RIMS Kôkyûroku Bessatsu
B32 (2012), 3–18

Ramification correspondence of finite flat group schemes and canonical subgroups — a survey

By

Shin Hattori *

Abstract

Let $p > 2$ be a rational prime, k be a perfect field of characteristic p and K be a finite totally ramified extension of the fraction field of the Witt ring of k. Let $\mathcal{G}$ and $\mathcal{H}$ be finite flat (commutative) group schemes killed by p over $\mathcal{O}_K$ and $k[[u]]$, respectively. In this survey article, we explain the coincidence of ramification subgroups of $\mathcal{G}$ and $\mathcal{H}$ in the sense of Abbes-Mokrane when they are associated to the same Kisin module. We also give a survey of its application to an existence theorem of canonical subgroups of truncated Barsotti-Tate groups of higher dimension.

§ 1. Introduction: canonical subgroups

This article is a survey of the author's result on a correspondence of ramification subgroups of finite flat group schemes over complete discrete valuation rings of mixed and equal characteristics ([19]) and its application to an existence theorem of canonical subgroups of truncated Barsotti-Tate groups ([20]).

Let p be a rational prime and $N \geq 5$ be an integer which is prime to p. Serre ([29]) defined p-adic elliptic modular forms of level N as p-adic limits of q-expansions of usual elliptic modular forms of level N, and Katz ([22]) introduced their modular description as a function on the ordinary locus of the modular curve $X_1(N)$ over $\mathbb{Q}_p$. Namely, a p-adic modular form f can be identified with a rule functorially associating to any triplet (E, ω, ι) over a p-adically complete ring B an element $f(E, \omega, \iota)$ of B, where the triplet consists of an elliptic curve E over B such that $\bar{E} = E \times \mathrm{Spec}(B/pB)$

Received March 29, 2011. Revised July 4, 2011.
2000 Mathematics Subject Classification(s): 11S23
Key Words: ramification, finite flat group schemes, Breuil-Kisin classification
Supported by Grant-in-Aid for Young Scientists B-21740023.
*Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan.
e-mail: `shin-h@math.kyushu-u.ac.jp`

is ordinary, a nowhere vanishing differential form ω on E and a level $\Gamma_1(N)$-structure $\iota : \mu_N \to E$.

The space of p-adic modular forms admits actions of Hecke operators T_l $(l \nmid N)$ and U_l $(l \mid N)$ for $l \neq p$ as usual, and the U operator which affects the q-expansions as

$$f(q) = \sum a_n q^n \mapsto Uf(q) = \sum a_{pn} q^n.$$

The definition of the U operator is as follows. Let f be a p-adic elliptic modular form and (E, ω, ι) be a triplet as above. Since $\bar{E}$ is ordinary, we have a unique subgroup scheme H of E which lifts the Frobenius kernel $\mathrm{Ker}(F_{\bar{E}})$ by Hensel's lemma. Consider the projection $\pi : E \to E/H$ and its dual map $\pi^\vee$. Then using the Frobenius operator

$$(\varphi f)(E, \omega, \iota) = f(E/H, (\pi^\vee)^* \omega, \pi(\iota)),$$

the U operator is defined to be $U = p^{-1}\mathrm{tr}_\varphi$. However, the eigenspaces of the U operator on the whole space of p-adic elliptic modular forms are all isomorphic to each other and infinite dimensional ([17, Section II.3]).

To obtain a reasonable spectral theory of the U operator, we need to restrict U to the subspace consisting of the p-adic modular forms which are also defined on some peripheral area outside the ordinary locus. The resulting class of p-adic modular forms is called overconvergent. Katz ([22]) showed that the U operator acts complete continuously on the space of overconvergent modular forms and thus has meaningful eigenspaces via the theory of the Fredholm determinant.

The key point for studying the action of the U operator on the subspace of overconvergent modular forms is an existence theorem of canonical subgroups. When E does not have ordinary reduction, we do not have any Frobenius lift in general. Nevertheless, we do have a canonical Frobenius lift C when E has supersingular reduction but lies sufficiently near to the ordinary locus in $X_1(N)$. This subgroup scheme C is called the canonical subgroup of E, and we can analyze the Frobenius operator and the U operator by controlling C, instead of H in the ordinary case.

This whole story suits better for the rigid-analytic setting (see for example [9]). We can realize the locus of elliptic curves "sufficient near to ordinary reduction" as an admissible open of the associated rigid-analytic space $X_1(N)^{\mathrm{rig}}$ and the overconvergent modular forms can be identified with the sections of an invertible sheaf over this admissible open. Moreover, we can patch the canonical subgroup into a family. Namely, there exists an admissible open subgroup of the p-torsion of the universal elliptic curve over this locus which gives the canonical subgroup on each fiber.

Katz attributed the existence theorem of canonical subgroups to Lubin ([22, Theorem 3.1]), and the proof is accomplished by a calculation of formal power series of one variable for the formal completion of E. In fact, when B is a complete discrete valuation ring over $\mathbb{Z}_p$, an elliptic curve E over B has the canonical subgroup if and only if the Newton polygon of the multiplication-by-p formula $[p](X)$ of the formal completion of E has a vertex at $x = p$. Thus the proof of Katz-Lubin heavily relies on the one-dimensionality of elliptic curves, and a similar consideration for higher dimensional abelian schemes was too hard to carry out, at that time.

This had been one of the obstacles to establish the theory of overconvergent Siegel modular forms, until Abbes-Mokrane ([1]) achieved a breakthrough. Let K be a complete discrete valuation field of residue characteristic p. They defined, for a finite flat generically etale group scheme $\mathcal{G}$ over $\mathcal{O}_K$, a filtration $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ by finite flat closed subgroup schemes of $\mathcal{G}$ using a ramification theory of Abbes-Saito ([2], [3]), which is called the upper ramification filtration of $\mathcal{G}$. The canonical subgroup of Katz-Lubin of an elliptic curve E over $\mathcal{O}_K$ appears in the upper ramification filtration of the p-torsion subgroup scheme $E[p]$. Then they proved that, for an abelian scheme A of arbitrary relative dimension over $\mathcal{O}_K$ which is "sufficiently close to ordinary reduction", a subgroup scheme which appears in the filtration of $A[p]$ satisfies similar properties to the canonical subgroup of Katz-Lubin.

Since then several improvements of the result have been obtained, such as [4], [10], [15], [16], [25], [28], [31] and most recently [12] and [32]. Let us summarize some of main points of these improvements. First, the canonical subgroup theorem is generalized for truncated Barsotti-Tate groups ([21]) over $\mathcal{O}_K$ instead of abelian schemes. In particular, a higher analogue of the canonical subgroup theorem, namely the existence of a similar canonical subgroup in $A[p^n]$ instead of $A[p]$, is also known. Moreover, this improvement means that we can construct the canonical subgroup for an abelian scheme A without deep geometric techniques such as p-adic vanishing cycles but just from the finite flat group scheme $A[p^n]$. Second, the original condition on "sufficient closeness to ordinarity" in Abbes-Mokrane's work is much relaxed.

One of the two main theorems of this survey article is also an improvement of the canonical subgroup theorem along these two lines. To state the result, we fix some notations. Let $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. For a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ and its module of invariant differentials $\omega_\mathcal{G}$ over $\mathcal{O}_K$, write $\omega_\mathcal{G} \simeq \oplus_i \mathcal{O}_K/(a_i)$ with some $a_i \in \mathcal{O}_K$ and put $\deg(\mathcal{G}) = \sum_i v_p(a_i)$, where v_p is the normalized p-adic valuation. Put $\tilde{\mathcal{O}}_K = \mathcal{O}_K/p\mathcal{O}_K$. For a truncated Barsotti-Tate group $\mathcal{G}$ of level n, height h and dimension $d < h$ over $\mathcal{O}_K$, let us consider its Cartier dual $\mathcal{G}^\vee$ and the p-torsion subgroup scheme $\mathcal{G}^\vee[p]$. Then the Lie algebra $\mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathrm{Spec}(\tilde{\mathcal{O}}_K))$ over $\tilde{\mathcal{O}}_K$ is a free $\tilde{\mathcal{O}}_K$-module of rank $h-d$. We define the Hasse invariant $\mathrm{Ha}(\mathcal{G})$ of $\mathcal{G}$ to be the truncated p-adic valuation $v_p(\det(V_{\mathcal{G}^\vee[p]})) \in [0,1]$ of the determinant of the action of the Verschiebung $V_{\mathcal{G}^\vee[p]}$ on this $\tilde{\mathcal{O}}_K$-module. This invariant measures the distance of $\mathcal{G}$ from ordinary reduction. In fact, $\mathcal{G}$ is an extension of a finite etale group scheme by a finite flat group scheme of multiplicative type if and only if $\mathrm{Ha}(\mathcal{G}) = 0$. Finally, we put $\mathcal{G}^{j+}$ to be the scheme-theoretic closure in $\mathcal{G}$ of the subgroup $\cup_{j'>j}\mathcal{G}^{j'}(\mathcal{O}_{\bar{K}})$. Then our canonical subgroup theorem is as follows.

Theorem 1.1 ([20], Theorem 1.1). *Let $p > 2$ be a rational prime, $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields and e be the absolute ramification index of K. Put $m_K^{\geq i} = \{x \in K \mid ev_p(x) \geq i\}$. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level n, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hasse invariant $w = \mathrm{Ha}(\mathcal{G})$. If $w < 1/(2p^{n-1})$, then the upper ramification subgroup scheme $\mathcal{C}_n = \mathcal{G}^{j+}$ for*

$$pew(p^n - 1)/(p - 1)^2 \leq j < pe(1 - w)/(p - 1)$$

satisfies $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^d$. Moreover, the group scheme $\mathcal{C}_n$ has the following properties:

(a) $\deg(\mathcal{G}/\mathcal{C}_n) = w(p^n - 1)/(p - 1)$.

(b) $\mathcal{C}_n \times \mathrm{Spec}(\mathcal{O}_K/m_K^{\geq e(1-p^{n-1}w)})$ *coincides with the kernel of the n-th iterated Frobenius homomorphism F^n of $\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_K/m_K^{\geq e(1-p^{n-1}w)})$.*

(c) The scheme-theoretic closure of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^i]$ in $\mathcal{C}_n$ coincides with the subgroup scheme $\mathcal{C}_i$ of $\mathcal{G}[p^i]$ for $1 \leq i \leq n - 1$.

Since it is known that the upper ramification filtration can be patched into a family, we have the following corollary.

Corollary 1.2 ([20], Corollary 1.2). *Let $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields and j be a positive rational number. Let $\mathfrak{X}$ be an admissible formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$ which is quasi-compact and $\mathfrak{G}$ be a truncated Barsotti-Tate group of level n over $\mathfrak{X}$ of constant height h and dimension d with $0 < d < h$. We let G and X denote the Raynaud generic fibers of the formal schemes $\mathfrak{X}$ and $\mathfrak{G}$, respectively. For a finite extension L/K and $x \in X(L)$, we put $\mathfrak{G}_x = \mathfrak{G} \times_{\mathfrak{X},x} \mathrm{Spf}(\mathcal{O}_L)$, where we let x also denote the map $\mathrm{Spf}(\mathcal{O}_L) \to \mathfrak{X}$ induced from x by taking the scheme-theoretic closure and the normalization. Let G^{j+} be the admissible open subgroup of G over X such that for any $x \in X(L)$ as above, the fiber G_x^{j+} coincides with the upper ramification subgroup $\mathfrak{G}_x^{je(L/K)+}(\bar{K})$. For a non-negative rational number r, let $X(r)$ be the admissible open of the rigid-analytic space X defined by*

$$X(r)(\bar{K}) = \{x \in X(\bar{K}) \mid \mathrm{Ha}(\mathfrak{G}_x) < r\}.$$

Suppose $p > 2$. Then the finite etale rigid-analytic group $G^{j+}|_{X(r)}$ over $X(r)$ is etale locally isomorphic to the constant group $(\mathbb{Z}/p^n\mathbb{Z})^d$ for $r = 1/(2p^{n-1})$ and $j = (2p^{n-1} - 1)/(2p^{n-2}(p - 1))$.

In particular, using this corollary, we can generalize results of [1] to the case of $\mathrm{Ha}(A) < 1/2$.

The idea of the proof of Theorem 1.1 is as follows. By an elementary argument as in [12] and [32], it suffices to treat the case of level one. Moreover, by a base change, we may assume that the residue field k of K is perfect. Then, the key point is that we can reduce ourselves to showing a similar statement to Theorem 1.1 for the lower ramification filtration of a finite flat generically etale group scheme over a complete discrete valuation ring of equal characteristic with residue field k. This equal characteristic counterpart can be shown by an easy calculation in a spirit of the Elkik approximation ([11, Section I]). The reduction to the equal characteristic case is a consequence of a ramification correspondence theorem between finite flat group schemes over complete discrete valuation rings of mixed and equal characteristics, which is the other main theorem of this survey article (Theorem 3.2).

§ 2. Breuil-Kisin classification

In this section, we recall classification theories of finite flat generically etale (commutative) group schemes over a complete discrete valuation ring with perfect residue field k of characteristic p. The case of equal characteristic is classical: Let p be a rational prime and T be a scheme with $p\mathcal{O}_T = 0$. We let ϕ denote the absolute Frobenius morphism of T. We define a ϕ-module over T to be a pair $(\mathcal{M}, \phi_{\mathcal{M}})$ of an $\mathcal{O}_T$-module $\mathcal{M}$ and a ϕ-semilinear homomorphism $\phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$. A ϕ-module $(\mathcal{M}, \phi_{\mathcal{M}})$ is said to be finite locally free if $\mathcal{M}$ is a locally free $\mathcal{O}_T$-module of finite rank. Then we have the following classification theorem for finite locally free group schemes over T.

Theorem 2.1 ([14], Théorème 7.4)**.** *Let T be a scheme with $p\mathcal{O}_T = 0$. For a finite locally free group scheme $\mathcal{H}$ over T, we regard the Lie algebra of the Cartier dual* $\mathrm{Lie}(\mathcal{H}^\vee)$ *as a ϕ-module over T via the map* $\mathrm{Lie}(V_{\mathcal{H}^\vee})$ *induced by the Verschiebung $V_{\mathcal{H}^\vee}$. Then we have an anti-equivalence $\mathcal{H}(-)$ from the category of finite locally free ϕ-modules over T to the category of finite locally free group schemes over T killed by their Verschiebung. Its quasi-inverse is given by the functor $\mathcal{H} \mapsto$* $\mathrm{Lie}(\mathcal{H}^\vee)$*, and these anti-equivalences are compatible with any base change.*

Let us specialize to the case of $T = \mathrm{Spec}(k[[u]])$. Put $\mathfrak{S}_1 = k[[u]]$ and we also let ϕ denote the absolute Frobenius endomorphism of this ring. Let $(\mathfrak{M}, \phi_{\mathfrak{M}})$ be a finite (locally) free ϕ-module over $k[[u]]$. Namely, this is a pair of a free $\mathfrak{S}_1$-module of finite rank with a ϕ-semilinear map $\phi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. Put $\phi^*\mathfrak{M} = \mathfrak{S}_1 \otimes_{\phi, \mathfrak{S}_1} \mathfrak{M}$. We say $(\mathfrak{M}, \phi_{\mathfrak{M}})$ is a Kisin module of u-height $\leq r$ if the cokernel of the map $1 \otimes \phi_{\mathfrak{M}} : \phi^*\mathfrak{M} \to \mathfrak{M}$ is killed by u^r, though it was introduced by Breuil ([5]). Then by Theorem 2.1 we also have an anti-equivalence $\mathcal{H}(-)$ from the category of Kisin modules of u-height $\leq r$ to the category of finite flat group schemes $\mathcal{H}$ over $k[[u]]$ killed by $V_{\mathcal{H}}$ such that the ϕ-module $\mathrm{Lie}(\mathcal{H}^\vee)$ is of u-height $\leq r$. For a Kisin module $\mathfrak{M}$ of u-height $\leq r$, we can explicitly write defining equations of the associated group scheme $\mathcal{H}(\mathfrak{M})$. Indeed, choose a basis $e_1, \ldots, e_d$ of the $\mathfrak{S}_1$-module $\mathfrak{M}$ and define a matrix $A = (a_{i,j})$ by

$$\phi_{\mathfrak{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A.$$

Then the group scheme $\mathcal{H}(\mathfrak{M})$ is naturally isomorphic to the additive group scheme defined by the equations

$$X_i^p - \sum_{j=1}^{d} a_{j,i} X_j \quad (i = 1, \ldots, d).$$

Next we suppose that K is of mixed characteristic $(0, p)$ with $p > 2$ and perfect residue field k. In this case, the classification theorem for finite flat group schemes is due to Breuil ([5], [7], [8]) and Kisin ([23]). For simplicity, we concentrate on the classification of finite flat group schemes killed by p. Let e be the absolute ramification index of K. Let $\bar{K}$ be an algebraic closure of K, $\hat{\bar{K}}$ be its completion and put $G_K = \mathrm{Gal}(\bar{K}/K)$. We let $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ denote the category of Kisin modules of u-height $\leq e$. Then we have the following theorem.

Theorem 2.2 ([8], Theorem 3.3.2)**.** *Let $p > 2$ and K as above. Then there exists an anti-equivalence $\mathcal{G}(-)$ from the category* $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ *to the category of finite flat group schemes over $\mathcal{O}_K$ killed by p.*

Remark. Breuil ([7]) proved a similar classification in terms of slightly different linear algebraic data, which are now called Breuil modules, and he showed Theorem 2.2 in [8] by constructing an equivalence of categories between those of Breuil and Kisin modules. Kisin ([23]) gave a much simpler proof independent of Breuil's earlier classification: he first proved directly a classification of Barsotti-Tate groups over $\mathcal{O}_K$ via Breuil modules by an elementary argument using the deformation theory of Messing ([27]) and then derived the classification of finite flat group schemes by taking a resolution by Barsotti-Tate groups, the idea which he attributed to Beilinson. These two construction of the anti-equivalence $\mathcal{G}(-)$ are naturally isomorphic to each other.

From a Kisin module $\mathfrak{M} \in \mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$, we can decode the action of G_K on the finite module $\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})$ associated to the finite flat group scheme $\mathcal{G}(\mathfrak{M})$ corresponding to $\mathfrak{M}$ as follows. Let us fix a uniformizer π of K and a system of its p-power roots $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ in $\bar{K}$ such that $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$. Put $K_n = K(\pi_n)$ and $K_\infty = \cup_{n \in \mathbb{Z}_{\geq 0}} K_n$. We define a ring R to be

$$R = \varprojlim_n (\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \leftarrow \cdots),$$

where the arrows are the p-th power map, and also an element $\underline{\pi}$ of R to be $\underline{\pi} = (\pi, \pi_1, \pi_2, \ldots)$. The ring R is a complete valuation ring whose valuation v_R is given as follows: for $x = (x_0, x_1, \ldots) \in R$, choose a lift $\hat{x}_n$ of x_n in $\mathcal{O}_{\bar{K}}$ and put $x^{(0)} = \lim_{n\to\infty} \hat{x}_n^{p^n} \in \mathcal{O}_{\hat{\bar{K}}}$. Then the valuation of the element x is defined as $v_R(x) = v_K(x^{(0)})$.

The ring R admits a natural action of the Galois group G_K and an $\mathfrak{S}_1$-algebra structure defined by $u \mapsto \underline{\pi}$ which is compatible with the action of $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$. Moreover, the ring R is considered as a ϕ-module over $\mathfrak{S}_1$ via the absolute Frobenius endomorphism ϕ. For a Kisin module $\mathfrak{M} \in \mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$, we put

$$T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S}_1,\phi}(\mathfrak{M}, R),$$

where the module on the right-hand side consists of the $\mathfrak{S}_1$-linear homomorphisms compatible with ϕ's. The Galois group G_{K_∞} acts naturally on the module $T^*_{\mathfrak{S}}(\mathfrak{M})$. Then there is a natural isomorphism of G_{K_∞}-modules $\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M}) \to T^*_{\mathfrak{S}}(\mathfrak{M})$ ([8, *loc. cit.*]).

The field $\mathrm{Frac}(R)$ can be identified with the completion of an algebraic closure of the subfield $\mathcal{X} = k((u))$. This identification induces the "field-of-norms" isomorphism of Galois groups $G_{K_\infty} \simeq G_{\mathcal{X}}$ which is compatible with the upper ramification subgroups of both sides up to a shift by the Herbrand function of K_∞/K ([6, Subsection 4.2], [33]). On the other hand, we also have a natural isomorphism of $G_{\mathcal{X}}$-modules $\mathcal{H}(\mathfrak{M})(R) \to T^*_{\mathfrak{S}}(\mathfrak{M})$, by which we identify both sides. Thus we have a natural isomorphism of G_{K_∞}-modules

$$\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M})(R).$$

We can show that the greatest upper ramification jumps of the Galois modules in the classical sense ([30]) of both sides are no more than $pe/(p-1)$, where we follow the normalization of the upper ramification subgroups in [13]. Since the greatest upper ramification jump of the extension K_1/K is equal to $1 + pe/(p-1)$, we see by using the isomorphism $\varepsilon_{\mathfrak{M}}$ that both sides of the isomorphism have exactly the same greatest upper ramification jump.

§ 3. Ramification correspondence

In this section, we briefly recall the ramification theory of finite flat group schemes ([1]) and state the ramification correspondence theorem mentioned before. For a while, let p be a rational prime which may be two and K be a complete discrete valuation field with residue field of characteristic p which may be imperfect. We fix a uniformizer π and a separable closure K^{sep} of K, and put $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We let v_K denote the valuation on K^{sep} with $v_K(\pi) = 1$ and put $m^{\geq i}_{K^{\mathrm{sep}}} = \{x \in \mathcal{O}_{K^{\mathrm{sep}}} \mid v_K(x) \geq i\}$ for $i \in \mathbb{Q}_{\geq 0}$.

Let B be a finite flat $\mathcal{O}_K$-algebra locally of complete intersection which is generically etale, that is, $B \otimes_{\mathcal{O}_K} K$ is etale over K. Put $F(B) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(B, \mathcal{O}_{K^{\mathrm{sep}}})$. This is a finite G_K-set, namely a finite set where the Galois group G_K acts continuously. Fix a presentation

$$B \simeq \mathcal{O}_K[X_1, \ldots, X_r]/(f_1, \ldots, f_s)$$

and consider the K-affinoid variety

$$X_K^j(B) = \{(x_1, \ldots, x_r) \in \mathcal{O}_{K^{\mathrm{sep}}}^r \mid v_K(f_i(x_1, \ldots, x_r)) \geq j \quad (i = 1, \ldots, s)\}$$

for $j \in \mathbb{Q}_{>0}$. The set of geometric connected components of $X_K^j(B)$ turns out to be independent of the choice of a presentation in an appropriate sense and is denoted by $F^j(B)$. This set is also a finite G_K-set and, since the set $F(B)$ is identified with the set of zeros of the equations $f_1, \ldots, f_s$, we have a G_K-equivariant functorial surjection $F(B) \to F^j(B)$.

Let $\mathcal{G} = \mathrm{Spec}(B)$ be a finite flat generically etale group scheme over $\mathcal{O}_K$. Then the affine algebra B is locally of complete intersection ([7, Proposition 2.2.2]) and thus we can apply the above formalism to $\mathcal{G}$. Put $F(\mathcal{G}) = F(B) = \mathcal{G}(\mathcal{O}_{K^{\mathrm{sep}}})$ and $F^j(\mathcal{G}) = F^j(B)$. Then, by a functoriality, the set $F^j(\mathcal{G})$ is shown to have a structure of a G_K-module, namely a module where the Galois group G_K acts continuously and compatibly with the module structure. Moreover, the natural map $\mathcal{G}(\mathcal{O}_{K^{\mathrm{sep}}}) = F(\mathcal{G}) \to F^j(\mathcal{G})$ turns out to be a surjective homomorphism. Its kernel is denoted by $\mathcal{G}^j(\mathcal{O}_{K^{\mathrm{sep}}})$ and called the j-th upper ramification subgroup of $\mathcal{G}$. We also put $\mathcal{G}^{j+}(\mathcal{O}_{K^{\mathrm{sep}}}) = \cup_{j'>j}\mathcal{G}^{j'}(\mathcal{O}_{K^{\mathrm{sep}}})$ and let $\mathcal{G}^j$ and $\mathcal{G}^{j+}$ denote their scheme-theoretic closures in $\mathcal{G}$.

As in the classical ramification theory of local fields, we also have a "lower" variant of the upper ramification filtration. Consider the reduction map $\mathcal{G}(\mathcal{O}_{K^{\mathrm{sep}}}) \to \mathcal{G}(\mathcal{O}_{K^{\mathrm{sep}}}/m_{K^{\mathrm{sep}}}^{\geq i})$ for $i \in \mathbb{Q}_{\geq 0}$ and let $\mathcal{G}_i(\mathcal{O}_{K^{\mathrm{sep}}})$ denote its kernel, which is called the i-th lower ramification subgroup of $\mathcal{G}$. We define $\mathcal{G}_{i+}(\mathcal{O}_{K^{\mathrm{sep}}})$, $\mathcal{G}_i$ and $\mathcal{G}_{i+}$ similarly. These two filtrations are compatible with base extensions: for an extension L/K of complete discrete valuation fields with relative ramification index $e(L/K)$, we have natural isomorphisms

$$\mathcal{G}^j \times \mathrm{Spec}(\mathcal{O}_L) \to (\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_L))^{je(L/K)},$$
$$\mathcal{G}_i \times \mathrm{Spec}(\mathcal{O}_L) \to (\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_L))_{ie(L/K)}.$$

Example 3.1. If the affine algebra of $\mathcal{G}$ is generated by one element, we can easily calculate the ramification filtrations ([18, Corollary 5]). For example, suppose that K is of mixed characteristic $(0, p)$ with absolute ramification index e and let E be an elliptic curve over $\mathcal{O}_K$. Consider its formal completion $\hat{E}$ along the zero section with formal parameter X and write the multiplication-by-p formula as

$$[p](X) = pX + \cdots + cX^p + \cdots .$$

Put $f = v_K(c)$ and define a subgroup scheme C of $E[p]$ to be the scheme-theoretic closure of the subgroup $\{x \in \hat{E}[p](m_{\bar{K}}) \mid v_K(x) \geq (e-f)/(p-1)\}$ of $E[p](\mathcal{O}_{\bar{K}})$. Then, for $f < pe/(p+1)$, we have

$$E[p]^j = \begin{cases} E[p] & (0 < j \leq pf/(p-1)) \\ C & (pf/(p-1) < j \leq (pe-f)/(p-1)) \\ 0 & ((pe-f)/(p-1) < j), \end{cases}$$

$$E[p]_i = \begin{cases} E[p] & (0 \leq i \leq f/(p^2-p)) \\ C & (f/(p^2-p) < i \leq (e-f)/(p-1)) \\ 0 & ((e-f)/(p-1) < i). \end{cases}$$

On the other hand, for $f \geq pe/(p+1)$, we have

$$E[p]^j = \begin{cases} E[p] & (0 < j \leq p^2 e/(p^2-1)) \\ 0 & (p^2 e/(p^2-1) < j), \end{cases}$$

$$E[p]_i = \begin{cases} E[p] & (0 \leq i \leq e/(p^2-1)) \\ 0 & (e/(p^2-1) < i). \end{cases}$$

The subgroup scheme C coincides with the canonical subgroup of E in the sense of Katz-Lubin.

Now we can state the other main theorem of this survey article, which establishes a correspondence of ramification filtrations between finite flat group schemes over complete discrete valuation rings of mixed and equal characteristics.

Theorem 3.2 ([19], Theorem 1.1)**.** *Let $p > 2$ be a rational prime and K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field k and absolute ramification index e. Consider the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ of Kisin modules as before and let $\mathfrak{M}$ be its object. Then the natural isomorphism $\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M})(R)$ induces isomorphisms of ramification subgroups*

$$\mathcal{G}(\mathfrak{M})^j(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M})^j(R), \quad \mathcal{G}(\mathfrak{M})_i(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M})_i(R)$$

for any $j \in \mathbb{Q}_{>0}$ and $i \in \mathbb{Q}_{\geq 0}$.

This can be seen as a generalization of the coincidence of the classical greatest upper ramification jumps of both sides mentioned before.

§ 4. Sketch of the proofs

In this section, we give a brief sketch of the proofs of the main theorems. First we show compatibilities of the Breuil-Kisin classification with base extensions inside K_∞/K

and Cartier duality. For the compatibility with the base change, put $\mathfrak{S}'_1 = k[[v]]$ and let ϕ also denote the absolute Frobenius endomorphism of the ring $\mathfrak{S}'_1$. Fix a positive integer n and consider the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}'_1}$ of Kisin modules over $\mathfrak{S}'_1$ of v-height $\leq ep^n$. Using the k-algebra homomorphism $\mathfrak{S}'_1 \to R$ defined by $v \mapsto \underline{\pi}^{1/p^n}$, we define a similar functor $T^*_{\mathfrak{S}'}$ to $T^*_{\mathfrak{S}}$. The homomorphism of k-algebras $\mathfrak{S}_1 \to \mathfrak{S}'_1$ defined by $u \mapsto v^{p^n}$ induces a natural functor $(-)' : \mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1} \to \mathrm{Mod}^{1,\phi}_{/\mathfrak{S}'_1}$ by

$$\mathfrak{M} \mapsto \mathfrak{M}' = \mathfrak{S}'_1 \otimes_{\mathfrak{S}_1} \mathfrak{M}_1, \quad \phi_{\mathfrak{M}'} = \phi \otimes \phi_{\mathfrak{M}}.$$

Then we have a natural isomorphism of G_{K_∞}-modules $T^*_{\mathfrak{S}}(\mathfrak{M}) \to T^*_{\mathfrak{S}'}(\mathfrak{M}')$. On the other hand, by Theorem 2.2 the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}'_1}$ classifies finite flat group schemes over $\mathcal{O}_{K_n}$ killed by p, and we let $\mathcal{G}'(-)$ denote the anti-equivalences of the theorem over $\mathcal{O}_{K_n}$. Then we can show the following proposition.

Proposition 4.1 ([19], Proposition 4.3)**.** *Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and $\mathfrak{M}'$ be the associated object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}'_1}$. Then there exists a natural isomorphism*

$$\mathcal{G}'(\mathfrak{M}') \to \mathcal{G}(\mathfrak{M}) \times \mathrm{Spec}(\mathcal{O}_{K_n})$$

of finite flat group schemes over $\mathcal{O}_{K_n}$ which makes the following diagram commutative:

$$\begin{tikzcd}
\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})|_{G_{K_\infty}} \arrow[r, "\sim"'] \arrow[d, "\varepsilon_{\mathfrak{M}}"', "\wr"] & \mathcal{G}'(\mathfrak{M}')(\mathcal{O}_{\bar{K}})|_{G_{K_\infty}} \arrow[d, "\varepsilon_{\mathfrak{M}'}"', "\wr"] \\
T^*_{\mathfrak{S}}(\mathfrak{M}) \arrow[r, "\sim"'] & T^*_{\mathfrak{S}'}(\mathfrak{M}').
\end{tikzcd}$$

Remark. The author does not know if a similar base change compatibility of the Breuil-Kisin classification holds for wildly ramified extensions in general. Thus it seems difficult to prove Theorem 3.2 by killing the Galois action, taking scheme-theoretic closures on both sides of the isomorphism $\varepsilon_{\mathfrak{M}}$ and reducing to the case of rank one. Instead, by using the restricted compatibility as in Theorem 4.1, we reduce ourselves to comparing the defining equations of both sides, as we explain below.

For the compatibility with Cartier duality, we recall a duality of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ ([26, Section 3]). Let $W = W(k)$ be the Witt ring and $E(u)$ be the Eisenstein polynomial over W of the uniformizer π of K. Put $c_0 = p^{-1}E(0)$. We also fix a system of p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$ in $\bar{K}$ with $\zeta_p \neq 1$ and $\zeta_{p^n} = \zeta_{p^{n+1}}^p$. Then we have an element $\bar{\mathfrak{t}}$ of the ring R associated to the system on which the Galois group G_{K_∞} acts via the modulo p cyclotomic character ([26, *loc. cit.*]). Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and put $\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}_1}(\mathfrak{M}, \mathfrak{S}_1)$. Consider the natural pairing $\langle\ ,\ \rangle_{\mathfrak{M}} : \mathfrak{M} \times \mathfrak{M}^\vee \to \mathfrak{S}_1$. Then we can give $\mathfrak{M}^\vee$ a natural structure of an object of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ satisfying $\langle \phi_{\mathfrak{M}}(m), \phi_{\mathfrak{M}^\vee}(m^\vee)\rangle_{\mathfrak{M}} = c_0^{-1}u^e\phi(\langle m, m^\vee\rangle_{\mathfrak{M}})$ for any $m \in \mathfrak{M}$ and $m^\vee \in \mathfrak{M}^\vee$ ([26, Proposition 3.1.7]). This induces a natural perfect pairing of G_{K_∞}-modules $T^*_{\mathfrak{S}}(\mathfrak{M}) \times T^*_{\mathfrak{S}}(\mathfrak{M}^\vee) \to (\mathbb{Z}/p\mathbb{Z})\bar{\mathfrak{t}}$, which is also denoted by $\langle\ ,\ \rangle_{\mathfrak{M}}$. Then we can also show the following compatibility with Cartier duality.

Proposition 4.2 ([19], Proposition 4.4)**.** *Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and $\mathfrak{M}^\vee$ be its dual object. Then there exists a natural isomorphism $\mathcal{G}(\mathfrak{M})^\vee \to \mathcal{G}(\mathfrak{M}^\vee)$ of finite flat group schemes over $\mathcal{O}_K$ such that the induced map*

$$\delta_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})^\vee(\mathcal{O}_{\bar{K}}) \to \mathcal{G}(\mathfrak{M}^\vee)(\mathcal{O}_{\bar{K}}) \overset{\varepsilon_{\mathfrak{M}^\vee}}{\to} T^*_{\mathfrak{S}}(\mathfrak{M}^\vee)$$

fits into the commutative diagram of G_{K_∞}-modules

$$\begin{tikzcd} \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \times \mathcal{G}(\mathfrak{M})^\vee(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d, "\varepsilon_{\mathfrak{M}}"', "\wr"] \arrow[d, shift left=20, "\delta_{\mathfrak{M}}"', "\wr"] & \mathbb{Z}/p\mathbb{Z}(1) \arrow[d, "\wr"] \\ T^*_{\mathfrak{S}}(\mathfrak{M}) \times T^*_{\mathfrak{S}}(\mathfrak{M}^\vee) \arrow[r, "\langle\ ,\ \rangle_{\mathfrak{M}}"'] & (\mathbb{Z}/p\mathbb{Z})\bar{\mathfrak{t}}, \end{tikzcd}$$

where the top arrow is the Cartier pairing of $\mathcal{G}(\mathfrak{M})$ and the right vertical arrow is the isomorphism defined by $\zeta_p \mapsto \bar{\mathfrak{t}}$.

We also need the following duality result for upper and lower ramification subgroups. For $\mathcal{G}$, this is due to Tian and Fargues ([31, Theorem 1.6], [12, Proposition 6]), while the case of $\mathcal{H}(\mathfrak{M})$ is [19, Theorem 3.3].

Proposition 4.3. *Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with residue field k and absolute ramification index e. Put $l(j) = e/(p-1) - j/p$.*

1. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ killed by p. Then we have the equality $\mathcal{G}^j(\mathcal{O}_{\bar{K}})^\perp = (\mathcal{G}^\vee)_{l(j)+}(\mathcal{O}_{\bar{K}})$ for $j \leq pe/(p-1)$, where $\perp$ means the orthogonal subgroup with respect to the Cartier pairing.*

2. *Assume that the residue field k is perfect and let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$. Then we have the equality $\mathcal{H}(\mathfrak{M})^j(R)^\perp = (\mathcal{H}(\mathfrak{M}^\vee))_{l(j)+}(R)$ for $j \leq pe/(p-1)$, where $\perp$ means the orthogonal subgroup with respect to the pairing $\langle\ ,\ \rangle_{\mathfrak{M}}$.*

To prove Theorem 3.2, it is enough by Proposition 4.2 and Proposition 4.3 to show the assertion on lower ramification filtrations. By Proposition 4.1, we may replace K with K_1 and assume that the entries of a representing matrix of $\phi_{\mathfrak{M}}$ for some basis are contained in the subring $k[[u^p]]$ of $\mathfrak{S}_1$. In this case, we can write down defining equations of the group scheme $\mathcal{G}(\mathfrak{M})$ explicitly in terms of $\mathfrak{M}$, by using [7, Proposition 3.1.2]. Let us consider the isomorphism of k-algebras $k[[u]]/(u^e) \to \tilde{\mathcal{O}}_K = \mathcal{O}_K/p\mathcal{O}_K$ sending u to π, by which we identify both sides. Then we can check that the defining equations of the group schemes $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$ over $\tilde{\mathcal{O}}_K \simeq k[[u]]/(u^e)$ coincide with each other and this coincidence preserves zero sections. In other words, we can construct an isomorphism of schemes

$$\eta_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M}) \times \mathrm{Spec}(\tilde{\mathcal{O}}_K) \to \mathcal{H}(\mathfrak{M}) \times \mathrm{Spec}(k[[u]]/(u^e))$$

which preserves zero sections. Though this is is not compatible with group structures in general, we can show that this induces the commutative diagram of sets

$$\begin{tikzcd}
\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \arrow[r, "\varepsilon_{\mathfrak{M}}", "\sim"'] \arrow[d] & \mathcal{H}(\mathfrak{M})(R) \arrow[d] \\
\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}) \arrow[r, "\eta_{\mathfrak{M}}"'] & \mathcal{H}(\mathfrak{M})(R/m_R^{\geq i})
\end{tikzcd}$$

for any $i \leq e$, where we put $m_R^{\geq i} = \{x \in R \mid v_R(x) \geq i\}$. In the diagram, the arrows are homomorphism of groups except the bottom one, which is at least a bijection compatible with zero elements. Moreover, the i-th lower ramification subgroups on both sides of $\varepsilon_{\mathfrak{M}}$ are the inverse images of the zero elements by the vertical arrows. Hence the isomorphism $\varepsilon_{\mathfrak{M}}$ is compatible with the i-th lower ramification filtrations for $i \leq e$. Since we can easily show that the i-th lower ramification subgroups of both sides vanish for $i > e/(p-1)$, we can conclude the proof of Theorem 3.2.

Example 4.4. Put $\mathfrak{M} = \mathfrak{S}_1\mathbf{e}$ for a basis $\mathbf{e}$ with $\phi_{\mathfrak{M}}(\mathbf{e}) = c_0^{-1}u^e\mathbf{e}$. Then we have $\mathcal{G}(\mathfrak{M}) = \mu_p = \mathrm{Spec}(\mathcal{O}_K[X]/(X^p-1))$ ([5, Exemple 2.2.3]). On the other hand, the additive group scheme $\mathcal{H}(\mathfrak{M})$ is isomorphic to $\mathrm{Spec}(k[[u]][Y]/(Y^p - c_0^{-1}u^eY))$. Reducing modulo p and modulo u^e respectively, we have an isomorphism of schemes over $\tilde{\mathcal{O}}_K \simeq k[[u]]/(u^p)$

$$\tilde{\mathcal{O}}_K[X]/(X^p-1) \to (k[[u]]/(u^e))[Y]/(Y^p)$$

defined by $X \mapsto 1+Y$. This isomorphism is compatible with zero sections, but not with group structures. Nevertheless, the j-th upper (*resp.* i-th lower) ramification subgroups of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$ are zero if and only if $j > pe/(p-1)$ (*resp.* $i > e/(p-1)$).

As for Theorem 1.1, we may assume $n = 1$, as mentioned before. By Proposition 4.3, it suffices to show the following.

Proposition 4.5 ([19], Theorem 3.2)**.** *Let $p > 2$ and K be as in Theorem 1.1. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level one, height h and dimension d over $\mathcal{O}_K$ with $d < h$ and Hasse invariant $w = \mathrm{Ha}(\mathcal{G})$.*

1. *If $w < (p-1)/p$, then the lower ramification subgroup scheme $\mathcal{D} = \mathcal{G}_{e(1-w)/(p-1)}$ is of order p^d. The group scheme $\mathcal{D}$ has the following properties:*
   - *(a)* $\deg(\mathcal{G}/\mathcal{D}) = w$.
   - *(b) The reduction modulo $m_K^{\geq e(1-w)}$ of the closed subgroup scheme $(\mathcal{G}/\mathcal{D})^\vee$ of $\mathcal{G}^\vee$ coincides with the kernel of the Frobenius homomorphism of the reduction $\mathcal{G}^\vee \times$* $\mathrm{Spec}(\mathcal{O}_K/m_K^{\geq e(1-w)})$.
   - *(c)* $\mathcal{D} \times \mathrm{Spec}(\mathcal{O}_K/m_K^{\geq e(1-w)})$ *also coincides with the kernel of the Frobenius homomorphism of $\mathcal{G} \times$* $\mathrm{Spec}(\mathcal{O}_K/m_K^{\geq e(1-w)})$.
2. *If $w < 1/2$, then $\mathcal{D}$ coincides with the lower ramification subgroup scheme $\mathcal{G}_b$ for $ew/(p-1) < b \leq e(1-w)/(p-1)$.*

To prove the proposition, we may assume that the residue field k is perfect. Let $\mathfrak{M}$ be the object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ corresponding to $\mathcal{G}$ via the anti-equivalence $\mathcal{G}(-)$. We identify the k-algebras $\tilde{\mathcal{O}}_K$ and $k[[u]]/(u^e)$ as before. Then we can read off the Hasse invariant from the Kisin module $\mathfrak{M}$, as follows. Put $\mathfrak{M}_1 = \mathfrak{M} \otimes \tilde{\mathcal{O}}_K$. This can be considered as a finite (locally) free ϕ-module over $\tilde{\mathcal{O}}_K$. Consider the natural exact sequences of $\tilde{\mathcal{O}}_K$-modules

$$0 \to (1 \otimes \phi_{\mathfrak{M}_1})(\phi^*\mathfrak{M}_1) \to \mathfrak{M}_1 \to \mathfrak{M}_1/(1 \otimes \phi_{\mathfrak{M}_1})(\phi^*\mathfrak{M}_1) \to 0.$$

Since $\mathcal{G}$ is a truncated Barsotti-Tate group of level one, we can show from the construction of the functor $\mathcal{G}(-)$ that we have natural isomorphisms of $\tilde{\mathcal{O}}_K$-modules

$$\begin{aligned} \omega_{\mathcal{G} \times \tilde{\mathcal{O}}_K} &\simeq \mathrm{Ker}(1 \otimes \phi_{\mathfrak{M}_1} : \phi^*\mathfrak{M}_1 \to \mathfrak{M}_1), \\ \mathrm{Lie}(\mathcal{G}^\vee \times \tilde{\mathcal{O}}_K) &\simeq (1 \otimes \phi_{\mathfrak{M}_1})(\phi^*\mathfrak{M}_1) \end{aligned}$$

and thus the first term of the exact sequence above is a free $\tilde{\mathcal{O}}_K$-module of rank $h-d$. Moreover we can show that this term is stable under $\phi_{\mathfrak{M}_1}$ and the exact sequence of $\tilde{\mathcal{O}}_K$-modules above splits. Then the Hasse invariant $\mathrm{Ha}(\mathcal{G})$ is equal to the truncated p-adic valuation of the determinant of the $\phi_{\mathfrak{M}_1}$-action on the module of the first term of the exact sequence.

We choose a basis $e_1 \ldots, e_h$ of $\mathfrak{M}$ such that $e_1, \ldots, e_{h-d}$ (*resp.* $e_{h-d+1}, \ldots, e_h$) is a lift of a basis of the $\tilde{\mathcal{O}}_K$-module of the first (*resp.* third) term of the exact sequence above. Then we have

$$\phi_{\mathfrak{M}}(e_1, \ldots, e_h) = (e_1, \ldots, e_h) \begin{pmatrix} P_1 & P_2 \\ u^e P_3 & u^e P_4 \end{pmatrix}$$

for some matrices P_i with entries in $k[[u]]$, where P_4 is a $d \times d$-matrix. Set $A \in M_h(k[[u]])$ to be the matrix on the right-hand side and put $A\mathfrak{M}_1 = \mathrm{Span}_{\tilde{\mathcal{O}}_K}((e_1, \ldots, e_d)A)$. Then the exact sequence above is equal to the exact sequence

$$0 \to A\mathfrak{M}_1 \to \mathfrak{M}_1 \to \mathfrak{M}_1/A\mathfrak{M}_1 \to 0$$

of ϕ-modules over $\tilde{\mathcal{O}}_K$, where we have the equality $v_p(\det(\phi_{A\mathfrak{M}_1})) = w$. Let v_u be the u-adic valuation on $k[[u]]$ with $v_u(u) = 1$. Since $w < 1$, we also have the equality $v_u(\det(P_1)) = ew$ and thus we can define the object $\mathfrak{L}$ of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ by $\mathfrak{L} = \oplus_{i=1}^{h-d}\mathfrak{S}_1 e_i$ and

$$\phi_{\mathfrak{L}}(e_1, \ldots, e_{h-d}) = (e_1, \ldots, e_{h-d})P_1.$$

Now, using Theorem 3.2, we switch to $\mathcal{H}(\mathfrak{M})$ for calculating the lower ramification filtrations. Then the proofs of the following two lemmas are straightforward.

Lemma 4.6 ([20], Lemma 3.3). *Let l be a positive integer, U be an element of $M_l(k[[u]])$ and w' be a rational number such that $v_u(\det(U)) = ew'$. Let T be the scheme over $k[[u]]$ defined by the system of equations*

$$(X_1^p, \ldots, X_l^p) = (X_1, \ldots, X_l)U.$$

Suppose that we have the inequality $w' < (p-1)/p$. Then the natural map

$$T(R) \to \mathrm{Im}(T(R/m_R^{\geq e}) \to T(R/m_R^{\geq b}))$$

is a bijection for $ew'/(p-1) < b \leq e(1-w')$.

Lemma 4.7 ([20], Lemma 3.4). *There exists a unique injection $\iota : \mathfrak{L} \to \mathfrak{M}$ of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ such that the $\mathfrak{S}_1$-submodule $\mathfrak{L}$ is a direct summand of $\mathfrak{M}$ and the reduction modulo $u^{e(1-w)}$ of the injection ι coincides with the inclusion*

$$A\mathfrak{M}_1 \otimes (k[[u]]/(u^{e(1-w)})) \to \mathfrak{M}_1 \otimes (k[[u]]/(u^{e(1-w)})).$$

Let b be a rational number with $ew/(p-1) < b \leq e(1-w)$. Then the map $\iota \otimes (R/m_R^{\geq b})$ also coincides with the inclusion $A\mathfrak{M}_1 \otimes (R/m_R^{\geq b}) \to \mathfrak{M}_1 \otimes (R/m_R^{\geq b})$. Hence we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\mathfrak{M})(R) \arrow[r] \arrow[d] & \mathcal{H}(\mathfrak{L})(R) \arrow[d] \\
\mathcal{H}(\mathfrak{M}_1 \otimes (R/m_R^{\geq b}))(R/m_R^{\geq b}) \arrow[r] \arrow[dr] & \mathcal{H}(\mathfrak{L} \otimes (R/m_R^{\geq b}))(R/m_R^{\geq b}) \arrow[d, equal] \\
& \mathcal{H}(A\mathfrak{M}_1 \otimes (R/m_R^{\geq b}))(R/m_R^{\geq b})
\end{tikzcd}$$

whose upper right vertical arrow is an injection by Lemma 4.6. Put $\mathfrak{N} = \mathfrak{M}/\mathfrak{L}$. This is an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ which is free of rank d over $\mathfrak{S}_1$. Then the group $\mathcal{H}(\mathfrak{N})(R)$ is of order p^d and we have the equality

$$\mathcal{H}(\mathfrak{N})(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \to \mathcal{H}(A\mathfrak{M}_1)(R/m_R^{\geq b})).$$

Then, by an elementary calculation of the valuations of roots of the defining equations of $\mathcal{H}(\mathfrak{M})$, we can also show the following lemma, which settles the first assertion of Proposition 4.5 (1) and the assertion (2).

Lemma 4.8 ([20], Lemma 3.5). *The subgroup $\mathcal{H}(\mathfrak{N})(R)$ of $\mathcal{H}(\mathfrak{M})(R)$ is equal to the subgroup $\mathcal{H}(\mathfrak{M})_{e(1-w)/(p-1)}(R)$.*

By the description of the module of invariant differentials of truncated Barsotti-Tate groups stated before, we can show by taking a resolution that we have the same description for any finite flat group scheme over $\mathcal{O}_K$ killed by p. From this the assertion (1a) follows. Replacing K with K_1, the explicit description of defining equations of $\mathcal{G}(\mathfrak{L}^\vee) \times \mathrm{Spec}(\tilde{\mathcal{O}}_K)$ shows the assertion (1b), which in turn implies the assertion (1c).

References

[1] A. Abbes and A. Mokrane: *Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes*, Publ. Math. Inst. Hautes Etudes Sci. **99** (2004), 117–162.

[2] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields I*, Amer. J. Math. **124** (2002), 879–920.

[3] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields II*, Kazuya Kato's fiftieth birthday, Doc. Math. (2003), Extra Vol., 5–72 (electronic).

[4] F. Andreatta and C. Gasbarri: *The canonical subgroup for families of abelian varieties*, Compos. Math. **143** (2007), no. 3, 566–602.

[5] C. Breuil: *Schémas en groupes et corps des normes*, unpublished.

[6] C. Breuil: *Une application de corps des normes*, Compos. Math. **117** (1999), no. 2, 189–203.

[7] C. Breuil: *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.

[8] C. Breuil: *Integral p-adic Hodge theory*, Algebraic geometry 2000, Azumino (Hotaka), 51–80, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002.

[9] R. F. Coleman: *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), no. 1-3, 215–241.

[10] B. Conrad: *Higher-level canonical subgroups in abelian varieties*, preprint.

[11] R. Elkik: *Solutions d'equations a coefficients dans un anneau henselien*, Ann. Sci. Ecole Norm. Sup. (4) **6** (1973), 553–603.

[12] L. Fargues: *La filtration canonique des points de torsion des groupes p-divisibles (avec la collaboration de Yichao Tian)*, preprint.

[13] J.-M. Fontaine:*Il n'y a pas de variete abelienne sur* $\mathbb{Z}$, Invent. Math. **81** (1985), no. 3, 515–538.

[14] P. Gabriel: *Étude infinitésimale des schémas en groupes et groupes formels*, Schémas en groupes I, Lecture Notes in Mathematics **151**, Exp. $\mathrm{VII_A}$, 411–475, Springer-Verlag, Berlin-New York, 1970.

[15] E. Z. Goren and P. L. Kassaei: *The canonical subgroup: a "subgroup-free" approach*, Comment. Math. Helv. **81** (2006), no. 3, 617–641.

[16] E. Z. Goren and P. L. Kassaei: *Canonical subgroups over Hilbert modular varieties*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 17–18, 985–990.

[17] F. Q. Gouvêa: *Arithmetic of p-adic modular forms*, Lecture Notes in Math. **1304**, Springer, Berlin, 1988.

[18] S. Hattori: *Ramification of finite flat group schemes over local fields*, J. Number Theory **118** (2006), no. 2, 145–154.

[19] S. Hattori: *Ramification correspondence of finite flat group schemes over equal and mixed characteristic local fields*, preprint.

[20] S. Hattori: *Canonical subgroups via Breuil-Kisin modules*, preprint.

[21] L. Illusie: *Déformations de groupes de Barsotti-Tate (d'après A. Grothendieck)*, Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84), Asterisque No. **127** (1985), 151–198.

[22] N. M. Katz: *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69–190. Lecture Notes in Math. **350**, Springer, Berlin, 1973.

[23] M. Kisin: *Crystalline representations and F-crystals*, Algebraic geometry and number theory, 459–496, Progr. Math. **253**, Birkhauser Boston, Boston, MA, 2006.

[24] M. Kisin: *Moduli of finite flat group schemes and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.

[25] M. Kisin and K. F. Lai: *Overconvergent Hilbert modular forms*, Amer. J. Math. **127** (2005), no. 4, 735–783.

[26] T. Liu: *Torsion p-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. Ecole Norm. Sup. (4) **40** (2007), no. 4, 633–674.

[27] W. Messing: *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Math. **264**, Springer-Verlag, Berlin-New York, 1972.

[28] J. Rabinoff: *Higher-level canonical subgroups for p-divisible groups*, to appear in J. Inst. Math. Jussieu.

[29] J.-P. Serre: *Formes modulaires et fonctions zeta p-adiques*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), pp. 191–268. Lecture Notes in Math., **350**, Springer, Berlin, 1973.

[30] J.-P. Serre: *Local fields*, Graduate Texts in Mathematics **67**, Springer-Verlag, New York-Berlin, 1979.

[31] Y. Tian: *Canonical subgroups of Barsotti-Tate groups*, Ann. of Math. (2) **172** (2010), no. 2, 955–988.

[32] Y. Tian: *An upper bound on the Abbes-Saito filtration for finite flat group schemes and applications*, preprint, arXiv:1004.4205v2.

[33] J.-P. Wintenberger: *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. Ecole Norm. Sup. (4) **16** (1983), no. 1, 59–89.