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Ramification and cleanliness (joint work with Takeshi Saito)

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§ 1. Introduction

1.1 This is a report on a joint article with Takeshi Saito, with the same title [5], devoted to studying the ramification of Galois torsors and of ℓ-adic sheaves in characteristic $p > 0$ (with $\ell \neq p$), developing the project started in [1, 2, 3, 4, 10].

1.2 Let k be a perfect field of characteristic $p > 0$, X be a smooth, separated and quasi-compact k-scheme, D be a simple normal crossing divisor on X, $U = X - D$; we say for short that (X, D) is an *snc-pair* over k. We fix a prime number ℓ different from p and a finite local $\mathbb{Z}_\ell$-algebra Λ. Let $\mathscr{F}$ be a locally constant constructible sheaf of Λ-modules on U. The main problems in ramification theory are the following :

(A) to describe the ramification of $\mathscr{F}$ along D;

(B) to give a Riemann-Roch type formula for $\mathscr{F}$, that is, to compute the Euler-Poincaré characteristic with compact support of $\mathscr{F}$ on U in terms of its invariants of ramification (provided by (A)).

In [3], we gave cohomological answers to both problems that rely on the notion of *characteristic class* of $\mathscr{F}$. In the article under review, we develop a more geometric approach to problem (A) and give a conjectural formula for (B), generalizing the one proved in ([10] 3.7) and based on the finer notion of *characteristic cycle* of $\mathscr{F}$. For this purpose, we start by studying the ramification of Galois torsors over U, that is, torsors over U for the étale topology, under finite constant groups. Our approach is inspired by

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the ramification theory of local fields with imperfect residue fields that we developed in [1, 2, 10]. Its *leitmotiv* is to eliminate the ramification by blow-up.

§ 2. Review of ramification theory of local fields with imperfect residue fields

2.1 Let K be a discrete valuation field, $\mathscr{O}_K$ be the valuation ring of K, F be the residue field of $\mathscr{O}_K$, $\overline{K}$ be a separable closure of K and $\mathscr{G}$ be the Galois group of $\overline{K}/K$. We assume that $\mathscr{O}_K$ is henselian and that F has characteristic p. In ([1] 3.12), we defined a decreasing filtration $\mathscr{G}^r_{\log}$ ($r \in \mathbb{Q}_{\geqslant 0}$) of $\mathscr{G}$ by closed normal subgroups, called the *logarithmic ramification filtration.* For a rational number $r \geqslant 0$, we put

$$\mathscr{G}^{r+}_{\log} = \overline{\bigcup_{s>r} \mathscr{G}^s_{\log}},$$
$$\mathrm{Gr}^r_{\log}(\mathscr{G}) = \mathscr{G}^r_{\log}/\mathscr{G}^{r+}_{\log}.$$

This filtration satisfies the following properties, among others :

(i) The group $\mathscr{P} = \mathscr{G}^{0+}_{\log}$ is the wild inertia subgroup of $\mathscr{G}$, *i.e.*, the p-Sylow subgroup of the inertia subgroup $\mathscr{G}^0_{\log}$ ([1] 3.15).

(ii) For every rational number $r > 0$, the group $\mathrm{Gr}^r_{\log}(\mathscr{G})$ is abelian and is contained in the center of the pro-p-group $\mathscr{P}/\mathscr{G}^{r+}_{\log}$ ([2] Theorem 1).

Further properties are stated below.

2.2 Let L be a finite separable extension of K, r be a rational number $\geqslant 0$. Then $\mathscr{G}$ acts on $\mathrm{Hom}_{K\text{-Alg}}(L, \overline{K})$ via its action on $\overline{K}$. We say that the ramification of L/K is *bounded* by r (resp. by $r+$) if $\mathscr{G}^r_{\log}$ (resp. $\mathscr{G}^{r+}_{\log}$) acts trivially on $\mathrm{Hom}_K(L, \overline{K})$. We define the *conductor* c of L/K as the infinimum of rational numbers $r > 0$ such that the ramification of L/K is bounded by r. Then c is a rational number, and the ramification of L/K is bounded by $c+$ ([1] 9.5). If $c > 0$, the ramification of L/K is not bounded by c.

In fact, we define first the property that L/K has a bounded ramification by a rational number $r > 0$, then we deduce the filtration $\mathscr{G}^r_{\log}$ ($r \in \mathbb{Q}_{\geqslant 0}$) of $\mathscr{G}$. I will not recall the definition here, as I will introduce a geometric generalization for Galois torsors.

2.3 For any finite discrete Λ-representation M of $\mathscr{G}$, we have a canonical *slope decomposition*

$$M = \oplus_{r \in \mathbb{Q}_{\geqslant 0}} M^{(r)}, \tag{2.3.1}$$

characterised by the following properties : $M^{(0)} = M^{\mathscr{P}}$ and for every $r > 0$,

$$(M^{(r)})^{\mathscr{G}^r_{\log}} = 0 \quad \text{and} \quad (M^{(r)})^{\mathscr{G}^{r+}_{\log}} = M^{(r)}. \tag{2.3.2}$$

The values $r \geqslant 0$ for which $M^{(r)} \neq 0$ are called the *slopes* of M. We say that M is *isoclinic* if it has only one slope. If M is isoclinic of slope $r > 0$, we have a canonical *central character decomposition*

$$M = \oplus_\chi M_\chi, \tag{2.3.3}$$

where the sum runs over finite characters $\chi\colon \mathrm{Gr}^r_{\log}\mathscr{G} \to \Lambda_\chi^\times$ such that Λ_χ is a finite étale Λ-algebra.

2.4 We assume in the following that F is of finite type over k. We denote by $\Omega^1_F(\log)$ the quotient of $\Omega^1_F \oplus (F \otimes_{\mathbb{Z}} K^\times)$ by the sub-F-module generated by elements of the form $(d\overline{a}, 0) - (0, \overline{a} \otimes a)$, for $a \in \mathscr{O}_K - \{0\}$ and $\overline{a}$ its residue class in F. For every $a \in K^\times$, we denote by $d\log a$ the class of $(0, 1 \otimes a)$. Then we have an exact sequence

$$0 \longrightarrow \Omega^1_F \longrightarrow \Omega^1_F(\log) \xrightarrow{\mathrm{res}} F \longrightarrow 0, \tag{2.4.1}$$

where $\mathrm{res}((0, a \otimes b)) = a \cdot \mathrm{ord}(b)$ for $a \in F$ and $b \in K^\times$. In particular, $\Omega^1_F(\log)$ is an F-vector space of finite dimension, namely the transcendental degree of F over k plus one.

We denote by $\mathscr{O}_{\overline{K}}$ the integral closure of $\mathscr{O}_K$ in $\overline{K}$, by $\overline{F}$ the residue field of $\mathscr{O}_{\overline{K}}$, by ord the valuation of $\overline{K}$ normalized by $\mathrm{ord}(K^\times) = \mathbb{Z}$ and, for any rational number r, by $\mathfrak{m}^r_{\overline{K}}$ (resp. $\mathfrak{m}^{r+}_{\overline{K}}$) the $\mathscr{O}_{\overline{K}}$-module of elements $x \in \overline{K}$ such that $\mathrm{ord}(x) \geqslant r$ (resp. $\mathrm{ord}(x) > r$).

The following important property of the logarithmic ramification filtration was proved in ([10] 1.24) if K has characteristic p, and in [11] if K has characteristic 0. For any rational number $r > 0$, the group $\mathrm{Gr}^r_{\log}\mathscr{G}$ is an $\mathbb{F}_p$-vector space, and we have a canonical injective homomorphism

$$\mathrm{rsw}\colon \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Gr}^r_{\log}\mathscr{G}, \mathbb{F}_p) \to \mathrm{Hom}_{\overline{F}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_F(\log) \otimes_F \overline{F}), \tag{2.4.2}$$

called the *refined Swan conductor*. Our definition of characteristic cycle is a generalization of this notion.

§ 3. Ramification of Galois torsors

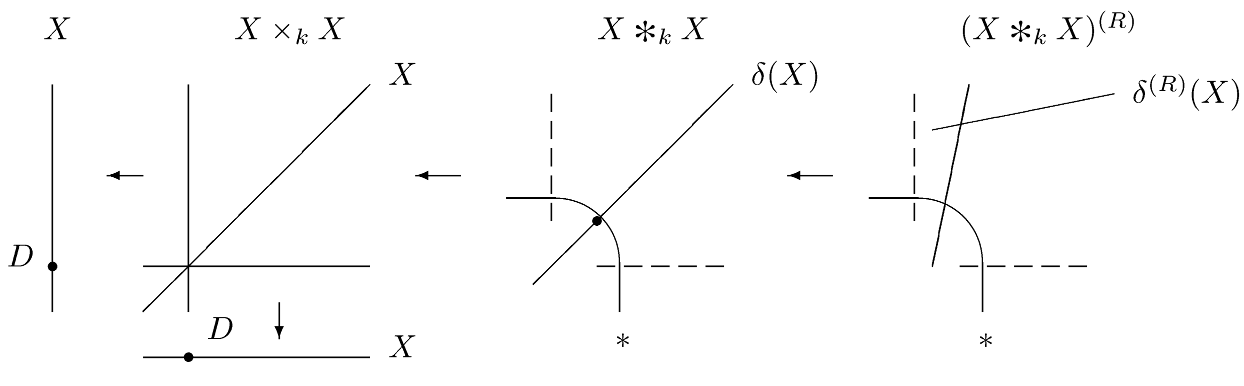
3.1 Our approach is based on a geometric construction introduced in [3, 4, 10]. Let (X, D) be an *snc-pair* over k, $U = X - D$, $D_1, \ldots, D_m$ be the irreducible components of D. We denote by $(X \times_k X)'$ the blow-up of $X \times_k X$ along $D_i \times_k D_i$ for all $1 \leqslant i \leqslant m$.

We define the *framed self-product* $X \circledast_k X$ of (X, D) over k as the open sub-scheme of $(X \times_k X)'$ obtained by removing the strict transforms of $D \times_k X$ and $X \times_k D$ (called the logarithmic self-product in [10]). We give in ([5] 5.20) an equivalent definition using logarithmic geometry, that extends to more general situations. The diagonal morphism $\delta_X \colon X \to X \times_k X$ lifts uniquely to a morphism $\delta \colon X \to X \circledast_k X$, called the *framed diagonal* of (X, D) (and the logarithmic diagonal in [10]). We consider $X \circledast_k X$ as an X-scheme by the second projection.

3.2 Let R be an effective rational divisor on X with support in D (*i.e.*, a sum of non-negative rational multiples of the irreducible components of D). We define in ([5] 5.26) the *dilatation* $(X \circledast_k X)^{(R)}$ *of* $X \circledast_k X$ *along* δ *of thickening* R as an affine scheme over $X \circledast_k X$ that fits in a canonical Cartesian diagram

$$\begin{tikzcd} U \arrow[r, "\delta_U"] \arrow[d, "j"'] & U \times_k U \arrow[d, "j^{(R)}"] \\ X \arrow[r, "\delta^{(R)}"] & (X \circledast_k X)^{(R)} \end{tikzcd} \tag{3.2.1}$$

where $j^{(R)}$ is a canonical open immersion, $\delta^{(R)}$ is the unique morphism lifting δ, j is the canonical injection and δ_U is the diagonal morphism. If R has integral coefficients, then $(X \circledast_k X)^{(R)}$ is a dilatation in the sense of Raynaud, more precisely, $(X \circledast_k X)^{(R)}$ is the maximal open sub-scheme of the blow-up of $X \circledast_k X$ along $\delta(R)$, where the exceptional divisor is equal to the pull-back of R by the second projection to X (cf. [5] 4.1).



(∗) The two curved lines should be removed (*i.e.*, dotted).

3.3 Let V be a Galois torsor over U of group G, R be an effective rational divisor on X with support in D. We consider $V \times_k V$ as a Galois torsor over $U \times_k U$ of group $G \times G$, and denote by W the quotient of $V \times_k V$ by $\Delta(G)$, where $\Delta\colon G \to G \times G$ is the diagonal homomorphism. The diagonal morphism $\delta_V : V \to V \times_k V$ induces a morphism $\varepsilon_U : U \to W$ lifting the diagonal morphism $\delta_U : U \to U \times_k U$. Note that W represents the sheaf of isomorphisms of G-torsors from $U \times_k V$ to $V \times_k U$ over $U \times_k U$, and that ε_U corresponds to the identity isomorphism of V (identified with the pull-backs of $U \times_k V$ and $V \times_k U$ by δ_U). We denote by Z the integral closure of $(X \circledast_k X)^{(R)}$ in W, by $\pi\colon Z \to (X \times_k X)^{(R)}$ the canonical morphism and by $\varepsilon\colon X \to Z$ the morphism induced by $\varepsilon_U : U \to W$. We have $\pi \circ \varepsilon = \delta^{(R)}$.

(3.3.1)
$$\begin{tikzcd}
 & W \arrow[r] \arrow[d] & Z \arrow[d, "\pi"] & \\
U \arrow[ur, bend left, "\varepsilon_U"] \arrow[r, "\delta_U"] & U \times_k U \arrow[r] & (X \circledast_k X)^{(R)} & X \arrow[l, "\delta^{(R)}"'] \arrow[ul, bend right, "\varepsilon"']
\end{tikzcd}$$

Let $x \in X$. We say that the ramification of V/U at x is *bounded* by $R+$ if the morphism π is étale at $\varepsilon(x)$, and that the ramification of V/U along D is *bounded* by $R+$ if π is étale over an open neighborhood of $\varepsilon(X)$ ([5] 7.3).

We establish several properties of this notion. First, we prove that it satisfies descent for faithfully flat and log-smooth morphisms ([5] 7.7). The second property plays a key role in [5] : if R has integral coefficients, we prove that the ramification of V/U along D is bounded by $R+$ if and only if there exists an open neighborhood Z_0 of $\varepsilon(X)$ in Z which is étale over $(X \circledast_k X)^{(R)}$ and such that $\pi(Z_0)$ contains $(X \circledast_k X)^{(R)} \times_X R$ ([5] 7.13). Third, we relate this notion to its analogue for finite separable extensions of local fields : let ξ be a generic point of D, $\overline{\xi}$ be a geometric point of X above ξ, S be the strict localization of X at $\overline{\xi}$, K be the fraction field of $\Gamma(S, \mathscr{O}_S)$, r be the multiplicity of R at ξ. We put $V \times_U \mathrm{Spec}(K) = \mathrm{Spec}(L)$, where $L = \prod_{i=1}^n L_i$ is a finite product of finite separable extensions of K. We prove in ([5] 7.18) that the ramification of V/U at ξ is bounded by $R+$ if and only if, for every $1 \leqslant i \leqslant n$, the logarithmic ramification of L_i/K is bounded by $r+$ in the sense of (2.2).

3.4 Let V be a Galois torsor over U of group G, Y be the integral closure of X in V, R be an effective rational divisor on X with support in D. Assume that the following conditions are satisfied :

(i) for every geometric point $\overline{y}$ of Y, the inertia group $I_{\overline{y}} \subset G$ of $\overline{y}$ has a normal p-Sylow subgroup ;

(ii) for every generic point ξ of D, the ramification of V/U at ξ is bounded by $R+$.

Then we prove that the ramification of V/U along D is bounded by $R+$ ([5] 7.19). This result is an analogue of the Zariski-Nagata purity theorem.

3.5 Let V be a Galois torsor over U of finite group G. We define the *conductor* of V/U relatively to X to be the minimum effective rational divisor R on X with support in D such that for every generic point ξ of D, the ramification of V/U at ξ is bounded by $R+$. This terminology may be slightly misleading as the ramification of V/U along D may not be bounded by $R+$ in general. However, we prove in ([5] 7.22), as a consequence of 3.4, that under a strong form of resolution of singularities, there exists an snc-pair (X', D') over k and a proper morphism $f\colon X' \to X$ inducing an isomorphism $X' - D' \xrightarrow{\sim} U$, such that if we denote by R' the conductor of V/U relatively to X', the ramification of V/U along D' is bounded by $R'+$.

§ 4. Ramification of ℓ-adic sheaves

4.1 Let (X, D) be an *snc-pair* over k, $U = X - D$, $\mathscr{F}$ be a locally constant constructible sheaf of Λ-modules on U, R be an effective rational divisor on X with support in D, $x \in X$, $\overline{x}$ be a geometric point of X above x. Recall that Λ is a finite local $\mathbb{Z}_\ell$-algebra (1.2). We denote by $\mathrm{pr}_1, \mathrm{pr}_2\colon U \times_k U \to U$ the canonical projections and put

$$\mathscr{H}(\mathscr{F}) = \mathscr{H}om(\mathrm{pr}_2^*\mathscr{F}, \mathrm{pr}_1^*\mathscr{F}). \tag{4.1.1}$$

We prove in ([5] 8.2) that the base change morphism

$$\alpha\colon \delta^{(R)*} j_*^{(R)}(\mathscr{H}(\mathscr{F})) \to j_*\delta_U^*(\mathscr{H}(\mathscr{F})) = j_*(\mathscr{E}nd(\mathscr{F})) \tag{4.1.2}$$

relatively to the Cartesian diagram (3.2.1) is injective. Furthermore, the following conditions are equivalent :

(i) The stalk $\alpha_{\overline{x}}$ of the morphism α at $\overline{x}$ is an isomorphism.

(ii) There exists a Galois torsor V over U trivializing $\mathscr{F}$ such the ramification of V/U at x is bounded by $R+$.

We say that the *ramification of $\mathscr{F}$ at $\overline{x}$ is bounded by* $R+$ if $\mathscr{F}$ satisfies these equivalent conditions. We say that the *ramification of $\mathscr{F}$ along D is bounded by* $R+$ if the ramification of $\mathscr{F}$ at $\overline{x}$ is bounded by $R+$ for every geometric point $\overline{x}$ of X. We establish several properties of this notion similar to those for Galois torsors. In particular, we relate it to the analogue notion for Galois representations of local fields (with possibly imperfect residue fields) ([5] 8.8).

4.2 Let $\mathscr{F}$ be a locally constant constructible sheaf of Λ-modules on U. We define the *conductor of $\mathscr{F}$ relatively to X* to be the minimum of the set of effective rational divisors R on X with support in D such that for every geometric point $\overline{\xi}$ of X above a generic point of D, the ramification of $\mathscr{F}$ at $\overline{\xi}$ is bounded by $R+$. As for Galois torsors, this terminology may be slightly misleading as the ramification of $\mathscr{F}$ along D may not be bounded by $R+$ in general. However, we prove that under a strong form of resolution of singularities, there exists an snc-pair (X', D') over k and a proper morphism $f\colon X' \to X$ inducing an isomorphism $X' - D' \xrightarrow{\sim} U$, such that if we denote by R' the conductor of $\mathscr{F}$ relatively to X', the ramification of $\mathscr{F}$ along D' is bounded by $R'+$ ([5] 8.11).

4.3 The last part of [5] is devoted to studying important specialization properties that lead to the fundamental notion of *cleanliness* and to the definition of the *characteristic cycle*. Let R be an effective divisor on X with support in D.[1] We prove ([5] 4.6) that $(X \circledast_k X)^{(R)}$ is smooth over X and that

$$E^{(R)} = (X \circledast_k X)^{(R)} \times_X R \tag{4.3.1}$$

is canonically isomorphic to the twisted logarithmic tangent bundle

$$\mathbf{V}(\Omega^1_{X/k}(\log D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(R)) \times_X R$$

over R. We denote by $\check{E}^{(R)}$ the dual vector bundle. Consider the commutative diagram with Cartesian squares

$$\begin{tikzcd} E^{(R)} \arrow[r] \arrow[d] & (X \circledast_k X)^{(R)} \arrow[d, "\mathrm{pr}_2"] & U \times_k U \arrow[l, "j^{(R)}"'] \arrow[d] \\ R \arrow[r] & X & U \arrow[l] \end{tikzcd} \tag{4.3.2}$$

Let $\mathscr{G}$ be a sheaf of Λ-modules on $U \times_k U$. We call *R-specialization* of $\mathscr{G}$ and denote by $\nu_R(\mathscr{G}, X)$, the sheaf on $E^{(R)}$ defined by

$$\nu_R(\mathscr{G}, X) = j_*^{(R)}(\mathscr{G})|E^{(R)}. \tag{4.3.3}$$

Let $\mathscr{F}$ be a locally constant constructible sheaf of Λ-modules on U such that its ramification along D is bounded by $R+$, $\mathscr{H}(\mathscr{F})$ be the sheaf on $U \times_k U$ defined in (4.1.1). We prove in ([5] 8.15) that $\nu_R(\mathscr{H}(\mathscr{F}), X)$ is *additive*, which means that its restrictions to the fibers of $E^{(R)}$ over R are invariant by translation (cf. [5] 3.1). This important property was first proved in ([10] 2.25); we give a new proof.

[1]We consider rational divisors on X with support in D and integral coefficients as Cartier divisors on X.

We fix a non-trivial additive character $\psi\colon \mathbb{F}_p \to \Lambda^\times$ and denote by $S \subset \check{E}^{(R)}$ the support of the Fourier-Deligne transform of $\nu_R(\mathscr{H}(\mathscr{F}), X)$ relatively to ψ (cf. [5] 3.4); more precisely, S is the subset of points of $\check{E}^{(R)}$ where the stalks of the cohomology sheaves of the Fourier-Deligne transform are not all zero (cf. [5] 3.5). The additivity of $\nu_R(\mathscr{H}(\mathscr{F}), X)$ is equivalent to the fact that, for every $x \in R$, the set $S \cap \check{E}^{(R)}_x$ is finite (cf. [5] 3.6). We call S the *Fourier dual support* of $\nu_R(\mathscr{H}(\mathscr{F}), X)$. We prove in fact that S is the underlying space of a closed sub-scheme of $\check{E}^{(R)}$ which is finite over R (cf. [5] 8.18). Note that S is a priori a constructible subset of $\check{E}^{(R)}$ and that it is not obvious that it is closed in $\check{E}^{(R)}$. We say that $\nu_R(\mathscr{H}(\mathscr{F}), X)$ is *non-degenerate* if S does not meet the zero section of $\check{E}^{(R)}$ over R.

4.4 Let $\mathscr{F}$ be a locally constant constructible sheaf of Λ-modules on U, ξ be a generic point of D, $X_{(\xi)}$ be the henselization of X at ξ, η_ξ be the generic point of $X_{(\xi)}$, $\overline{\eta}_\xi$ be a geometric generic point of $X_{(\xi)}$, $\mathscr{G}_\xi$ be the Galois group of $\overline{\eta}_\xi$ over η_ξ. We say that $\mathscr{F}$ is *isoclinic* at ξ if the representation $\mathscr{F}_{\overline{\eta}_\xi}$ of $\mathscr{G}_\xi$ is isoclinic (2.3), and that $\mathscr{F}$ is *isoclinic along* D if it is isoclinic at all generic points of D.

Assume first that $\mathscr{F}$ is isoclinic along D, and let R be its conductor relatively to X. We say ([5] 8.23) that $\mathscr{F}$ is *clean* along D if the following conditions are satisfied :

(i) the ramification of $\mathscr{F}$ along D is bounded by $R+$;

(ii) there exists a log-smooth morphism of snc-pairs $f\colon (X', D') \to (X, D)$ over k such that the morphism $X' \to X$ is faithfully flat, that $R' = f^*(R)$ has integral coefficients, and if we put $U' = X' - D'$ and $\mathscr{F}' = \mathscr{F}|U'$, that the R'-specialization $\nu'_{R'}(\mathscr{H}(\mathscr{F}'), X')$ of $\mathscr{H}(\mathscr{F}')$ in the sense of (4.3.3) relatively to (X', D'), is additive and non-degenerate.

Note that we may replace (ii) by the stronger condition that it holds for any morphism f satisfying the same assumptions (cf. [5] 8.24).

This notion can be extended to general sheaves as follows. Let $\overline{x}$ be a geometric point of X. We say that $\mathscr{F}$ is *clean* at $\overline{x}$ if there exists an étale neighborhood X' of $\overline{x}$ in X such that, if we put $U' = U \times_X X'$ and denote by D' the pull-back of D over X', there exists a finite decomposition

$$\mathscr{F}|U' = \oplus_{1 \leqslant i \leqslant n} \mathscr{F}'_i \tag{4.4.1}$$

of $\mathscr{F}|U'$ into a direct sum of locally constant constructible sheaves of Λ-modules $\mathscr{F}'_i$ $(1 \leqslant i \leqslant n)$ on U' which are isoclinic and clean along D' in the previous sense. We say that $\mathscr{F}$ is *clean* along D if it is clean at all geometric points of X (cf. [5] 8.25). Note that for isoclinic sheaves, the two definitions are equivalent (cf. [5] 8.27).

The notion of cleanliness was first introduced by Kato for rank 1 sheaves in [6].

Our definition extends his. It was extended to isoclinic sheaves by the second author (T. S.) in ([10] §3.2).

Roughly speaking, if $\mathscr{F}$ is clean along D, then its ramification along D is controlled by its ramification at the generic points of D. This is the main idea behind the following definition of the *characteristic cycle* of $\mathscr{F}$.

4.5 We assume that X is connected and denote by d the dimension of X, by $\mathrm{T}^*_X(\log D) = \mathbf{V}(\Omega^1_{X/k}(\log D))$ the logarithmic cotangent bundle of X and by $\xi_1, \ldots, \xi_n$ the generic points of D. For each $1 \leqslant i \leqslant n$, we denote by F_i the residue field of X at ξ_i, by $S_i = \mathrm{Spec}(\mathscr{O}_{K_i})$ the henselization of X at ξ_i and by $\eta_i = \mathrm{Spec}(K_i)$ the generic point of S_i. We fix a separable closure $\overline{K}_i$ of K_i and denote by $\mathscr{G}_i$ the Galois group of $\overline{K}_i/K_i$.

Let $\mathscr{F}$ be a locally constant constructible sheaf of free Λ-modules on U which is *clean* along D. We denote by M_i the $\Lambda[\mathscr{G}_i]$-module corresponding to $\mathscr{F}|\eta_i$, by

$$M_i = \oplus_{r \in \mathbb{Q}_{\geqslant 0}} M_i^{(r)} \tag{4.5.1}$$

its slope decomposition and, for each rational number $r > 0$, by

$$M_i^{(r)} = \oplus_\chi M_{i,\chi}^{(r)} \tag{4.5.2}$$

the central character decomposition of $M_i^{(r)}$. Note that $M_{i,\chi}^{(r)}$ is a free Λ-module of finite type for all $r > 0$ and all χ. By enlarging Λ, we may assume that for all rational numbers $r > 0$ and all central characters χ of $M_i^{(r)}$ (*i.e.*, all characters $\chi\colon \mathrm{Gr}^r_{\log}\mathscr{G}_i \to \Lambda_\chi^\times$ that appear in the decomposition (4.5.2)), we have $\Lambda_\chi = \Lambda$. Since $\mathrm{Gr}^r_{\log}\mathscr{G}_i$ is abelian and killed by p (2.4), χ factors uniquely as $\mathrm{Gr}^r_{\log}\mathscr{G}_i \to \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times$, where ψ is the non-trivial additive character fixed in 4.3. We denote also by $\chi\colon \mathrm{Gr}^r_{\log}\mathscr{G}_i \to \mathbb{F}_p$ the induced character and by

$$\mathrm{rsw}(\chi)\colon \mathfrak{m}^r_{\overline{K}_i}/\mathfrak{m}^{r+}_{\overline{K}_i} \to \Omega^1_{F_i}(\log) \otimes \overline{F}_i \tag{4.5.3}$$

its refined Swan conductor (2.4.2). Let F_χ be the field of definition of $\mathrm{rsw}(\chi)$, which is a finite extension of F_i contained in $\overline{F}_i$. The refined Swan conductor $\mathrm{rsw}(\chi)$ defines a line L_χ in $\mathrm{T}^*_X(\log D) \otimes_X F_\chi$. Let $\overline{L}_\chi$ be the closure of the image of L_χ in $\mathrm{T}^*_X(\log D)$. For each $1 \leqslant i \leqslant n$, we put

$$CC_i(\mathscr{F}) = \sum_{r \in \mathbb{Q}_{>0}} \sum_\chi \frac{r \cdot \mathrm{rk}_\Lambda(M_{i,\chi}^{(r)})}{[F_\chi : F_i]} [\overline{L}_\chi], \tag{4.5.4}$$

which is a d-cycle on $\mathrm{T}^*_X(\log D) \times_X D_i$. It follows from the proof of ([10] 1.26) that the coefficient of $[\overline{L}_\chi]$ is an element of $\mathbb{Z}[\frac{1}{p}]$, and hence gives an element of Λ.

Let $\sigma\colon X \to \mathrm{T}^*_X(\log D)$ be the zero-section of $\mathrm{T}^*_X(\log D)$ over X. We define the *characteristic cycle* of $\mathscr{F}$ and denote by $CC(\mathscr{F})$, the d-cycle on $\mathrm{T}^*_X(\log D)$ defined by

$$CC(\mathscr{F}) = \mathrm{rk}_\Lambda(\mathscr{F})[\sigma] - \sum_{1\leqslant i\leqslant n} CC_i(\mathscr{F}). \tag{4.5.5}$$

Recall ([3] 2.1.1) that we associated to $j_!\mathscr{F}$ a *characteristic class*, denoted by $C(j_!\mathscr{F})$, which is a section of $\mathrm{H}^0(X, \mathscr{K}_X)$, where $\mathscr{K}_X = f^!\Lambda$ and $f\colon X \to \mathrm{Spec}(k)$ is the structural morphism.

Conjecture 4.6. *Under the assumptions of* (4.5)*, we have in* $\mathrm{H}^0(X, \mathscr{K}_X)$

$$C(j_!\mathscr{F}) = (CC(\mathscr{F}), [\sigma]), \tag{4.6.1}$$

where the right hand side is the intersection pairing relatively to $\mathrm{T}^*_X(\log D)$.

Kato defined the characteristic cycle of a clean sheaf of rank 1 in [7]. The second author (T. S.) extended the definition to isoclinic and clean sheaves in ([10] 3.6) and proved conjecture 4.6 for these sheaves in (*loc. cit.* 3.7).

4.7 We may optimistically expect that for any locally constant constructible sheaf $\mathscr{F}$ of Λ-modules on U, there exists an snc-pair (X', D') over k and a proper morphism of snc-pairs $(X', D') \to (X, D)$ inducing an isomorphism $X' - D' \xrightarrow{\sim} U$ such that $\mathscr{F}$ is clean along D'. Kato proved this property for rank 1 sheaves on surfaces ([7] 4.1).

References

[1] A. ABBES, T. SAITO, *Ramification of local fields with imperfect residue fields*, Amer. J. of Math. **124** (2002), 879-920.

[2] A. ABBES, T. SAITO, *Ramification of local fields with imperfect residue fields II*, Documenta Math., Extra Volume K. Kato (2003), 3-70.

[3] A. ABBES, T. SAITO, *The characteristic class and ramification of an ℓ-adic étale sheaf*, Invent. Math. **168** (2007) 567-612.

[4] A. ABBES, T. SAITO, *Analyse micro-locale ℓ-adique en caractéristique $p > 0$. Le cas d'un trait*, Publ. RIMS, Kyoto Univ. **45** (2009), 25-74.

[5] A. ABBES, T. SAITO, *Ramification and cleanliness*, preprint (2010), arXiv:1007.3873.

[6] K. KATO, *Swan conductors for characters of degree one in the imperfect residue field case*, dans Algebraic K-theory and algebraic number theory, Contemp. Math. **83** Amer. Math. Soc. (1989), 101-131.

[7] K. KATO, *Class field theory, $\mathscr{D}$-modules and ramification of higher dimensional schemes, Part I*, Amer. J. of Math. **116** (1994), 757-784.

[8] K. KATO, T. SAITO, *On the conductor formula of Bloch* Publ. Math. IHES **100** (2004), 5-151.

[9] G. LAUMON, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Publ. Math. IHES **65** (1987), 131-210.

[10] T. SAITO, *Wild ramification and the characteristic cycle of an ℓ-adic sheaf*, J. Inst. Math. Jussieu **8** (2009), 769-829.
[11] T. SAITO, *Ramification of local fields with imperfect residue fields III*, preprint (2010).
[12] T. SAITO, *Wild ramification of schemes and sheaves*, Proceedings of the International Congress of Mathematicians, Hyderabad, India (2010).