RIMS Kôkyûroku Bessatsu
B30 (2012), 001–023

Integrability of soliton equations: from discrete to ultradiscrete

Ryogo Hirota
Emeritus Professor Waseda University

Abstract

We consider a $1+1$ dimensional discrete bilinear equation,

$$\cosh(\frac{1}{2}\alpha D_m)[\sinh^2(\frac{1}{2}D_m) - \delta^2\sinh^2(\frac{1}{2}D_n)]f\cdot f = 0, \tag{1}$$

where α is a non-negative integer. Equation (1) is the discrete-time Toda equation and the discrete-time Toda equation of BKP type for $\alpha = 0$ and for $\alpha = 1$ respectively. However it is not integrable for $\alpha = 2$.
Equation (1) can be transformed into a discrete nonlinear wave equatin for $\alpha = 0, 1, 2$, which can be mapped explicitly. It can be transformed into a ultradiscrete form for $\alpha = 0, 1$ except for $\alpha = 2$.
We show a convexity of $\tau-$ function of Eq. (1) for $\alpha = 2$ in the ultradiscrete limit. Then we show that in the ultradiscrete limit, Eq.(1) for $\alpha = 2$ is transformed into an integrable equation by virtue of the convexity of the ultradiscrete τ-function.

1 Introduction

It is known that a system of differential equations is integrable if it has an enough number of conserved quantities. Consider a second order differential equation. It is integrable if it has a conserved quantity. However it is incorrect for the discrete case. Consider a coupled first order discrete equations for example,[1]

$$\begin{cases} x_{n+1} = & (x_n^2 + c\, y_n^2)c(1-x_n)/y_n, \\ y_{n+1} = & (x_n^2 + c\, y_n^2)x_n(1-x_n)/y_n^2, \end{cases} \tag{2}$$

c being a constant, which has a conserved quantity,

$$H = \frac{x_n^2 + c\, y_n^2}{x_n y_n}. \tag{3}$$

Nevertheless the equation is not integrable because that Eq.(2) is reduced, using the conserved quantity, to

$$x_{n+1} = H\ cx_n(1 - x_n), \tag{4}$$

which is the well-known equation showing chaos.
The situation is worse when we go from discrete to ultradiscrete systems. Number of independent conserved quantities of a utradiscrete system is less than that of a discrete system. The independent conserved quantities in the discrete soliton equations such as

$$\begin{aligned} H^{(1)} &= V_1 + V_2 + \cdots + V_N, & (5)\\ H^{(2)} &= V_1^2 + V_2^2 + \cdots + V_N^2, & (6)\\ &\cdots\cdots\cdots, \end{aligned}$$

are reduced, in the ultradiscrete limit, to

$$\begin{aligned} h^{(1)} &= \max(v_1, v_2, \cdots, v_N), & (7)\\ h^{(2)} &= 2\max(v_1, v_2, \cdots, v_N), & (8)\\ &\cdots\cdots\cdots. \end{aligned}$$

These conserved quantities are equivalent to each other.
Under the circumstances it is difficult, using the conserved quantities, to discuss the integrability of discrete soliton equations. Instead of the conserved quantities we use muti-soliton solution to discrete soliton equations. We use the following simple hyposesis.

Discrete and ultradiscrete soliton equations are integrable if they exhibits 3-soliton solutions.

In Section 2 we shall review the method of finding 3-soliton solution to a discrete soliton equation.

Since the discovery of the method of transforming discrete systems into ultradiscrete systems [2] in 1996, many ultradiscrete soliton equations have been studied [3]-[6]. However these soliton equations are limited to the discrete KP equation, which is expressed by the following bilinear form

$$[z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3)] f \cdot f = 0, \tag{9}$$

where D_j for $j = 1, 2, 3$ are the bilinear operators [7] and z_j for $j = 1, 2, 3$ satisfies the relation $z_1 + z_2 + z_3 = 0$ for soliton solutions.

On the other hand we have the discrete BKP equation found by Miwa [9], which is expressed by

$$\begin{aligned}&(a+b)(a+c)(b-c)\tau(l+1,m,n)\tau(l,m+1,n+1)\\&+(b+c)(b+a)(c-a)\tau(l,m+1,n)\tau(l+1,m,n+1)\\&+(c+a)(c+b)(a-b)\tau(l,m,n+1)\tau(l+1,m+1,n)\\&+(a-b)(b-c)(c-a)\tau(l+1,m+1,n+1)\tau(l,m,n)=0.\end{aligned}\tag{10}$$

Ultradiscretization of the discrete BKP equation is very difficult in its present form because of negative terms in Eq.(10).
Equation (10) is expressed, using the bilinear operators, by

$$\begin{aligned}&\{z_1\exp[\frac{1}{2}(D_l-D_m-D_n)]+z_2\exp[\frac{1}{2}(-D_l+D_m-D_n)]+z_3\exp[\frac{1}{2}(-D_l-D_m+D_n)]\\&+z_4\exp[\frac{1}{2}(D_l+D_m+D_n)]\}\tau(l,m,n)\cdot\tau(l,m,n)=0,\end{aligned}\tag{11}$$

where

$$\begin{aligned}&z_1=(a+b)(a+c)(b-c),\quad z_2=(b+c)(b+a)(c-a),\\&z_3=(c+a)(c+b)(a-b),\quad z_4=(a-b)(b-c)(c-a).\end{aligned}\tag{12}$$

We consider a generalized form of Eq.(11) which is expressed by

$$[z_1\exp(D_1)+z_2\exp(D_2)+z_3\exp(D_3)+z_4\exp(D_4)]f\cdot f=0\tag{13}$$

where D_j for $j=1,2,3,4$ are linear combinations of the operators D_l, D_m and D_n satisfying the relation

$$D_1+D_2+D_3+D_4=0,\tag{14}$$

and the coefficients z_j for $j=1,2,3,4$ are arbitrary constants. For soliton solutions they satisfy the relation

$$z_1+z_2+z_3+z_4=0.\tag{15}$$

We call Eq.(13) g-dBKP(generalized discrete BKP) equation. Recently the Bäcklund transformation for g-dBKP equation was obtained [10] which is an strong indication of the integrability of g-dBKP equation.
We shall transform a special case of g-dBKP equation into an ultradiscrete form in Section 3.

2 Perturbational method of solving bilinear equations

We note that the bilinear equation,

$$F(D_m, D_n)f \cdot f = 0, \tag{16}$$

where $F(D_m, D_n)$ is an even function of D_m, D_n, namely $F(D_m, D_n) = F(-D_m, -D_n)$ and $F(0,0) = 0$, has the following properties.

1. Equation (16) is invariant under the transformation,

$$f \to f e^{\eta_0}, \quad \eta_0 = a_0 m + b_0 n + c_0, \tag{17}$$

   where a_0, b_0, c_0 are constants, which is the gauge invariance of the bilinear equation. One of the main reasons why perturbational approach works is the gauge invariance of the bilinear equation.

2. Equation (16) is invariant by changing the order of f and g,

$$F(D_m, D_n)f \cdot g = F(D_m, D_n)g \cdot f. \tag{18}$$

N-soliton solution f to the bilinear equation is obtained by using the perturbational method. We expand f in a power series of s

$$f = 1 + sf_1 + s^2 f_2 + s^3 f_3 + \cdots, \tag{19}$$

where s is a bookkeeping parameter.
Substituting f into eq.(16) we have

$$\begin{aligned}
&F(D_m, D_n)(1 + sf_1 + s^2 f_2 + s^3 f_3 + \cdots) \cdot (1 + sf_1 + s^2 f_2 + s^3 f_3 + \cdots) \\
&= F(D_m, D_n)[1 \cdot 1 + s(f_1 \cdot 1 + 1 \cdot f_1) + s^2(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) \\
&+ s^3(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) + \cdots] \\
&= 0.
\end{aligned} \tag{20}$$

We have the following linear equations for $f_1, f_2, f_3 \cdots$ at each order of s:

$$s: \ F(D_m, D_n)(f_1 \cdot 1 + 1 \cdot f_1) \quad = 2F(\partial_m, \partial_n)f_1 = 0, \tag{21}$$

$$\begin{aligned}
&s^2: \ F(D_m, D_n)(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) \\
&\quad = 2F(\partial_m, \partial_n)f_2 + F(D_m, D_n)f_1 \cdot f_1 = 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&s^3: \ F(D_m, D_n)(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) \\
&\quad = 2F(\partial_m, \partial_n)f_3 + 2F(D_m, D_n)f_2 \cdot f_1] = 0,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&s^4: \ F(D_m, D_n)(f_4 \cdot 1 + f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3 + 1 \cdot f_4) \\
&\quad = 2F(\partial_m, \partial_n)f_4 + 2F(D_m, D_n)f_3 \cdot f_1 + F(D_m, D_n)f_2 \cdot f_2 = 0,
\end{aligned} \tag{24}$$

$\cdots\cdots.$

2.1 One-soliton solution to the discrete equation

One-soliton solution to Eq.(16) is obtained assuming the following form of f,

$$f = 1 + e^{\eta_1(m,n)}, \quad \eta_1(m,n) = \omega_1 m + k_1(n - n_1), \tag{25}$$

where ω_1 and k_1 are nonlinear frequency and wave number, respectively, n_1 being an arbitrary parameter related to a position of a soliton.
We have

$$\frac{\partial}{\partial m} e^{\eta_1(m,n)} = \omega_1\, e^{\eta_1(m,n)}, \quad \frac{\partial}{\partial n} e^{\eta_1(m,n)} = k_1\, e^{\eta_1(m,n)}. \tag{26}$$

Accordingly Eq.(16) becomes

$$\begin{aligned} & F(D_m, D_n)[1 + e^{\eta_1(m,n)}] \cdot [1 + e^{\eta_1(m,n)}] \\ &= F(D_m, D_n)[e^{\eta_1(m,n)} \cdot 1 + 1 \cdot e^{\eta_1(m,n)}] \\ &= 2F(\omega_1, k_1) e^{\eta_1(m,n)} = 0. \end{aligned} \tag{27}$$

which gives the dispersion relation,

$$F(\omega_1, k_1) = 0. \tag{28}$$

2.2 2-soliton solution

Equation(21) at order s is a linear equation of f_1:

$$F(D_m, D_n)(f_1 \cdot 1 + 1 \cdot f_1) = 0, \tag{29}$$

so that the superposition formula of solutions holds. 2-soliton solution is given by assuming f_1 to be a sum of e^{η_1} and e^{η_2}:

$$f_1 = e^{\eta_1} + e^{\eta_2}, \tag{30}$$

$$\eta_1 = \omega_1 t + k_1(n - n_1),\ \eta_2 = \omega_2 t + k_2(n - n_2). \tag{31}$$

Substituting f_1 into Eq.(29) we have

$$F(D_m, D_n)(e^{\eta_1} \cdot 1 + e^{\eta_2} \cdot 1) = 0, \tag{32}$$

which gives the dispersion relation,

$$F(\omega_j, k_j) = 0,\ \text{for } j = 1, 2. \tag{33}$$

Equation(22) at order s^2 is a linear equation for f_2:

$$F(D_m, D_n)(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0. \tag{34}$$

Substituting f_1 into Eq.(34) we find

$$F(\partial_m, \partial_n)f_2 + F(\omega_1 - \omega_2, k_1 - k_2)e^{\eta_1+\eta_2} = 0. \tag{35}$$

We assume a special solution $f_2 = e^{a_{12}+\eta_1+\eta_2}$ to Eq.(35). The coefficient a_{12} is determined to be

$$e^{a_{12}} = -\frac{F(\omega_1 - \omega_2, k_1 - k_2)}{F(\omega_1 + \omega_2, k_1 + k_2)}. \tag{36}$$

Equation (23) at order s^3 is a linear equation for f_3:

$$F(D_m, D_n)(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0. \tag{37}$$

Equation(24) at order s^4 is a linear equation for f_4:

$$F(D_m, D_n)(f_4 \cdot 1 + f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3 + 1 \cdot f_4) = 0. \tag{38}$$

We may choose f_3 and f_4 to be zero because that the inhomogenious terms in Eqs. (37) and (38) vanish by virtue of the gauge invariance. Accordingly we find that 2-soliton solution is given by

$$\begin{aligned}
&f = 1 + e^{\eta_1} + e^{\eta_2} + e^{a_{12}+\eta_1+\eta_2},\\
&F(\omega_j, k_j) = 0, \text{ for } j = 1, 2,\\
&e^{a_{12}} = -\frac{F(\omega_1 - \omega_2, k_1 - k_2)}{F(\omega_1 + \omega_2, k_1 + k_2)}.
\end{aligned}$$

2.3 3-soliton solution

We have the linear equation (21) at order s:

$$F(D_m, D_n)f_1 \cdot 1 = 0. \tag{39}$$

3-soliton solution is given by assuming f_1 to be a sum of e^{η_1}, e^{η_2} and e^{η_3}:

$$f_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}, \tag{40}$$

$$\eta_j = \omega_j m + k_j(n - n_j), \text{ for } j = 1, 2, 3. \tag{41}$$

Substituting f_1 into Eq.(39) we have

$$F(D_m, D_n)(e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \cdot 1 = 0, \tag{42}$$

which gives the dispersion relation,

$$F(\omega_j, k_j) = 0, \text{ for } j = 1, 2, 3. \tag{43}$$

Equation(22) at order s^2 is a linear equation for f_2:

$$F(D_m, D_n)(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0. \tag{44}$$

Substituting f_1 into Eq.(44) the inhomogeneous term becomes

$$\begin{aligned}
&F(D_m, D_n)(e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \cdot (e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \\
&\quad = 2e^{2\eta_2} F(D_m, D_n) e^{\eta_1 - \eta_2} \cdot 1 + 2e^{2\eta_3} F(D_m, D_n) e^{\eta_1 - \eta_3} \cdot 1 \\
&\qquad + 2e^{2\eta_3} F(D_m, D_n) e^{\eta_2 - \eta_3} \cdot 1 \\
&\quad = 2F(\omega_1 - \omega_2, k_1 - k_2) e^{\eta_1 + \eta_2} + 2F(\omega_1 - \omega_3, k_1 - k_3) e^{\eta_1 + \eta_3} \\
&\qquad + 2F(\omega_2 - \omega_3, k_2 - k_3) e^{\eta_2 + \eta_3}.
\end{aligned} \tag{45}$$

Substituting (45) into Eq.(44) we obtain

$$\begin{aligned}
&F(\partial_m, \partial_n) f_2 + F(\omega_1 - \omega_2, k_1 - k_2) e^{\eta_1 + \eta_2} \\
&\quad + F(\omega_1 - \omega_3, k_1 - k_3) e^{\eta_1 + \eta_3} + F(\omega_2 - \omega_3, k_2 - k_3) e^{\eta_2 + \eta_3} = 0.
\end{aligned} \tag{46}$$

We assume f_2 to be

$$f_2 = e^{a_{12} + \eta_1 + \eta_2} + e^{a_{13} + \eta_1 + \eta_3} + e^{a_{23} + \eta_2 + \eta_3}. \tag{47}$$

Substituting this expression into Eq.(44) the phase-shifts a_{12}, a_{13}, a_{23} are determined to be

$$e^{a_{ij}} = -\frac{F(\omega_i - \omega_j, k_i - k_j)}{F(\omega_i + \omega_j, k_i + k_j)}, \text{ for } 1 \le i < j \le 3. \tag{48}$$

Equation(23) at order s^3 is a linear equation for f_3:

$$F(D_m, D_n)(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0. \tag{49}$$

We calculate the inhomogeneous term in Eq.(49):

$$\begin{aligned}
&F(D_m, D_n)(f_2 \cdot f_1 + f_1 \cdot f_2) \\
&= 2F(D_m, D_n)(e^{a_{12} + \eta_1 + \eta_2} + e^{a_{13} + \eta_1 + \eta_3} + e^{a_{23} + \eta_2 + \eta_3}) \cdot (e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \\
&= 2e^{a_{12}} F(D_m, D_n) e^{\eta_1 + \eta_2} \cdot e^{\eta_3} \\
&\quad + 2e^{a_{13}} F(D_m, D_n) e^{\eta_1 + \eta_3} \cdot e^{\eta_2} \\
&\quad + 2e^{a_{23}} F(D_m, D_n) e^{\eta_2 + \eta_3} \cdot e^{\eta_1} \\
&= 2e^{\eta_1 + \eta_2 + \eta_3} \{ e^{a_{12}} F(\omega_1 + \omega_2 - \omega_3, k_1 + k_2 - k_3) \\
&\quad + e^{a_{13}} F(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3) \\
&\quad + e^{a_{23}} F(-\omega_1 + \omega_2 + \omega_3, -k_1 + k_2 + k_3) \}.
\end{aligned} \tag{50}$$

We assume f_3 to be

$$f_3 = e^{a_{123}+\eta_1+\eta_2+\eta_3}. \tag{51}$$

Substituting these expressions into Eq.(49) we find that a_{123} is determined by the relation:

$$\begin{aligned}
&e^{a_{123}}F(\omega_1+\omega_2+\omega_3, k_1+k_2+k_3)\\
&+e^{a_{12}}F(\omega_1+\omega_2-\omega_3, k_1+k_2-k_3)\\
&+e^{a_{13}}F(\omega_1-\omega_2+\omega_3, k_1-k_2+k_3)\\
&+e^{a_{23}}F(-\omega_1+\omega_2+\omega_3, -k_1+k_2+k_3) = 0.
\end{aligned} \tag{52}$$

Here we make an important assumption that a_{123} is equal to a sum of
a_{12}, a_{13}, a_{23},

$$a_{123} = a_{12} + a_{13} + a_{23}. \tag{53}$$

Hence we have an identity Δ to be satisfied by the parameters $\omega_1, k_1, \omega_2, k_2$ and ω_3, k_3,

$$\begin{aligned}
&\Delta(\omega_1,\omega_2,\omega_3,k_1,k_2,k_3) \equiv\\
&\quad e^{a_{12}+a_{13}+a_{23}}F(\omega_1+\omega_2+\omega_3, k_1+k_2+k_3)\\
&+e^{a_{12}}F(\omega_1+\omega_2-\omega_3, k_1+k_2-k_3) + e^{a_{13}}F(\omega_1-\omega_2+\omega_3, k_1-k_2+k_3)\\
&+e^{a_{23}}F(-\omega_1+\omega_2+\omega_3, -k_1+k_2+k_3) = 0,
\end{aligned} \tag{54}$$

where

$$e^{a_{ij}} = -\frac{F(\omega_i-\omega_j, k_i-k_j)}{F(\omega_i+\omega_j, k_i+k_j)}, \quad \text{for } \; 1 \le i < j \le 3. \tag{55}$$

The identity(54) is crucial for the bilinear equation to exhibit 3-soliton solution.

2.4 N-soliton solution

In general, N-soliton solution to Eq.(16) is expressed by

$$f = \sum_{\mu=0,1} \exp\,[\sum_{i=1}^{N} \mu_i\eta_i + \sum_{1\le i<j\le N} a_{ij}\mu_i\mu_j] \tag{56}$$

where $\sum_{\mu=0,1}$ means the summation over all possible combinations of $\mu_1 = 0,1,\ \mu_2 = 0,1,\cdots,\mu_N = 0,1$, provided that $F(D_m.D_n)$ satisfies the condition

$$\sum_{\sigma=\pm1} \exp\,[\sum_{1\le i<j\le N} \frac{1}{2}(1+\sigma_i\sigma_j)a_{ij}]\; F(\sum_{i=1}^{N}\sigma_i\omega_i, \sum_{i=1}^{N}\sigma_i k_i) = 0, \tag{57}$$

where $\sum_{\sigma=\pm 1}$ means the summation over all possible combinations of $\sigma_1 = \pm 1, \ \sigma_2 = \pm 1, \cdots, \sigma_N = \pm 1$.[1] We have obtained the condition (54) using the fact that $F(D_m, D_n)$ is an even fuction of D_m and D_n. Accordingly the total number of terms in (54) is one half of that in (57).

3 Extended discrete Toda equation

We consider a $1+1$ dimensional discrete bilinear equation,

$$\cosh(\frac{1}{2}\alpha D_m)[\sinh^2(\frac{1}{2}D_m) - \delta^2 \sinh^2(\frac{1}{2}D_n)]f \cdot f = 0, \tag{58}$$

where m and n are the discrete time and space respectively and α is a non-negative integer and δ is a non-negative parameter less than unity which is related to a time-interval.

Equation (58) is the discrete-time Toda equation of type I for $\alpha = 0$, which is a member of the discrete KP equation. And it is the discrete-time Toda equation of BKP type for $\alpha = 1$ [8], which is a member of Eq.(13), as we show in the next subsection.
We shall show in Subsection 3.4 that Eq.(58) for $\alpha = 2$ does not satisfy the integrability condition stated in Subsection 2.3. Accordingly it is not integrable for $\alpha = 2$.

Hereafter we call Eq. (58) "E-Toda equation" (Extended Toda equation).
We show that E-Toda equation can be transformed into an ordinary wave equation.

3.1 Transformation of E-Toda equation into the ordinary form

We write E-Toda equation as

$$\begin{aligned}
&4\cosh(\alpha\frac{1}{2}D_m)[\sinh^2(\frac{1}{2}D_m) - \delta^2 \sinh^2(\frac{1}{2}D_n)]f \cdot f \\
&= 2\cosh(\alpha\frac{1}{2}D_m)[\cosh(D_m) - \delta^2\cosh(D_n) - 1 + \delta^2]f \cdot f \\
&= \{\cosh[\frac{1}{2}(\alpha+2)D_m] + \cosh[\frac{1}{2}(\alpha-2)D_m] \\
&-\delta^2[\cosh(\frac{1}{2}\alpha D_m + D_n) + \cosh(\frac{1}{2}\alpha D_m - D_n)]
\end{aligned}$$

[1] "The Direct Method in Soliton Theory", Cambridge Univ.Press(2004), p. 55 under Eq.(1.250), "$\sigma_1 = 0, 1, \ \sigma_2 = 0, 1, \cdots, \sigma_N = 0, 1$." should be read "$\sigma_1 = \pm 1, \ \sigma_2 = \pm 1, \cdots, \sigma_N = \pm 1$".

$$-2(1-\delta^2)\cosh(\frac{1}{2}\alpha D_m)\}f\cdot f=0,$$

which is equal to

$$\begin{aligned}&\{\exp[(\frac{1}{2}\alpha+1)D_m]+\exp[(\frac{1}{2}\alpha-1)D_m]\}f\cdot f\\&=\{\delta^2\exp[\frac{1}{2}\alpha D_m+D_n]+\delta^2\exp[\frac{1}{2}\alpha D_m-D_n]+2(1-\delta^2)\exp[\frac{1}{2}\alpha D_m]\}f\cdot f.\end{aligned}\tag{59}$$

Accordingly E-Toda equation is expressed without using the bilinear operators as

$$\begin{aligned}&f_n^{m+\frac{1}{2}\alpha+1}f_n^{m-\frac{1}{2}\alpha-1}+f_n^{m+\frac{1}{2}\alpha-1}f_n^{m-\frac{1}{2}\alpha+1}\\&=\delta^2[f_{n+1}^{m+\frac{1}{2}\alpha}f_{n-1}^{m-\frac{1}{2}\alpha}+f_{n-1}^{m+\frac{1}{2}\alpha}f_{n+1}^{m-\frac{1}{2}\alpha}]+2(1-\delta^2)f_n^{m+\frac{1}{2}\alpha}f_n^{m-\frac{1}{2}\alpha}.\end{aligned}\tag{60}$$

We note that Eq.(59) for $\alpha=1$ is reduced to

$$\begin{aligned}&\exp[(3/2)D_m]f\cdot f\\&=\{\delta^2\exp[\frac{1}{2}D_m+D_n]+\delta^2\exp[\frac{1}{2}D_m-D_n]+(1-2\delta^2)\exp[\frac{1}{2}D_m]\}f\cdot f,\end{aligned}\tag{61}$$

which is a g-dBKP equation.

In order to transform Eq.(60) into a discrete soliton equation of ordinary form, we divide Eq.(60) by $f_n^{m+\frac{1}{2}\alpha}f_n^{m-\frac{1}{2}\alpha}$ and shift m by $\frac{1}{2}\alpha$, namely $m\to m+\frac{1}{2}\alpha$. Then we have

$$\frac{f_n^{m+\alpha+1}f_n^{m-1}}{f_n^{m+\alpha}f_n^m}+\frac{f_n^{m+\alpha-1}f_n^{m+1}}{f_n^{m+\alpha}f_n^m}=\delta^2\frac{f_{n+1}^{m+\alpha}f_{n-1}^m}{f_n^{m+\alpha}f_n^m}+\delta^2\frac{f_{n-1}^{m+\alpha}f_{n+1}^m}{f_n^{m+\alpha}f_n^m}+2(1-\delta^2).\tag{62}$$

Let us introduce dependent variables, u_n^m, v_n^m, w_n^t, and x_n^m by

$$u_n^m=\frac{f_n^{m+\alpha+1}f_n^{m-1}}{f_n^{m+\alpha}f_n^m},\quad v_n^m=\frac{f_n^{m+\alpha-1}f_n^{m+1}}{f_n^{m+\alpha}f_n^m},\tag{63}$$

$$w_n^m=\frac{f_{n+1}^{m+\alpha}f_{n-1}^m}{f_n^{m+\alpha}f_n^m},\quad x_n^m=\frac{f_{n-1}^{m+\alpha}f_{n+1}^m}{f_n^{m+\alpha}f_n^m}.\tag{64}$$

Then Eq.(62) is simply expressed by

$$u_n^m+v_n^m=\delta^2(w_n^m+x_n^m)+2(1-\delta^2).\tag{65}$$

3.2 Relations among dependent variables

In order to find relations among dependent variables u, v, w, x,we introduce shift operators p and q which operate on an arbitrary function, h_n^m of m and n as follows

$$p^\alpha h_n^m = h_n^{m+\alpha}, \quad q^\beta h_n^m = h_{n+\beta}^m, \tag{66}$$

for arbitrary constants, α and β.
We have

$$u_n^m = \frac{f_n^{m+\alpha+1} f_n^{m-1}}{f_n^{m+\alpha} f_n^m}, \quad v_n^m = \frac{f_n^{m+\alpha-1} f_n^{m+1}}{f_n^{m+\alpha} f_n^m},$$
$$w_n^m = \frac{f_{n+1}^{m+\alpha} f_{n-1}^m}{f_n^{m+\alpha} f_n^m}, \quad x_n^m = \frac{f_{n-1}^{m+\alpha} f_{n+1}^m}{f_n^{m+\alpha} f_n^m}.$$

The relations among the variables $u_n^m, v_n^m, w_n^m, x_n^m$ depend on the values of α

1. E-Toda equation ($\alpha = 0$):
   We have $v_n^m = u_n^m, \quad x_n^m = w_n^m$ and the logarithm of u and w are expressed by the shift operators as

$$\log u = (p + p^{-1} - 2) \log f, \tag{67}$$
$$\log w = (q + q^{-1} - 2) \log f, \tag{68}$$

   which give the relation

$$(p + p^{-1} - 2) \log w = (q + q^{-1} - 2) \log u. \tag{69}$$

   Accordingly we find

$$\frac{w_n^{m+1} w_n^{m-1}}{(w_n^m)^2} = \frac{u_{n+1}^m u_{n-1}^m}{(u_n^m)^2}. \tag{70}$$

2. E-Toda equation ($\alpha = 1$):
   We have $v_n^m = 1$ and the logarithm of u, w and x are expressed by the shift operators as

$$\log u = (p^2 + p^{-1} - p - 1) \log f, \tag{71}$$
$$\log w = (pq + q^{-1} - p - 1) \log f, \tag{72}$$
$$\log x = (pq^{-1} + q - p - 1) \log f, \tag{73}$$

   which give the relation

$$(p^2 + p^{-1} - p - 1) \log w = (pq + q^{-1} - p - 1) \log u \tag{74}$$

and

$$(p^2+p^{-1}-p-1)\log x=(pq^{-1}+q-p-1)\log u. \tag{75}$$

Accordingly we find

$$\frac{w_n^{m+2}w_n^{m-1}}{w_n^{m+1}w_n^m}=\frac{u_{n+1}^{m+1}u_{n-1}^m}{u_n^{m+1}u_n^m} \quad \text{and} \quad \frac{x_n^{m+2}x_n^{m-1}}{x_n^{m+1}x_n^m}=\frac{u_{n-1}^{m+1}u_{n+1}^m}{u_n^{m+1}u_n^m}. \tag{76}$$

3. E-Toda equation ($\alpha=2$):
   The logarithm of u, v, w and x are expressed by the shift operators as

$$\log u=(p^3+p^{-1}-p^2-1)\log f=(p-1)^2(p+1+p^{-1})\log f, \tag{77}$$
$$\log v=(2p-p^2-1)\log f=-(p-1)^2\log f, \tag{78}$$
$$\log w=(p^2q+q^{-1}-p^2-1)\log f, \tag{79}$$
$$\log x=(p^2q^{-1}+q-p^2-1)\log f, \tag{80}$$

which give the relations

$$(p+1+p^{-1})\log v=-\log u, \tag{81}$$
$$(p^3+p^{-1}-p^2-1)\log w=(p^2q+q^{-1}-p^2-1)\log u, \tag{82}$$
$$(p^3+p^{-1}-p^2-1)\log x=(p^2q^{-1}+q-p^2-1)\log u. \tag{83}$$

Accordingly we find

$$v_n^{m+1}v_n^m v_n^{m-1}=\frac{1}{u_n^m}, \quad \frac{w_n^{m+3}w_n^{m-1}}{w_n^{m+2}w_n^m}=\frac{u_{n+1}^{m+2}u_{n-1}^m}{u_n^{m+2}u_n^m}, \quad \frac{x_n^{m+3}x_n^{m-1}}{x_n^{m+2}x_n^m}=\frac{u_{n-1}^{m+2}u_{n+1}^m}{u_n^{m+2}u_n^m}. \tag{84}$$

3.3 Explicit mapping of E-Toda equation

Using the relations among the dependent variables we find the explicit mapping of E-Toda equation in the following forms,

1. E-Toda equation ($\alpha=0$):

$$u_n^m=\delta^2 w_n^m+(1-\delta^2), \tag{85}$$
$$\frac{w_n^{m+1}w_n^{m-1}}{(w_n^m)^2}=\frac{u_{n+1}^m u_{n-1}^m}{(u_n^m)^2}, \tag{86}$$

which give the explicit mapping of w_n^m,

$$w_n^{m+1}=\frac{u_{n+1}^m u_{n-1}^m}{(u_n^m)^2}\frac{(w_n^m)^2}{w_n^{m-1}}, \tag{87}$$

where u_n^m is given by Eq.(85).

2. E-Toda equation ($\alpha = 1$):

$$u_n^m = \delta^2(w_n^m + x_n^m) + 1 - 2\delta^2, \tag{88}$$

$$\frac{w_n^{m+1} w_n^{m-2}}{w_n^m w_n^{m-1}} = \frac{u_{n+1}^m u_{n-1}^{m-1}}{u_n^m u_n^{m-1}}, \tag{89}$$

$$\frac{x_n^{m+1} x_n^{m-2}}{w_n^m w_n^{m-1}} = \frac{u_{n-1}^m u_{n+1}^{m-1}}{u_n^m u_n^{m-1}}, \tag{90}$$

which give the explicit mapping of w_n^m and x_n^m,

$$w_n^{m+1} = \frac{u_{n+1}^m u_{n-1}^{m-1}}{u_n^m u_n^{m-1}} \frac{w_n^m w_n^{m-1}}{w_n^{m-2}}, \tag{91}$$

$$x_n^{m+1} = \frac{u_{n-1}^m u_{n+1}^{m-1}}{u_n^m u_n^{m-1}} \frac{x_n^m x_n^{m-1}}{x_n^{m-2}}, \tag{92}$$

where u_n^m is given by Eq.(88).

3. E-Toda equation ($\alpha = 2$):

$$u_n^m = -v_n^m + \delta^2(w_n^m + x_n^m) + 2(1 - \delta^2), \tag{93}$$

$$v_n^{m+1} v_n^m v_n^{m-1} = \frac{1}{u_n^m}, \tag{94}$$

$$\frac{w_n^{m+1} w_n^{m-3}}{w_n^m w_n^{m-2}} = \frac{u_{n+1}^m u_{n-1}^{m-2}}{u_n^m u_n^{m-2}}, \tag{95}$$

$$\frac{x_n^{m+1} x_n^{m-3}}{x_n^m x_n^{m-2}} = \frac{u_{n-1}^m u_{n+1}^{m-2}}{u_n^m u_n^{m-2}}, \tag{96}$$

which give the explicit mapping of v_n^m, w_n^m and x_n^m,

$$v_n^{m+1} = \frac{1}{u_n^m v_n^m v_n^{m-1}}, \tag{97}$$

$$w_n^{m+1} = \frac{u_{n+1}^m u_{n-1}^{m-2}}{u_n^m u_n^{m-2}} \frac{w_n^m w_n^{m-2}}{w_n^{m-3}}, \tag{98}$$

$$x_n^{m+1} = \frac{u_{n-1}^m u_{n+1}^{m-2}}{u_n^m u_n^{m-2}} \frac{x_n^m x_n^{m-2}}{x_n^{m-3}}, \tag{99}$$

where u_n^m is given by Eq.(93).

3.4 Integrability of E-Toda equation

Following the procedure stated in Section 2 we obtain 3-soliton solution to E-Toda equation,

$$\cosh(\frac{1}{2}\alpha D_m)[\sinh^2(\frac{1}{2}D_m) - \delta^2 \sinh^2(\frac{1}{2}D_n)]f \cdot f = 0,$$

which is expressed explicitly by

$$\begin{aligned} f_3 = 1 &+ s\{e^{\eta_1} + e^{\eta_2} + e^{\eta_3}\} + s^2\{e^{a_{12}+\eta_1+\eta_2} + e^{a_{13}+\eta_1+\eta_3} + e^{a_{23}+\eta_2+\eta_3}\} \\ &+ s^3 e^{a_{123}+\eta_1+\eta_2+\eta_3}, \end{aligned} \tag{100}$$

where $\eta_i = \omega_i m + k_i(n - n_i)$, for $i = 1, 2, 3$. The dispersion relation and the phase shifts are given by

$$[\sinh(\frac{1}{2}\omega_i)]^2 - \delta^2[\sinh(\frac{1}{2}k_i)]^2 = 0, \tag{101}$$

$$\begin{aligned} e^{a_{ij}} &= -\frac{F(\omega_i - \omega_j, k_i - k_j)}{F(\omega_i + \omega_j, k_i + k_j)} \\ &= -\frac{\cosh[\frac{1}{2}\alpha(\omega_i - \omega_j)]}{\cosh[\frac{1}{2}\alpha(\omega_i + \omega_j)]} \frac{\{[\sinh(\frac{1}{2}(\omega_i - \omega_j))]^2 - \delta^2[\sinh(\frac{1}{2}(k_i - k_j))]^2\}}{\{[\sinh(\frac{1}{2}(\omega_i + \omega_j))]^2 - \delta^2[\sinh(\frac{1}{2}(k_i + k_j))]^2\}}. \end{aligned} \tag{102}$$

We have checked the integrability condition,

$$\begin{aligned} &\Delta(\omega_1, \omega_2, \omega_3, k_1, k_2, k_3) \equiv \\ &\quad e^{a_{12}+a_{13}+a_{23}} F(\omega_1 + \omega_2 + \omega_3, k_1 + k_2 + k_3) \\ &+ e^{a_{12}} F(\omega_1 + \omega_2 - \omega_3, k_1 + k_2 - k_3) + e^{a_{13}} F(\omega_1 - \omega_2 + \omega_3, k_1 - k_2 + k_3) \\ &+ e^{a_{23}} F(-\omega_1 + \omega_2 + \omega_3, -k_1 + k_2 + k_3) = 0, \end{aligned} \tag{103}$$

using REDUCE (free software) and found that

1. E-Toda equation (for $\alpha = 0, 1$) are integrable.
2. E-Toda equation (for $\alpha = 2$) is not integable.

4 Transition of E-Toda equation from discrete to ultradiscrete

Numerical simulation of collisions of solitons in a non-integrable system shows that solitons generate ripples after colliding with each other. On the other hand in an ultradiscrete system we may choose all input values to be intergers so that all output values are integers.

Hence no ripple is generated. We have found that E-toda equation for $\alpha = 2$ is not integable. Accordingly it is of strong interest to see collisions of solitons in an ultradiscrete E-Toda equation for $\alpha = 2$.
The explicit mapping forms of E-Toda equation for $\alpha = 2$ indicate that ultradiscretization is difficult in its present form because of the negative term $-v_n^m$ in Eq.(93).
We shall investigate transition of u_n^m of Eq.(93) from discrete to ultradiscrete. For the purpose we introduce intermediate dependent variables U_n^m, V_n^m, W_n^m and X_n^m,

$$
\begin{aligned}
u_n^m &= \exp(U_n^m/\epsilon), \quad v_n^m = \exp(V_n^m/\epsilon), \\
w_n^m &= \exp(W_n^m/\epsilon), \quad x_n^m = \exp(X_n^m/\epsilon),
\end{aligned}
$$

and an intermediate soliton solution f_ϵ,

$$
f_\epsilon = \sum_{\mu=0,1} \exp\,[\sum_{i=1}^{N} \mu_i \xi_i/\epsilon + \sum_{1\le i<j\le N} A_{ij}\mu_i\mu_j/\epsilon] \tag{104}
$$

using an ultradiscrete paramete ϵ, where

$$
\xi_j = \omega_j m + k_j(n - n_j), \quad \text{for} \quad j = 1, 2, \cdots, N. \tag{105}
$$

$$
A_{ij} = \epsilon\, a_{ij}. \quad \text{for} \quad i, j = 1, 2, \cdots, N. \tag{106}
$$

The intermediate solution is obtained simply by replacing the nonlinear frequencies and wave numbers ω_j and k_j in f by ω_j/ϵ and k_j/ϵ, respectively, for $j = 1, 2, \ldots, N$. Then E-Toda equation ($\alpha = 2$) becomes the following intermediate equations

$$
V_n^{m+1} = -U_n^m - V_n^m - V_n^{m-1}, \tag{107}
$$

$$
W_n^{m+1} = U_{n+1}^m + U_{n-1}^{m-2} - U_n^m - U_n^{m-2} + W_n^m + W_n^{m-2} - W_n^{m-3}, \tag{108}
$$

$$
X_n^{m+1} = U_{n-1}^m + U_{n+1}^{m-2} - U_n^m - U_n^{m-2} + X_n^m + X_n^{m-2} - X_n^{m-3}, \tag{109}
$$

$$
\begin{aligned}
U_n^m = \epsilon \log\{&-\exp(V_n^m/\epsilon) \\
&+ \exp[(W_n^m - 2)/\epsilon] + \exp[(X_n^m - 2)/\epsilon] + 2(1 - \exp(-2/\epsilon))\}.
\end{aligned} \tag{110}
$$

Equations (107)-(110) describe E-Toda equation for $\epsilon = 1$ and the ultradiscrete E-Toda equation in the limit, $\epsilon = 0$.
We have mapped Eqs.(107)-(110) changing ϵ from 1 to 0.1 and observed no change of U_n^m. It implies that the negative term $-\exp(V_n^m/\epsilon)$ in Eq.(110) plays no role in determining U_n^m in the ultradiscrete limit, $\epsilon = 0$.

We notice the following fact. The dispersion relation between new ω_i and k_i is obtained by Eq.(28),

$$
f(\omega_i/\epsilon, k_i/\epsilon) = 0. \tag{111}
$$

We have E-Toda equation,

$$\{\exp[(\frac{1}{2}\alpha+1)D_m]+\exp[(\frac{1}{2}\alpha-1)D_m]\}f\cdot f$$
$$=\{\delta^2\exp[\frac{1}{2}\alpha D_m+D_n]+\delta^2\exp[\frac{1}{2}\alpha D_m-D_n]+2(1-\delta^2)\exp[\frac{1}{2}\alpha D_m]\}f\cdot f,$$

whose dispersion relation is given by

$$\cosh[(\frac{1}{2}\alpha+1)\omega/\epsilon]+\cosh[(\frac{1}{2}\alpha-1)\omega/\epsilon]=$$
$$\delta^2 2\cosh[\frac{1}{2}\alpha\omega/\epsilon]\cosh[k/\epsilon]+2(1-\delta^2)\cosh[\frac{1}{2}\alpha\omega/\epsilon]. \tag{112}$$

Obvioully we have for $\alpha>1$,

$$\cosh[(\frac{1}{2}\alpha-1)\omega/\epsilon]\le\cosh[\frac{1}{2}\alpha\omega/\epsilon]\le\cosh[(\frac{1}{2}\alpha+1)\omega/\epsilon], \tag{113}$$

which suggests the inequalities,

$$\cosh[(\frac{1}{2}\alpha-1)D_m/\epsilon]f\cdot f\le\cosh[\frac{1}{2}\alpha D_m/\epsilon]f\cdot f\le\cosh[(\frac{1}{2}\alpha+1)D_m/\epsilon]f\cdot f, \tag{114}$$

in UD (ultradiscrete) limit.
Let UD limit $\epsilon\to 0$ of f_ϵ be $\tau(m,n)$,

$$\lim_{\epsilon\to 0}\epsilon\log f_\epsilon=\tau(m,n). \tag{115}$$

Then $\tau(m,n)$ is expressed by

$$\tau(m,n)=\max(0,\xi_1,\xi_2,\cdots,\xi_N,A_{12}+\xi_1+\xi_2,A_{13}+\xi_1+\xi_3,\cdots,$$
$$A_{N-1,N}+\xi_{N-1}+\xi_N,\cdots,\cdots,A_{12\cdots N}+\xi_1+\xi_2+\cdots+\xi_N). \tag{116}$$

We shall prove in Appendix A that $\tau(m,n)$ is a convex function of m for fixed n.

5 E-Toda equation for $\alpha=2$ in UD limit

Let

$$\tau(m,n)=\lim_{\epsilon\to 0}\epsilon\log f_\epsilon \quad\text{and}\quad \delta=\exp(-1/\epsilon). \tag{117}$$

Then E-Toda equation in the bilinear form, Eq.(60) is expressed, in UD limit, by

$$\begin{aligned}
&\max(\ \tau(m+\frac{1}{2}\alpha+1,n)+\tau(m-\frac{1}{2}\alpha-1,n),\\
&\qquad \tau(m+\frac{1}{2}\alpha-1,n)+\tau(m-\frac{1}{2}\alpha+1,n\ )\\
&=\max(\ \tau(m+\frac{1}{2}\alpha,n+1)+\tau(m-\frac{1}{2}\alpha,n-1)-2,\\
&\qquad \tau(m+\frac{1}{2}\alpha,n-1)+\tau(m-\frac{1}{2}\alpha,n+1)-2,\\
&\qquad \tau(m+\frac{1}{2}\alpha,n)+\tau(m-\frac{1}{2}\alpha,n)\ ).
\end{aligned} \tag{118}$$

Let

$$U_0(m,n)=\lim_{\epsilon\to 0}\epsilon\log u_n^m,\quad V_0(m,n)=\lim_{\epsilon\to 0}\epsilon\log v_n^m, \tag{119}$$

$$W_0(m,n)=\lim_{\epsilon\to 0}\epsilon\log w_n^m,\quad X_0(m,n)=\lim_{\epsilon\to 0}\epsilon\log x_n^m. \tag{120}$$

Then E-Toda equation for $\alpha=2$, Eqs.(93),(97),(98) and (99) is expressed,in UD limit, by

$$\max(\ U_0(m,n),V_0(m,n)\ )=\max(\ W_0(m,n)-2,X_0(m,n)-2,\ 0\ ), \tag{121}$$

$$V_0(m+1,n)=-U_0(m,n)-V_0(m,n)-V_0(m-1,n), \tag{122}$$

$$\begin{aligned}W_0(m+1,n)&=U_0(m,n+1)+U_0(m-2,n-1)-U_0(m,n)-U_0(m-2,n)\\&\quad+W_0(m,n)+W_0(m-2,n)-W_0(m-3,n),\end{aligned} \tag{123}$$

$$\begin{aligned}X_0(m+1,n)&=U_0(m,n-1)+U_0(m-2,n+1)-U_0(m,n)-U_0(m-2,n)\\&\quad+X_0(m,n)+X_0(m-2,n)-X_0(m-3,n),\end{aligned} \tag{124}$$

where

$$U_0(m,n)=\tau(m+3,n)+\tau(m-1,n)-\tau(m+2,n)-\tau(m,n), \tag{125}$$

$$V_0(m,n)=\tau(m+1,n)+\tau(m+1,n)-\tau(m+2,n)-\tau(m,n), \tag{126}$$

$$W_0(m,n)=\tau(m+2,n+1)+\tau(m,n-1)-\tau(m+2,n)-\tau(m,n), \tag{127}$$

$$X_0(m,n)=\tau(m+2,n-1)+\tau(m,n+1)-\tau(m+2,n)-\tau(m,n). \tag{128}$$

We remark that Eq.(121) does not determine $U_0(m,n)$ uniquely if $U_0(m,n)<V_0(m,n)$. Therefore it is not a proper form of the ultradiscretized E-Toda equation. In the following we shall prove that $U_0(m,n)\ge V_0(m,n)$ using the convexity of $\tau(m,n)$ function.
We obtain using Eqs.(125) and (126)

$$U_0(m-1,n)-V_0(m-1,n)=\tau(m+2,n)+\tau(m-2,n)-2\tau(m,n). \tag{129}$$

The inequality shown in Appendix A gives, for $\alpha = 2$,

$$\tau(m+2,n) + \tau(m-2,n) \geq 2\tau(m,n). \tag{130}$$

Hence we find that

$$U_0(m,n) \geq V_0(m,n). \tag{131}$$

Accordingly Eq.(118) is reduced to

$$\begin{aligned}
&\tau(m+\frac{1}{2}\alpha+1,n) + \tau(m-\frac{1}{2}\alpha-1,n) \\
&= \max(\ \tau(m+\frac{1}{2}\alpha,n+1) + \tau(m-\frac{1}{2}\alpha,n-1) - 2, \\
&\qquad\quad \tau(m+\frac{1}{2}\alpha,n-1) + \tau(m-\frac{1}{2}\alpha,n+1) - 2, \\
&\qquad\quad \tau(m+\frac{1}{2}\alpha,n) + \tau(m-\frac{1}{2}\alpha,n)\ ).
\end{aligned} \tag{132}$$

And Eq.(121) is transformed into

$$U_0(m,n) = \max(\ W_0(m,n) - 2, X_0(m,n) - 2, 0\ ), \tag{133}$$

which determine $U_0(m,n)$ uniquely. Equations.(133),(123) and (124) give the ultradiscrete form of E-Toda equation for $\alpha = 2$.

5.1 Mapping of E-Toda UD equation for $\alpha = 2$

We have found that the following 3-soliton solution solves Eq.(132).

$$\begin{aligned}
\tau(m,n) = \max(&0, \xi_1, \xi_2, \xi_3, A_{12}+\xi_1+\xi_2, A_{13}+\xi_1+\xi_3, \\
&A_{23}+\xi_2+\xi_3, A_{12}+A_{13}+A_{23}+\xi_1+\xi_2+\xi_3),
\end{aligned} \tag{134}$$

$$\xi_i = \omega_i m - k_i(n - n_1), \tag{135}$$

$$k_i = \epsilon_i(|\omega_i| + 2), \quad \epsilon_i = \pm 1, \quad \text{for } i = 1,2,3, \tag{136}$$

$$A_{ij} = -2\epsilon_i\epsilon_j - (3 + \epsilon_i\epsilon_j)\min(\omega_i, \omega_j), \quad \text{for } i,j = 1,2,3. \tag{137}$$

In Fig.1 the solid lines express the theoretical values of $U_0(m,n)$ as a function of a continuous variable, n. While the dots indicate the numerical mapping of $U(m,n)$ by Eqs.(133), (122), (123) and (124) for integer n. The parameters used are $\alpha = 2, \omega_1 = 2$ $(\epsilon_1 = 1), \omega_2 = 3$ $(\epsilon_2 = 1)$, and $\omega_3 = 5$ $(\epsilon_3 = 1)$, and $n_1 = n_2 = n_3 = 0$. All dots are on the solid lines. In Fig.2 the solid lines express the theoretical values of $U_0(m,n)$ as a function of a continuous variable, n. While the dots indicate the numerical mapping

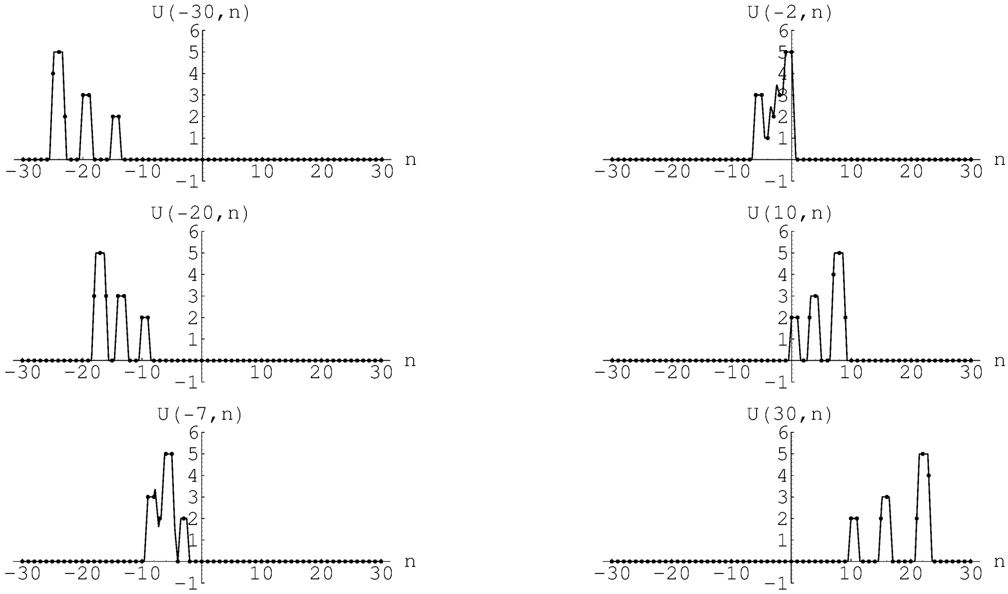

Figure 1: Collisions of 3-solitons of E-Toda equation in UD limit for $\alpha = 2$

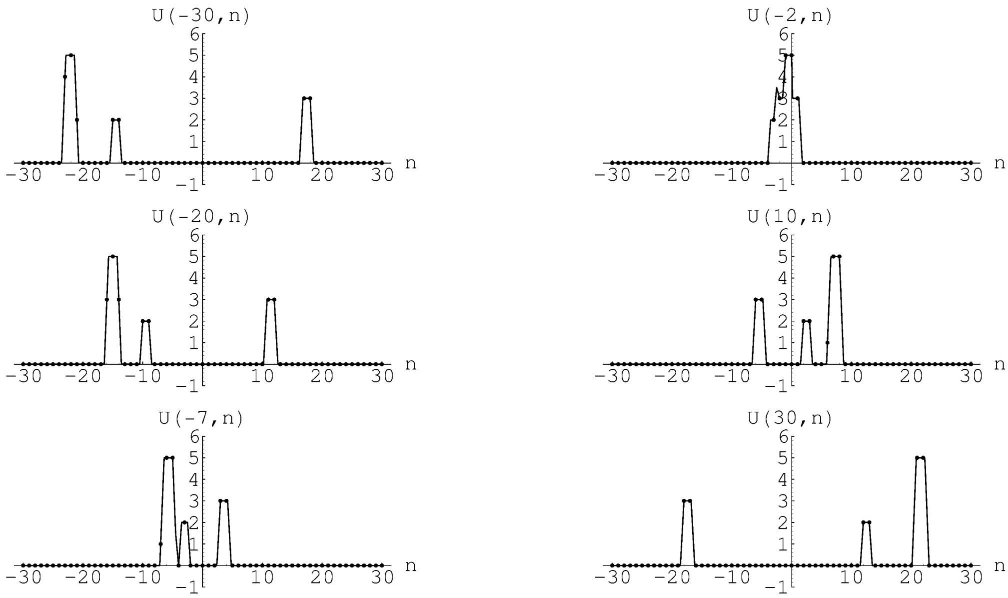

Figure 2: Head-on collisions of solitons of E-Toda equation in UD limit for $\alpha = 2$

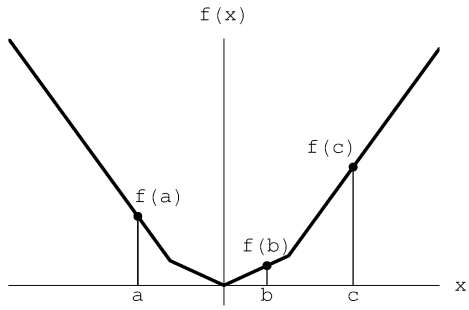

Figure 3: A convex function $f(x)$

of $U(m,n)$ by Eqs.(133), (122), (123) and (124) for integer n. The parameters used are $\alpha = 2, \omega_1 = 2\ (\epsilon_1 = 1), \omega_2 = 3\ (\epsilon_2 = -1)$, and $\omega_3 = 5\ (\epsilon_3 = 1)$, and $n_1 = n_2 = n_3 = 0$. All dots are on the solid lines.

In conclusion, we have found that E-Toda discrete equation for $\alpha = 2$ is not integrable but it is integrable in UD limit.

A Convexity of $\tau-$ function

We shall prove the following convexity of $\tau-$ function,

$$\begin{aligned}
&\tau(m+\frac{1}{2}\alpha-1,n)+\tau(m-\frac{1}{2}\alpha+1,n)\\
&\leq \tau(m+\frac{1}{2}\alpha,n)+\tau(m-\frac{1}{2}\alpha,n) &(138)\\
&\leq \tau(m+\frac{1}{2}\alpha+1,n)+\tau(m-\frac{1}{2}\alpha-1,n), \quad \text{for } \alpha>1. &(139)
\end{aligned}$$

A.1 Introduction to a convex function

A convex function $f(x)$ is defined by

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} \geq 0, \quad \text{for } a<b<c. \tag{140}$$

Equation (140) is transformed into

$$\frac{f(a)-f(b)}{a-b} \leq \frac{f(b)-f(c)}{b-c}. \tag{141}$$

For a convex function $f(x)$, we have the following inequality

$$f(x+\alpha)+f(x-\alpha) \leq f(x+\beta)+f(x-\beta). \quad \text{for } 0<\alpha<\beta. \tag{142}$$

Proof. From the definition (141) we obtain, for $0<\alpha<\beta$,

$$\begin{aligned}
\frac{f(x)-f(x+\alpha)}{x-(x+\alpha)} &\leq \frac{f(x+\alpha)-f(x+\beta)}{(x+\alpha)-(x+\beta)}, \\
\frac{f(x-\alpha)-f(x)}{(x-\alpha)-x} &\leq \frac{f(x)-f(x+\alpha)}{x-(x+\alpha}, \\
\frac{f(x-\beta)-f(x-\alpha)}{x-\beta-(x-\alpha)} &\leq \frac{f(x-\alpha)-f(x)}{(x-\alpha)-x}.
\end{aligned}$$

Hence we have

$$\frac{f(x-\beta)-f(x-\alpha)}{-\beta-(-\alpha)} \leq \frac{f(x+\alpha)-f(x+\beta)}{\alpha-\beta}, \tag{143}$$

which is equal to

$$f(x+\alpha)+f(x-\alpha) \leq f(x+\beta)+f(x-\beta). \tag{144}$$

A.2 Symbol of lists

In order to generate a list we introduce a symbol $\sum\{\cdots\}$. For example,we define

$$\begin{aligned}
&\sum_{i=1}^{n}\{a_i\} \equiv \{a_1,a_2,a_3,\ldots,a_n\}, \\
&\sum_{i=1}^{n}\{\sigma_i a_i\} \equiv \{\sigma_1 a_1,\sigma_2 a_2,\ldots,\sigma_n a_n\}, \\
&\sum_{\sigma=\pm1}\{\sigma_1 a_1,\sigma_2 a_2,\ldots,\sigma_n a_n\} \equiv \{a_1,-a_1,a_2,-a_2,\ldots,a_n,-a_n\}.
\end{aligned}$$

Accordingly we have

$$\max(\sum_{\sigma=\pm1}\sum_{i=1}^{n}\{\sigma_i a_i\}) = \max(\sum_{i=1}^{n}\{|a_i|\}).$$

A.3 Fundamental theorem in Max-plus algebra

Consider a couple of lists A and B,

$$A = \{a_1, a_2, \cdots, a_n\}, \tag{145}$$
$$B = \{b_1, b_2, \cdots, b_n\}. \tag{146}$$

Then we have

$$\max(A) \leq \max(B) \quad \text{if } a_i \leq b_i \quad \text{for } i = 1, 2, \cdots, n. \tag{147}$$

Proof.

Let $\quad a_m = \max(A). \quad$ Then $a_m \leq b_m \leq \max(B)$.

Hence

$$\max(A) \leq \max(B). \qquad Q.E.D.$$

A.4 Convexity of $\tau-$function

We have the $\tau-$ function, Eq.(116), which is a function of m for fixed n. We express it as

$$\tau(m,n) = \max(\sum_{i=1}\{s_i(m,n)\}),$$

where $s_i(m,n) = \Omega_i m + b_i + c_i$ and Ω_i, b_i, c_i are constant. We shall prove the convexity of the $\tau-$ function, for $a \leq b$,

$$\tau(m+a,n) + \tau(m-a,n) \leq \tau(m+b,n) + \tau(m-b,n).$$

Noting that $s_i(m+a,n) = s_i(m,n) + a\ \Omega_i$ and $s_i(m-a,n) = s_i(m,n) - a\ \Omega_i$, we have

$$\begin{aligned}
&\tau(m+a,n) + \tau(m-a,n)\\
&= \max(\sum_{i=1}^{N}\{s_i(m,n) + a\ \Omega_i\}) + \max(\sum_{j=1}^{N}\{s_j(m,n) - a\ \Omega_j\}),\\
&= \max(\sum_{i=1}^{N}\sum_{j=1}^{N}\{s_i(m,n) + s_j(m,n) + a\ |\Omega_i - \Omega_j|\}).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\tau(m+b,n) + \tau(m-b,n)\\
&= \max(\sum_{i=1}^{N}\sum_{j=1}^{N}\{s_i(m,n) + s_j(m,n) + b\ |\Omega_i - \Omega_j|\}).
\end{aligned}$$

Hence we find by virtue of the theorem that

$$\tau(m+a,n)+\tau(m-a,n)\le\tau(m+b,n)+\tau(m-b,n),\quad \text{if } a\le b. \tag{148}$$

References

[1] Ryogo Hirota and Hideyuki Yahagi, ""Recurrence Equation", An Integrable System", J.Phys.Soc.Jpn **71**(2002) 2867-2872.

[2] T. Tokihiro, D. Takahashi, J.Matsukidaira and J. Satsuma,"From Soliton Equations to Integrable Cellular Automata through Limiting Procedure",Phys.Rev.Lett. Vol.**76**(1996) 3247-3250.

[3] D.Takahashi and J.Matsukidaira,"Box and ball system with a carrier and ultradiscrete modified KdV equations",J. Phys. A: Math. Gen. **30**(1997)L733-L739.

[4] T.Tokihiro, D.Takahashi and J.Matsukidaira,"Box and ball system as a realization of ultradiscrete nonautonomous KP equation".J. Phys. A: Math. Gen. **33**(2000)607-619.

[5] Daisuke Takahashi and Ryogo Hirota," Ultradiscrete Soliton Solution of Permanent Type", J.Phys.Soc.Jpn **76**(2007),104407(6pp).

[6] Hidetomo Nagai,"A New Expression of Soliton Solution to the Ultradiscrete Toda Equation",J. Phys. A: Math. Theor. **41**(2008),235204(12pp).

[7] Ryogo Hirota,"The Direct Method in Soliton Theory", Cambridge Univ.Press(2004).

[8] Ryogo Hirota and Masataka Iwao,"Soliton Equations Exhibiting Pfaffian Solutions",*Glasgow Math, J.***43A** (2001) 33-41.

[9] Miwa T."On Hirota's difference equations", Proc. Japan Acad. Ser. A. Math. Sci. **58** (1982) 9-12.

[10] 広田良吾,"一般化 BKP 差分方程式のバックルンド変換", 九州大学応用力学研究所研究集会報告 **21ME-S7**(2010) 58-68.