  $j$  at  $t = t_{\max}$ . As we can see from Figure 3,  $G(a_i, b_j)$  depend only on the difference  $i - j$ , and we thus have

$$(4.13) \quad Z_{\text{crystal}}^{\mathbb{C}^3}(q) = \det_{i,j} (G_{i-j}(q)).$$

It turns out to be easier to write a generating function for  $G_n$  than to write each  $G_n$  separately:

$$(4.14) \quad f(z) := \sum_n G_n z^n = \prod_n (1 + z q^n) \prod_n (1 + z^{-1} q^{n+1}).$$

*Proof.* To see this, note that a term in the expansion of the product is in one-to-one correspondence with a path. For example, for  $t < 0$  we take either 1 or  $zq^t$  from the product, and the choice corresponds to the two possibilities in (4.5). The change of the

horizontal coordinates is measured by  $z$ , and taking the coefficient in front of  $z^n$  means summing paths with height change  $n$ . The product in (4.14) is over all non-negative integers  $n$  when we send  $t_{\min} \rightarrow -\infty, t_{\max} \rightarrow \infty$ .  $\square$

Finally, we have the following theorem:

**Theorem 4.3** (Heine [58], Szegö [59]). *Suppose that  $f(z) = \sum_n G_n z^n$ . We then have*

$$(4.15) \quad \int_{U(N)} dU \det f(U) = \det_{1 \leq i, j \leq N} G_{i-j}$$

*Remark.* The RHS of the equation is often called a Toeplitz determinant of  $f$ .

*Proof.* Diagonalize the unitary matrix  $U$  to be  $(e^{\sqrt{-1}\phi_1}, \dots, e^{\sqrt{-1}\phi_N})$ . Then the integral  $\int dU$  reduces to  $\int \prod d\phi_i \det_{i,j}(e^{\sqrt{-1}i\phi_j}) \det_{i,j}(e^{-\sqrt{-1}i\phi_j})$ , while the integrand becomes a product  $\prod_i f(e^{\sqrt{-1}\phi_i})$ . After expanding the two determinants using the definition of the determinant, we can easily carry out the integral, and the result follows.  $\square$

This theorem, together with the form of  $f(z)$  in (4.14), completes the derivation of the matrix model for  $\mathbb{C}^3$ .

The analysis for the conifold is essentially the same, so let us summarize the result briefly. By defining  $h_k(n)$  again as in (4.1), we again have (4.2) and (4.3), except that (4.4), (4.5) are going to be replaced by

1. When  $t$  is odd,

$$(4.16) \quad h_k(t+1) - h_k(t) = \begin{cases} 0, & 1 & (t < 0), \\ 0, & -1 & (t \geq 0). \end{cases}$$

2. When  $t$  is even,

$$(4.17) \quad \dots \leq h_{k-1}(t+1) < h_k(t) \leq h_k(t+1) < h_{k+1}(t) \leq \dots$$

for  $t < 0$  and

$$(4.18) \quad \dots \leq h_{k-1}(t) < h_k(t+1) \leq h_k(t) < h_{k+1}(t+1) \leq \dots$$

for  $t \geq 0$ .

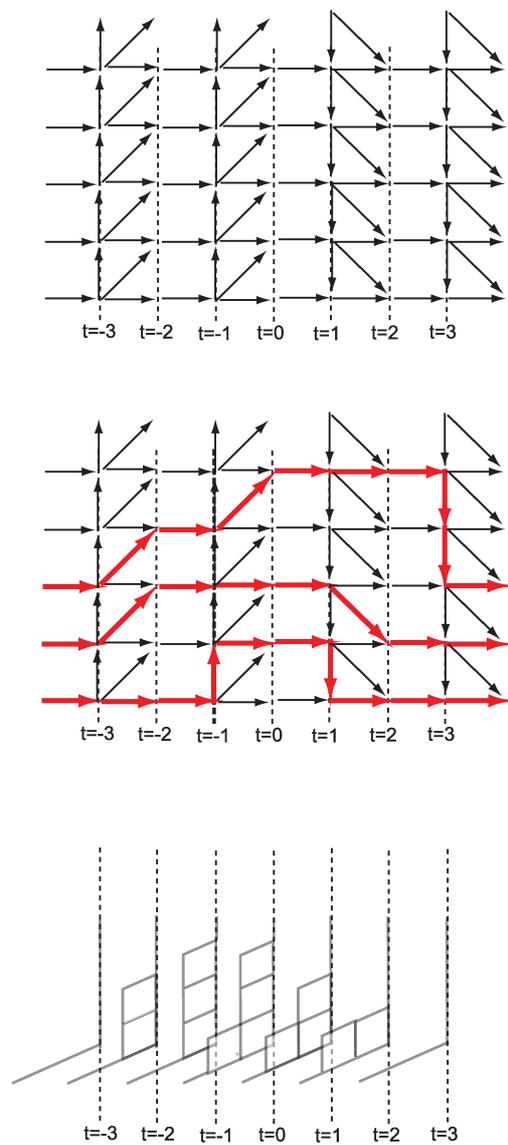


Figure 6. Top: an oriented graph for the conifold. Middle: an example of 3 non-intersecting paths on the graph shown in red. Bottom: the corresponding evolution of Young diagrams.

These conditions mean  $\{h_k(n)\}$  move on the graph shown in Figure 6. Note that the structure of the graph is different depending on whether  $t$  is even or odd.

Again by using Theorems 4.1 and 4.3, we have

$$(4.19) \quad Z = \int dU \det f(U),$$

where

$$(4.20) \quad f(U) = \prod_n \frac{1 + (q_0 q_1)^n z}{1 - (q_0 q_1)^n q_1 z} \prod_n \frac{1 + (q_0 q_1)^{n+1} z^{-1}}{1 - (q_0 q_1)^{n+1} q_1 z^{-1}}.$$

This is nothing but the expression (3.5) for  $n = 0$ .

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### § A. Generalized Donaldson-Thomas Invariants

In this appendix we briefly summarize the ingredients of the generalized Donaldson-Thomas invariants. Our discussion in this section is far from rigorous and at best schematic, since the main focus of this paper is more on combinatorial aspects presented in the main text.

For the definition of generalized DT invariants, we need the following:

- $X$ : a Calabi-Yau 3-fold.
- a “charge lattice”:

$$H_{\text{even}}(X; \mathbb{Z}) = H_0(X; \mathbb{Z}) \oplus H_2(X; \mathbb{Z}) \oplus H_4(X; \mathbb{Z}) \oplus H_6(X; \mathbb{Z}).$$

- complexified Kähler moduli of  $X$ :

$$t_i = B_i + \sqrt{-1} k_i, \quad (i = 1, \dots, \dim H_2(X; \mathbb{Z})),$$

where the real part  $B_i$  (imaginary part  $k_i$ ) denotes the B-field flux through (the volume of) the  $i$ -th 2-cycle.

- A central charge function  $Z_\gamma(t)$ , which depend on  $t := \{t_i\}$  and linearly on  $\gamma \in H_{\text{even}}(X; \mathbb{Z})$ <sup>16</sup>.

With these data we can define “generalized DT invariants”

$$\Omega(\gamma; t) \in \mathbb{Q}.$$

For concreteness, in this paper we restrict ourselves to the following situations:

- $X$ : a toric<sup>17</sup> Calabi-Yau 3-fold without compact 4-cycles. For example,  $X$  can be  $\mathbb{C}^3$  or the (resolved) conifold.
- In the charge lattice  $H_{\text{even}}(X; \mathbb{Z})$  we only consider the following set of charges<sup>18</sup>:

$$\begin{aligned} H_{\text{even}}(X; \mathbb{Z}) &= H_0(X; \mathbb{Z}) \oplus H_2(X; \mathbb{Z}) \oplus H_4(X; \mathbb{Z}) \oplus H_6(X; \mathbb{Z}) \\ \gamma &= ( n, \quad \beta = \{\beta_i\}, \quad 0, \quad 1 ). \end{aligned}$$

We then have a set of integer invariants

$$\Omega(\gamma = (n, \beta, 0, 1); t) \in \mathbb{Z}.$$

Instead of studying these invariants separately, it is useful to define their generating function:

$$(A.1) \quad Z_{\text{gDT}}(q, Q; t) = \sum_{n, \beta} \Omega(\gamma = (n, \beta, 0, 1); t) q^n Q^\beta,$$

where  $Q := \{Q_i\}$  denotes a set of parameters and  $Q^\beta := \prod_i Q_i^{\beta_i}$ . This is the partition function for generalized DT invariants studied in the main text.

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<sup>16</sup>This is part of the data for the stability conditions [60].

<sup>17</sup>Toric here means local toric, i.e. it is a canonical bundle over a complex 2-dimensional toric variety.

<sup>18</sup>Here  $H_0$  and  $H_2$  correspond to compact 0- and 2-cycles, respectively, and  $H_6$  to  $X$  itself, which is noncompact.

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