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On the parity of poly-Euler numbers

By

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Abstract

Poly-Euler numbers are introduced in [9] via special values of an L-function as a generalization of the Euler numbers. In this article, poly-Euler numbers with negative index are mainly treated, and the parity of them is shown as the main theorem. Furthermore the divisibility of poly-Euler numbers are also discussed.

§ 1. Introduction

For every integer k, we define *poly-Euler numbers* $E_n^{(k)}$ $(n = 0, 1, 2, \dots)$, which is introduced as a generalization of the Euler number, by

$$\frac{\mathrm{Li}_k(1 - e^{-4t})}{4t(\cosh t)} = \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n. \tag{1.1}$$

Here,

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad (|x| < 1,\ k \in \mathbb{Z})$$

is the k-th polylogarithm. When $k = 1$, $E_n^{(1)}$ is the Euler number defined by

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n^{(1)}}{n!} t^n. \tag{1.2}$$

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The reason why we refer to $E_n^{(k)}$'s as "poly-Euler numbers" will be mentioned in the next section from the point of view of the relation between the poly-Bernoulli number and Arakawa-Kaneko's zeta-function. In this article, we treat some number theoretical properties of poly-Euler numbers with negative index ($k \leq 0$). Our main theorem is Theorem 3.1 described in Section 3, which mentions the parity of poly-Euler numbers can be determined definitely. In Section 4, we discuss the divisibility of poly-Euler numbers via congruence relations of them. Tables 1 and 2 cited at the end of this article are the lists of numerical values of poly-Euler numbers. General properties of poly-Euler numbers including the case of positive index are treated in [8].

§ 2. The poly-Bernoulli numbers and Arakawa-Kaneko's zeta-function

For every integer k, the poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ and the modified poly-Bernoulli numbers $C_n^{(k)}$ introduced by Kaneko [4] are defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(k)}}{n!} t^n \quad \text{and} \quad \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} \frac{C_n^{(k)}}{n!} t^n, \tag{2.1}$$

respectively. When $k = 1$, the above generating functions become

$$\frac{te^t}{e^t-1} \qquad \text{and} \qquad \frac{t}{e^t-1},$$

respectively. Therefore $\mathbb{B}_n^{(k)}$ and $C_n^{(k)}$ are generalizations of the classical Bernoulli numbers. Some number theoretic properties of the poly-Bernoulli number were given by Kaneko [4], Arakawa and Kaneko [2] and others. Furthermore the combinatorial interpretations of $\mathbb{B}_n^{(-k)}$ were given by Brewbaker [3] and Launois [6]. Recently, Shikata [10] gives the alternative proof of the result of Brewbaker.

It is known that the poly-Bernoulli numbers are special values of Arakawa-Kaneko's zeta-function. Arakawa and Kaneko [1] introduced a zeta-function:

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\, dt \qquad (k \geq 1).$$

We refer to the above function as *Arakawa-Kaneko's zeta-function.* Arakawa-Kaneko's zeta-function satisfies $\xi_1(s) = s\zeta(s+1)$, and

$$\xi_k(-n) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \mathbb{B}_l^{(k)} = (-1)^n C_n^{(k)}$$

for any non-positive integer n, where $\zeta(s)$ is the Riemann zeta-function. Hence Arakawa-Kaneko's zeta-function is a kind of generalization of the Riemann zeta-function. Furthermore, we should mention that Arakawa-Kaneko's zeta-function is applied to research on multiple zeta values. For example, Kaneko and Ohno [5] showed a duality property of multiple zeta-star values by using this property.

From the point of view mentioned above, poly-Euler numbers should be defined as special values of an L-function generalized by using the method of Arakawa and Kaneko. Moreover it can be reasonably expected that such L-function has nice properties and applications similar to Arakawa-Kaneko's zeta-function.

The Euler number E_n is the generalized Bernoulli number associated with the Dirichlet character of conductor 4, and the L-function is

$$L(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{e^t + e^{-t}}\, dt \tag{2.2}$$

satisfies $L(-n) = E_n/2$ for any non-negative integer n. The second author gave in [9] a general method for defining L-functions that have similar properties to Arakawa-Kaneko's zeta-function. By using the method, a generalization of $L(s)$ is given by

$$L_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-4t})}{4(e^t + e^{-t})}\, dt \qquad (k \geq 1). \tag{2.3}$$

It is natural to define poly-Euler numbers as special values at non-positive integers of the above L-function. Therefore, we define poly-Euler numbers as above (1.1).

We should mention that the above generalized L-function is applicable to research on multiple L values. In fact, the second author [9] treated the generalization of Dirichlet L-functions for general Dirichlet characters and showed such generalized L-functions can be written in terms of multiple L-functions.

Remark. Although $L_1(s)$ does not reduce to $L(s)$ ($L_1(s) = sL(s+1)$), the above definition (2.3) is optimal as an analogue of Arakawa-Kaneko's zeta-function. In fact, we have $\xi_1(s) = s\zeta(s+1)$.

§ 3. The parity of poly-Euler numbers

In this section, we determine the parity of poly-Euler numbers $(n+1)E_n^{(-k)}$.

Theorem 3.1. *For any non-negative integer k, $(n+1)E_n^{(-k)}$ is even (odd, respectively) integer, when n is odd (even, respectively).*

Proof of Theorem 3.1. We first review an explicit formula:

Lemma 3.2 (Ohno-Sasaki [8]). *For any non-negative integers k and n, we have*

$$(n+1)E_n^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} \sum_{\substack{m=1, \\ m:odd}}^{n+1} \binom{n+1}{m} (4l+2)^{n+1-m}, \tag{3.1}$$

where the symbol $\left\{ {k \atop l} \right\}$ *is the Stirling number of the second kind defined by the recurrence relation*

$$\left\{ {k+1 \atop l} \right\} = \left\{ {k \atop l-1} \right\} + l \left\{ {k \atop l} \right\} \tag{3.2}$$

with

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {k \atop 0} \right\} = \left\{ {0 \atop l} \right\} = 0 \quad (k, l \neq 0)$$

for any integers k *and* l.

See [8] for the detailed proof of Lemma 3.2.

Remark. We easily see that the right-hand side of (3.1) is an integer, since the Stirling numbers are integers. Furthermore this fact indicates that the denominator of poly-Euler number $E_n^{(-k)}$ is at most $n+1$.

We prove Theorem 3.1 by using the above lemma. Note that

$$\sum_{\substack{m=1, \\ m:odd}}^{n+1} \binom{n+1}{m} (4l+2)^{n+1-m} \equiv \begin{cases} 1 \pmod 2 & n : \text{even}, \\ 0 \pmod 2 & n : \text{odd} \end{cases}$$

for any non-negative integer l. Hence, when n is odd, we have Theorem 3.1 immediately from Lemma 3.2.

On the other hand, when n is even, we have

$$(n+1)E_n^{(-k)} \equiv \sum_{l=0}^{k} l! \left\{ {k \atop l} \right\} \equiv \left\{ {k \atop 0} \right\} + \left\{ {k \atop 1} \right\} = \left\{ {k+1 \atop 1} \right\} = 1 \pmod 2$$

for any non-negative integer k. Here, we have used the recurrence relation (3.2). Thus the proof of Theorem 3.1 is completed. □

§ 4. Congruence relations of poly-Euler numbers

In the previous section, we definitely determined the parity of poly-Euler numbers. In this section, we treat the divisibility of poly-Euler numbers via congruence relations of them. In particular, we consider the case of

$$(n+1)E_n^{(-k)} \pmod{n+1},$$

which allows us to evaluate whether $E_n^{(-k)}$ is an integer. In [8], we treat the case when $n+1$ is an odd prime. Hence we discuss the composite cases here.

Theorem 4.1. *For any non-negative integer k, we have*

$$6E_5^{(-k)} \equiv \begin{cases} 4 \pmod 6 & \text{if } k \text{ is even,} \\ 0 \pmod 6 & \text{if } k \text{ is odd.} \end{cases}$$

Proof of Theorem 4.1. From Lemma 3.2, we have

$$6E_5^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} \sum_{j=0}^{2} \binom{6}{2j+1} (4l+2)^{5-2j}. \tag{4.1}$$

We see that $l! \equiv 0 \pmod 6$ for $l \geq 3$ and

$$\begin{aligned} \sum_{j=0}^{2} \binom{6}{2j+1} (4l+2)^{5-2j} &\equiv 2(4l+2)^3 \pmod 6 \\ &\equiv \begin{cases} 4 \pmod 6 & \text{if } l \equiv 0 \pmod 3, \\ 0 \pmod 6 & \text{if } l \equiv 1 \pmod 3, \\ 2 \pmod 6 & \text{if } l \equiv 2 \pmod 3. \end{cases} \end{aligned}$$

Thus (4.1) becomes

$$\begin{aligned} 6E_5^{(-k)} &\equiv (-1)^k 4 \left(\left\{ {k \atop 0} \right\} + \left\{ {k \atop 2} \right\} \right) \pmod 6 \\ &\equiv \begin{cases} 4 \pmod 6 & \text{if } k = 0, \\ 0 \pmod 6 & \text{if } k = 1, \\ (-1)^k \left\{ {k \atop 2} \right\} 4 \pmod 6 & \text{if } k \geq 2. \end{cases} \end{aligned}$$

Therefore, we claim

$$(-1)^k \left\{ {k \atop 2} \right\} \equiv \begin{cases} 1 \pmod 6 & \text{if } k \text{ is even,} \\ 3 \pmod 6 & \text{if } k \text{ is odd.} \end{cases} \tag{4.2}$$

Using the expression $\left\{ {k \atop 2} \right\} = 2^{k-1} - 1$ $(k \geq 2)$, we have

$$\begin{aligned} \left\{ {k \atop 2} \right\} &= 2^{k-1} - 1 = \sum_{j=0}^{k-2} 2^j = 1 + \sum_{\substack{j=1, \\ j:odd}}^{k-2} 2^j + \sum_{\substack{j=2, \\ j:even}}^{k-2} 2^j \\ &\equiv 1 + 2[(k-1)/2] + 4[(k-2)/2] \pmod 6, \end{aligned}$$

which gives (4.2). Thus we have Theorem 4.1. □

By the same way as above, we can also show the following theorem:

Theorem 4.2. *For any non-negative integer k, we have*

$$12E_{11}^{(-k)} \equiv \begin{cases} 4 \pmod{12} & \text{if } k \text{ is even,} \\ 0 \pmod{12} & \text{if } k \text{ is odd.} \end{cases}$$

In general, to understand the prime factors of the numerator of $E_n^{(-k)}$ is proper. In [8], we prove a congruence relation

$$(n+1)E_n^{(-k)} \equiv 0 \pmod{p}$$

holds for any odd prime p, odd positive integer n and non-positive integer k satisfying $k \equiv p-2 \pmod{p-1}$. The second assertion of Theorem 4.1 is also given by combining the above congruence relation with Theorem 3.1.

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Table 1. $E_n^{(k)}$ (positive index)

n \ k	1	2	3	4
0	1	1	1	1
1	0	-1	$-\frac{3}{2}$	$-\frac{7}{4}$
2	-1	$-\frac{1}{9}$	$\frac{41}{27}$	$\frac{221}{81}$
3	0	3	$\frac{7}{6}$	$-\frac{85}{36}$
4	5	$-\frac{51}{25}$	$-\frac{3493}{375}$	$-\frac{31079}{5625}$
5	0	-25	$\frac{521}{90}$	$\frac{77071}{2700}$
6	-61	$\frac{33221}{735}$	$\frac{8169601}{77175}$	$-\frac{41535229}{8103375}$
7	0	427	$-\frac{18313}{70}$	$-\frac{19160833}{44100}$
8	1385	$-\frac{1288391}{945}$	$-\frac{70339397}{33075}$	$\frac{33076559267}{31255875}$
9	0	-12465	$\frac{11557561}{1050}$	$\frac{748381847}{73500}$
10	-50521	$\frac{252042789}{4235}$	$\frac{986047910537}{14674275}$	$-\frac{3065162516767009}{50846362875}$
11	0	555731	$-\frac{4213240267}{6930}$	$-\frac{5414464148791}{16008300}$
12	2702765	$-\frac{706698126353}{195195}$	$-\frac{2983263446051093}{976950975}$	$\frac{530921430493573134689}{132020270006625}$
13	0	-35135945	$\frac{624220472627}{14014}$	$\frac{897755174765977787}{6312606300}$
14	-199360981	$\frac{66380136099899}{225225}$	$\frac{6316799116073503337}{3381753375}$	$-\frac{1910856585406763406371}{5641891880625}$

Table 2. $E_n^{(-k)}$ (negative index)

n \ $-k$	0	-1	-2	-3
0	1	1	1	1
1	2	6	14	30
2	$\frac{13}{3}$	$\frac{109}{3}$	$\frac{493}{3}$	$\frac{1837}{3}$
3	10	222	1798	10710
4	$\frac{121}{5}$	$\frac{6841}{5}$	$\frac{95161}{5}$	$\frac{865081}{5}$
5	$\frac{182}{3}$	8502	$\frac{594554}{3}$	2670350
6	$\frac{1093}{7}$	$\frac{372709}{7}$	$\frac{14331493}{7}$	$\frac{280592677}{7}$
7	410	335886	21078134	591278790
8	$\frac{9841}{9}$	$\frac{19200241}{9}$	$\frac{1951326961}{9}$	$\frac{77624198641}{9}$
9	$\frac{14762}{5}$	$\frac{68177406}{5}$	$\frac{11157142694}{5}$	124916013054
10	$\frac{88573}{11}$	$\frac{964249309}{11}$	$\frac{252966361693}{11}$	$\frac{19811958812317}{11}$
11	$\frac{66430}{3}$	566547774	$\frac{712300066738}{3}$	25898630029750
12	$\frac{797161}{13}$	$\frac{47834153641}{13}$	$\frac{31933012161961}{13}$	$\frac{4834065180508201}{13}$
13	$\frac{1195742}{7}$	$\frac{168029889306}{7}$	25466927239262	$\frac{37348182927733890}{7}$
14	$\frac{7174453}{15}$	$\frac{2358521965909}{15}$	$\frac{3968998515355093}{15}$	$\frac{1148212279510308757}{15}$