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Operator norms and the mean values of multiplicative functions

By

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Abstract

Elementary functional analysis is employed to give necessary and sufficient conditions for the asymptotic mean value of logarithmically sized multiplicative functions.

§ 1. Introduction

In the present paper I establish a weighted version of the Turán–Kubilius inequality and apply its dual to the study of multiplicative functions.

In general outline the arguments follow those in the relevant parts of my Cambridge Tract, [6]. However, the underlying spaces are no longer of finite dimension and the structure of their associated measures does not readily admit of a restriction that would yield the classical inequality of Turán and Kubilius.

A typical positive integer will be denoted by n, a prime by p and a prime-power by q. $f(x) \ll g(x)$ will denote that $|f(x)| \le cg(x)$, for some constant c, holds uniformly on a specified set of x-values.

Theorem 1.1. *Let β be a non-negative real-valued arithmetic function that satisfies $\beta(1) = 1$, $\beta(qn) \le \beta(q)\beta(n)$ if $(q, n) = 1$ and for which the series $\sum \beta(n)$, taken over all positive integers, converges.*

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If $\alpha \geq 2$ and f is a complex-valued additive function, then

$$\left(\exp\left(-\sum_q \beta(q)\right)\sum_{n=1}^{\infty}\beta(n)\left|f(n)-\sum_q f(q)\beta(q)\right|^{\alpha}\right)^{1/\alpha}$$

$$\ll \left(\sum_q |f(q)|^{\alpha}\beta(q)\right)^{1/\alpha} + \left(\sum_q |f(q)|^2\beta(q)\right)^{1/2}$$

provided the series in the upper bound converge. The implicit constant depends at most upon α.

If $1 \leq \alpha < 2$, then there is a similar inequality with upper bound

$$\inf_{\eta>0}\left\{\left(\sum_{|f(q)|>\eta}|f(q)|^{\alpha}\beta(q)\right)^{1/\alpha} + \left(\sum_{|f(q)|\leq\eta}|f(q)|^2\beta(q)\right)^{1/2}\right\}.$$

Each of the inequalities with $\alpha > 1$ in Theorem 1.1 has a dual. For ease of presentation only I confine myself to the cases $1 < \alpha \leq 2$; cf. [6], Chapter 3.

Theorem 1.2. *If $\alpha \geq 2$ and the function β is everywhere positive, then*

$$\left(\sum_q \beta(q)^{1-\alpha}\left|\sum_{\substack{n=1\\ n\cong o(q)}}^{\infty} a_n - \beta(q)\sum_{n=1}^{\infty} a_n\right|^{\alpha}\right)^{1/\alpha}$$

$$+\ (\textit{a similar expression with } \alpha = 2)$$

$$\ll \left(\exp\left((\alpha-1)\sum_q \beta(q)\right)\sum_{n=1}^{\infty}\beta(n)^{1-\alpha}|a_n|^{\alpha}\right)^{1/\alpha}$$

provided the series in the upper bound that involves the complex numbers a_n, converges. The condition $n \cong o(q)$ indicates that q exactly divides n: $q \mid n$ and $(nq^{-1}, q) = 1$.

I give two applications of these results.

§ 1.1. First application.

Theorem 1.3. *Let $\tau > 0$, $\alpha > 1$. If for the complex-valued multiplicative function, g,*

$$\limsup_{x\to\infty}(x(\log x)^{\tau^{\alpha}-1})^{-1}\sum_{n\leq x}|g(n)|^{\alpha}$$

$$\lim_{x\to\infty}(x(\log x)^{\tau-1})^{-1}\sum_{n\leq x} g(n)$$

exist and the second is non-zero, then the series

$$\sum_p p^{-1}(g(p)-\tau)$$

$$\sum_{|g(p)-\tau|\le\tau/2} p^{-1}|g(p)-\tau|^2, \qquad \sum_{|g(p)-\tau|>\tau/2} p^{-1}|g(p)-\tau|^\alpha, \qquad \sum_{p,m\ge 2} p^{-m}|g(p^m)|^\alpha$$

converge and no sum

$$1+\sum_{m=1}^{\infty} g(p^m)p^{-1}$$

vanishes.

Moreover, if $\tau \ge 1$, then the converse holds.

Theorem 1.3 exemplifies the results established in a recent extensive work of Zhang, [7].

The case $\tau = 1$ of Theorem 1.3, in which no logarithmic factors occur, was established by Elliott, [2], [4] and Daboussi, [1]. A considerable study of related theorems, with emphasis on method, may be found in the author's volume, [6].

For its own interest and to meet a challenge of Zhang, [7] §4, see also the review MR 2452645 (2010), I shall illustrate with an appropriate weighted version of the Turán–Kubilius inequality and its dual, that the general approach of that volume applies equally well to the cases $\tau \ne 1$.

Other weighted versions of the Turán–Kubilius inequality may be found in the author's paper on compositions of arithmetic operators, [5].

The second application is concerned only with inequalities.

§ 1.2. Second application

Theorem 1.4. *Let $\tau > 1$, $\alpha > 1$. In order that for positive constants c_j, $j = 1, 2, 3$, the non-negative multiplicative function, g, satisfy the inequalities*

$$\sum_{n\le x} g(n)^\alpha \le c_1 x(\log x)^{\tau^\alpha - 1}$$

$$c_2(\log x)^\tau \le \sum_{n\le x} n^{-1}g(n) \le c_3(\log x)^\tau$$

for all sufficiently large values of x, it is necessary and sufficient that the series

$$\sum_{|g(p)-\tau|\le\tau/2} p^{-1}|g(p)-\tau|^2, \qquad \sum_{|g(p)-\tau|>\tau/2} p^{-1}|g(p)-\tau|^\alpha, \qquad \sum_{p,m\ge 2} p^{-m}|g(p^m)|^\alpha$$

converge and that the partial sums

$$\sum_{p\le y} p^{-1}(g(p)-\tau), \quad y \ge 2,$$

be uniformly absolutely bounded.

If the condition on the mean-value of g is replaced by

$$\sum_{n \le x} n^{-1} g(n)^\alpha \le c_1 (\log x))^{\tau^\alpha},$$

then with no further change the theorem holds for $\tau > 0$.

§ 2. Background Functional Analysis

Let Q be the set of prime-powers. We define a measure, μ, on the power set of Q by $\mu(F) = \sum \beta(q)$, the sum taken over the prime-powers in the set F. Under the initial hypothesis of Theorem 1.1, Q itself has finite measure.

A function $f : Q \to \mathbb{C}$ may be viewed as a point $(f(q))$ in the space $\mathbb{C}^Q$. For $\alpha \ge 1$ there is an attached norm

$$||f||_\alpha = \left(\sum_q |f(q)|^\alpha \beta(q) \right)^{1/\alpha} = \left(\int |f|^\alpha d\mu \right)^{1/\alpha}.$$

$L^\alpha(Q)$ will denote the space of functions f with well defined norm $||f||_\alpha$.

With N the set of positive integers we define a measure

$$\nu(G) = \exp\left(-\sum_q \beta(q) \right) \sum_{n \in G} \beta(n)$$

on the power set of N. Lemma 3.1 which follows will show that ν assigns to N a measure of at most 1.

For $\alpha \ge 1$ and with $(h(n))$ a typical point in $\mathbb{C}^N$, there is a corresponding norm

$$||h||_\alpha = \left(\exp\left(-\sum_q \beta(q) \right) \sum_{n=1}^{\infty} |h(n)|^\alpha \beta(n) \right)^{1/\alpha} = \left(\int |h|^\alpha d\nu \right)^{1/\alpha}.$$

$M^\alpha(N)$ will denote the space of functions h with well defined norm $||h||_\alpha$.

To each complex-valued additive arithmetic function $f(n)$ there corresponds a map $A : \mathbb{C}^Q \to \mathbb{C}^N$ given by

$$(f(q)) \longrightarrow \left(\sum_{q||n} f(q) - \sum_q f(q)\beta(q) \right) = \left(f(n) - \int f d\mu \right).$$

The first part of Theorem 1.1 asserts that for $\alpha \ge 2$, and in an obvious notation, the map

$$L^2 \cap L^\alpha(Q) \xrightarrow{A\alpha} M^\alpha(N)$$

has norm $||A_\alpha||$ bounded in terms of α. The domain of A_α is the intersection of spaces $L^2(Q)$ and $L^\alpha(Q)$, with associated norm $||\ ||_2 + ||\ ||_\alpha$.

We may dualise the diagram, identifying the dual space $(M^\alpha(N))'$ with $M^{\alpha'}(N)$, where $1/\alpha + 1/\alpha' = 1$:

$$(L^2 \cap L^\alpha(Q))' \overset{A'_\alpha}{\longleftarrow} M^{\alpha'}(N).$$

A standard result in functional analysis guarantees that $||A'_\alpha|| = ||A_\alpha||$, hence $||A'_\alpha|| \ll 1$.

For $1 < \alpha \le 2$ there is a corresponding map

$$(L^2 \cap L^{\alpha'}(Q))' \overset{A\alpha}{\longrightarrow} M^\alpha(N)$$

with dual

$$(L^2 \cap L^{\alpha'}(Q))'' \overset{A'_\alpha}{\longleftarrow} M^{\alpha'}(N).$$

Since Q has finite measure we may identify the second dual in the final target space with the original space $L^2 \cap L^{\alpha'}(Q)$. Once again $\|A_\alpha\| \ll 1$, which is the second part of Theorem 1.1, and $\|A'_\alpha\|$ $(= \|A_\alpha\|) \ll 1$, which is the content of Theorem 1.2.

For $\alpha \ge 2$, the composition

$$L^2 \cap L^\alpha(Q) \overset{A_\alpha}{\longrightarrow} M^\alpha(N) \overset{A'_{\alpha'}}{\longrightarrow} L^2 \cap L^\alpha(Q)$$

corresponds to the map T in the author's volume, [6]. There is a similar composition between the spaces $(L^2 \cap L^{\alpha'}(Q))'$ if $1 < \alpha \le 2$.

Note that without severe restrictions upon the function β, $A'_{\alpha'} A_\alpha$ will not be the identity map.

§ 3. Proof of Theorem 1.1

In order that various infinite series arising in the argument should converge absolutely, it is convenient to establish Theorem 1.1 with $\beta(n)n^{-\sigma}$ in place of $\beta(n)$, uniformly for $0 < \sigma \le 2$, and let σ approach zero. The notation of Theorem 1.1 will otherwise remain in force.

For an arithmetic function t, let $E(t)$ denote the sum $\sum t(q)\beta(q)q^{-\sigma}$, taken over the prime-powers, q.

§ 3.1. Large values of f

Lemma 3.1.

$$\sum_{n=1}^{\infty} \beta(n) \le \exp\left(\sum_q \beta(q)\right)$$

provided one of the two series converges.

Proof of Lemma 3.1. Argument by induction shows that

$$\sum_{n=1}^{\infty} \beta(n) \le \prod_p \left(1 + \sum_{m=1}^{\infty} \beta(p^m)\right),$$

and the asserted inequality follows from applications of the inequality $e^b \ge 1 + b$, valid for all real non-negative b.

As a paradigm for later procedures we may replace the excursion through an Euler product by the notion of dominance. The function $J(\sigma) = \sum_{h=1}^{\infty} \beta(n) n^{-\sigma}$, defined for real positive σ, is differentiable and satisfies

$$-J'(\sigma) = \sum_{n=1}^{\infty} \beta(n) n^{-\sigma} \log n = \sum_{n=1}^{\infty} \beta(n) n^{-\sigma} \sum_{q \| n} \log q$$

$$= \sum_q \log q \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \beta(mq)(mq)^{-\sigma} \le \sum_q \beta(q) q^{-\sigma} \log q \cdot J(\sigma),$$

the condition $(m, q) = 1$ having been abandoned. Hence

$$(-\log J(y))' \le \sum_q \beta(q) q^{-y} \log q$$

and, integrating over the half-line $y \ge \sigma$,

$$\log J(\sigma) \le \sum_q \beta(q) q^{-\sigma}.$$

We let σ approach zero to complete the proof.

Define

$$D = \left(\sum_q |f(q)|^2 \beta(q) q^{-\sigma}\right)^{1/2} \ge 0.$$

For complex z, w define the additive function $h = zf_1 + w\bar{f}_1$, where f_1 is the additive function

$$f_1(n) = \sum_{\substack{q \| n \\ |f(q)| \le D}} f(q).$$

Define

$$\psi(z, w) = \sum_{n=1}^{\infty} \beta(n) n^{-\sigma} \exp(D^{-1}(h(n) - E(h))).$$

Lemma 3.2.

$$\psi(z,w) \ll \exp\left(\sum_q \beta(q)q^{-\sigma}\right)$$

uniformly for $|z| \le 1$, $|w| \le 1$, $0 < \sigma \le 2$.

Proof of Lemma 3.2. Define the multiplicative function $g(n) = \exp(D^{-1}h(n))$. Within the constraints of Lemma 3.2, $|g(q)| = \exp(D^{-1}\mathrm{Re}(h(q))) \le e^2$. By Lemma 3.1

$$\sum_{m=1}^{\infty} |g(n)|\beta(n)n^{-\sigma} \ll \exp\left(\sum_q |g(q)|\beta(q)q^{-\sigma}\right).$$

Thus

$$\psi(z,w)\exp\left(-\sum_q \beta(q)q^{-\sigma}\right)$$

$$\ll \exp\left(\sum_q \beta(q)q^{-\sigma}\{|g(q)| - 1 - \mathrm{Re}(D^{-1}h(q))\}\right).$$

In view of the inequality $|e^s - 1 - s| \le |s|^2 \max(1, e^{Re(s)})$, uniform in complex s, the sum in the bounding exponential is

$$\ll \sum_q \beta(q)q^{-\sigma}D^{-2}|\mathrm{Re}(h(q))|^2 \ll D^{-2}\sum_q \beta(q)q^{-\sigma}|f(q)|^2 \ll 1.$$

Lemma 3.2 is established.

Lemma 3.3. *For* $\alpha > 0$,

$$\sum_{n=1}^{\infty} \beta(n)n^{-\sigma}|f_1(n) - E(f_1)|^{\alpha} \ll D^{\alpha}\exp\left(\sum_q \beta(q)q^{-\sigma}\right).$$

Proof of Lemma 3.3. Application of Hölder's inequality shows that it will suffice to establish the bound for integral values $2k$ of α. In view of Lemma 3.2, the desired inequality follows from the representation

$$D^{-2k}\sum_{n=1}^{\infty} \beta(n)n^{-\sigma}|f_1(n) - E(f_1)|^{2k}$$

$$= -(k!)^2(4\pi^2)^{-1}\int_{|w|=1}\int_{|z|=1} \psi(z, z^{-1}w)z^{-1}w^{-k-1}dzdw.$$

This completes our treatment of the large values of f.

§ 3.2. Small values of f

Lemma 3.4. *If the non-negative real arithmetic function t satisfies $t(q) \le 1$ and $t(qm) \le t(q) + t(m)$ when $(m, q) = 1$, then*

$$\sum_{n=1}^{\infty} t(n)^k \beta(n) \le \left(k + \sum_q t(q)\beta(q) \right)^k \sum_{n=1}^{\infty} \beta(n)$$

for each non-negative integer, k.

Proof of Lemma 3.4. The proof goes by induction on k for all such functions, t.

The case $k = 0$ is clear.

Assume that the proposition holds for non-negative integers up to r. Then

$$\sum_{n=1}^{\infty} t(n)^{r+1} \beta(n) \le \sum_{n=1}^{\infty} t(n)^r \beta(n) \sum_{q \| n} t(q)$$

$$= \sum_q t(q) \sum_{n \cong o(q)} t(n)^r \beta(n) \le \sum_q t(q)\beta(q) \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} t(qm)^r \beta(m).$$

Here $t(qm)^r \le (t(q)+t(m))^r \le (1+t(m))^r$, which we expand. Suppressing the condition $(q, m) = 1$, we apply the induction hypothesis to each of the resulting sums. The upper bound is at most

$$\sum_q t(q)\beta(q) \left(r + 1 + \sum_q t(q)\beta(q) \right)^r \sum_{m=1}^{\infty} \beta(m),$$

from which the validity of the proposition with $k = r + 1$ follows.

Corollary 3.5. *The function*

$$t(n) = \sum_{q \| n, |f(q)| > D} 1$$

satisfies

$$\sum_{n=1}^{\infty} t(n)^{\alpha} \beta(n) n^{-\sigma} \le (\alpha + 2)^{\alpha+1} \sum_{n=1}^{\infty} \beta(n) n^{-\sigma}$$

uniformly for $\alpha \ge 0$, $\sigma \ge 0$.

A further application of Hölder's inequality shows that it will suffice to establish the corollary for α a positive integer k, with an upper bound factor $(k + 1)^k$. Since

$$\sum_{|f(q)| > D} \beta(q) q^{-\sigma} \le D^{-2} \sum_q |f(q)|^2 \beta(q) q^{-\sigma} = 1,$$

we may apply Lemma 3.4 with β replaced by the function $n \mapsto \beta(n)n^{-\sigma}$.

To complete the proof of Theorem 1.1 when $\alpha \geq 2$, define

$$f_2(n) = \sum_{q \| n, |f(q)| > D} f(q).$$

Lemma 3.6. *If $\alpha \geq 1$, then*

$$\sum_{n=1}^{\infty} \beta(n)n^{-\sigma}|f_2(n) - E(f_2)|^{\alpha} \ll \sum_q |f_2(q)|^{\alpha}\beta(q)q^{-\sigma} \sum_{n=1}^{\infty} \beta(n)n^{-\sigma}$$

uniformly for $\sigma \geq 0$.

Proof of Lemma 3.6. With the above definition of t,

$$|f_2(n)|^{\alpha} \leq \sum_{q \| n} |f_2(q)|^{\alpha} t(n)^{\alpha - 1},$$

and

$$\sum_{n=1}^{\infty} \beta(n)n^{-\sigma}|f_2(n)|^{\alpha} \leq \sum_q |f_2(q)|^{\alpha}\beta(q)q^{-\sigma} \sum_{(m,q)=1} t(qm)^{\alpha-1}\beta(m)m^{-\sigma}.$$

Again $t(qm)^{\alpha-1} \leq (1 + t(m))^{\alpha-1} \ll 1 + t(m)^{\alpha-1}$. We omit the condition $(m, q) = 1$ and apply the corollary to Lemma 3.4.

Moreover, an application of Hölder's inequality with $\alpha^{-1} + \gamma^{-1} = 1$, suitably interpreted, yields

$$|E(f_2)| \leq \left(\sum_q |f_2(q)|^{\alpha}\beta(q)q^{-\sigma} \right)^{1/\alpha} \left(\sum_{|f(q)|>D} \beta(q)q^{-\sigma} \right)^{1/\gamma}$$

with a final sum that does not exceed 1.

The proof of Lemma 3.6 is complete.

Since $f = f_1 + f_2$, from Lemma 3.3 and Lemma 3.6 combined with Lemma 3.1

$$\left(\exp\left(-\sum_q \beta(q)q^{-\sigma} \right) \sum_{n=1}^{\infty} \beta(n)n^{-\sigma}|f(n) - E(f)|^{\alpha} \right)^{1/\alpha}$$

$$\ll \left(\sum_q |f(q)|^{\alpha}\beta(q)q^{-\sigma} \right)^{1/\alpha} + \left(\sum_q |f(q)|^2\beta(q)q^{-\sigma} \right)^{1/2}$$

uniformly for $0 < \sigma \leq 2$, and we may let σ approach zero.

For the cases $1 \leq \alpha < 2$ of Theorem 1.1 the following modification suffices.

Lemma 3.7. *The function*

$$\lambda(y) = \left(\sum_{|f(q)|>y} |f(q)|^{\alpha} \beta(q) q^{-\sigma} \right)^{1/\alpha} + \left(\sum_{|f(q)|\le y} |f(q)|^{2} \beta(q) q^{-\sigma} \right)^{1/2}$$

defined for real $y > 0$, satisfies $\lambda(\lambda(y)) \le 2\lambda(y)$.

Proof of Lemma 3.7. If $\lambda(y) = w \ge y$, then

$$\begin{aligned}\lambda(w) \le &\left(\sum_{|f(q)|>y} |f(q)|^{\alpha} \beta(q) q^{-\sigma} \right)^{1/\alpha} + \left(\sum_{|f(q)|\le y} |f(q)|^{2} \beta(q) q^{-\sigma} \right)^{1/2} \\ &+ \left(\sum_{y<|f(q)|\le w} |f(q)|^{2} \beta(q) q^{-\sigma} \right)^{1/2} \\ &\le w + w^{1-\alpha/2} \left(\sum_{|f(q)|>y} |f(q)|^{\alpha} \beta(q) q^{-\sigma} \right)^{1/2} \le 2w.\end{aligned}$$

There is a similar argument if $w < y$.

This completes the proof of Lemma 3.7.

In particular, for any value, w, of $\lambda(y)$,

$$\sum_{|f(q)|>w} \beta(q) q^{-\sigma} \le w^{-\alpha} \sum_{|f(q)|>w} |f(q)|^{\alpha} \beta(q) q^{-\sigma} \le w^{-\alpha} \lambda(w)^{\alpha} \le 2^{\alpha},$$

$$\sum_{|f(q)|\le w} |f(q)|^{2} \beta(q) q^{-\sigma} \le \lambda(w)^2 \le 4w^2.$$

We may replace D by $\lambda(\eta)$ and follow the argument for the cases $\alpha \ge 2$.

This completes the proof of Theorem 1.1.

§ 3.3. Remarks

If $0 < \alpha < 1$, then for any $\eta > 0$ the argument supporting Theorem 1.1 delivers the inequality

$$\left(\exp\left(-\sum_{q} \beta(q) \right) \sum_{n=1}^{\infty} \beta(n) \left| f(n) - \sum_{|f(q)|\le\eta} f(q)\beta(q) \right|^{\alpha} \right)^{1/\alpha}$$

$$\ll \left(\sum_{|f(q)|>\eta} |f(q)|^{\alpha} \beta(q) \right)^{1/\alpha} + \left(\sum_{|f(q)|\le\eta} |f(q)|^{2} \beta(q) \right)^{1/2}.$$

the implicit constant depending at most upon α. Since they do not satisfy the triangle inequality, the sums involving α are not norms. This inequality is guided rather by an aesthetic from the theory of probability.

Over the range $0 < \alpha < 1$ similar modifications should be made to the inequalities of [6], Theorem 2.1, [5], Theorems 1 and 2 and therefore [6], Theorems 34.1 and 34.2. This does not affect any of the applications in those works since only inequalities with $\alpha > 1$ are employed.

For $0 < \alpha < 1$ other variants are possible, cf. [3].

With modest assumptions inequalities of Turán–Kubilius type may be extended to normed freely generated commutative semigroups, as is sketched in exercises 14–17 of the author's volume [6], Chapter 34.

This ends the remarks.

§ 4. Proof of Theorem 1.3

I shall concentrate on the necessity of the conditions involving the primes. I shall apply Theorem 1.2 with $\beta(n) = \tau^{\omega(n)} n^{-\sigma}$, where $\omega(n)$ denotes the number of distinct prime divisors of n and $1 < \sigma \le 2$. For ease of presentation only, I confine myself to the cases $\alpha \ge 2$. Theorem 1.2 then delivers the estimate

$$\left(\sum_q q^{(\alpha-1)\sigma} \left| \sum_{n \cong o(q)} a_n n^{-\sigma} - \tau q^{-\sigma} \sum_{n=1}^{\infty} a_n n^{-\sigma} \right|^{\alpha} \right)^{1/\alpha}$$

$$+ \text{(a similar expression with } \alpha = 2)$$

$$\ll (\sigma - 1)^{-\tau/\gamma} \left(\sum_{n=1}^{\infty} |a_n|^{\alpha} \tau^{-(\alpha-1)\omega(n)} n^{-\sigma} \right)^{1/\alpha}$$

with $1/\alpha + 1/\gamma = 1$, valid for all complex numbers a_n for which the final series converges.

§ 4.1. General remarks

Let $0 < w < 1/2$. Provided $0 < \delta < \min(\alpha^{-1}, (\alpha - 1)w/2)$, the uniform bound on $\sum |g(n)|^{\alpha}, n \le x$, guarantees uniform bounds $g(q) \ll q^{1-\delta}$ and

$$\sum_{q \le x, |g(q)| > q^w} |g(q)| \ll x^{1-\delta}.$$

An integration by parts shows the series $\sum |g(q)| q^{-1}$, taken over the prime-powers for which $|g(q)| > q^w$, to converge, consequently the series $\sum |g(p^m)| p^{-m}$, taken over the prime-powers with $m \ge 2$, also to converge.

For each prime p,

$$\sum_{m=1}^{\infty} |g(p^m)|p^{-m} \ll \sum_{m=1}^{\infty} p^{-\delta m} < 1/2,$$

provided p is sufficiently large.

The function

$$F(\sigma) = \sum_{n=1}^{\infty} |g(n)|^{\alpha} n^{-\sigma} = \sigma \int_1^{\infty} y^{-\sigma-1} \sum_{n \le y} |g(n)|^{\alpha} dy$$

satisfies

$$F(\sigma) \ll \sigma \int_1^{\infty} y^{-\sigma} (\log y)^{\tau^{\alpha}-1} dy \ll \Gamma(\tau^{\alpha})(\sigma-1)^{-\tau^{\alpha}}, \quad 1 < \sigma \le 2,$$

the last step after a change of variables $y = \exp(u(\sigma-1)^{-1})$.

Similarly,

$$G(\sigma) = \sum_{n=1}^{\infty} g(n) n^{-\sigma} = (1+o(1))A\Gamma(\tau)(\sigma-1)^{-\tau} \quad \text{as} \quad \sigma \to 1+,$$

where A is the value of the limit implicit in the hypothesis on the sum $\sum g(n)$, $n \le x$, and is not zero.

§ 4.2. Necessity, Case $0 < \tau \le 1$

We apply the dual Turán–Kubilius inequality to $|g(n)|$. Since $1 + ta \le (1+a)^t$ if $a \ge 0$, $t \ge 1$,

$$\prod_p \left(1 + \tau^{-(\alpha-1)} \sum_{m=1}^{\infty} |g(p^m)|^{\alpha} p^{-m\sigma}\right) \le F(\sigma)^{\tau^{-(\alpha-1)}} \ll (\sigma-1)^{-\tau},$$

hence

$$\left(\sum_q q^{-\sigma} \left| |g(q)| \sum_{(n,q)=1} |g(n)| n^{-\sigma} - \tau \sum_{n=1}^{\infty} |g(n)| n^{-\sigma} \right|^{\alpha}\right)^{1/\alpha}$$

$$+ \text{(a similar expression with } \alpha = 2)$$

$$\ll \zeta(\sigma)^{\tau/\gamma} (\sigma-1)^{-\tau/\alpha} \ll (\sigma-1)^{-\tau},$$

uniformly for $1 < \sigma \le 2$. Noting that

$$\sum_{n=1}^{\infty} |g(n)| n^{-\sigma} \ge \left| \sum_{n=1}^{\infty} g(n) n^{-\sigma} \right| \gg (\sigma-1)^{-\tau},$$

we derive the convergence of the series

$$\sum_q q^{-1}||g(q)|\theta(q)^{-1} - \tau|^\alpha \quad \text{and} \quad \sum_q q^{-1}||g(q)|\theta(q)^{-1} - \tau|^2,$$

where $\theta(q) = 1 + \sum_{m=1}^{\infty} |g(q_0^m)|q_0^{-m}$ and q_0 is the prime of which q is a power.

From our general remarks the $\theta(q)$ are uniformly bounded, the series $\sum q^{-1}\, |\theta(q) - 1|^\alpha$ and $\sum q^{-1}|\theta(q) - 1|^2$ converge, and so do the series

$$\sum_q q^{-1}||g(q)| - \tau|^\alpha \quad \text{and} \quad \sum_q q^{-1}||g(q)| - \tau|^2.$$

In particular, for any ε, $0 < \varepsilon < 1$, the series $\sum |g(q)|q^{-1}$, taken over the prime-powers for which $|g(q)| > \tau + \varepsilon$, converges. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} |g(n)n^{-\sigma} &= \prod_p \left(1 + \sum_{m=1}^{\infty} |g(p^m)|p^{-m\sigma}\right) \\ &\le \exp\left(\sum_q |g(q)|q^{-\sigma}\right) \ll \exp\left((\tau+\varepsilon)\sum_q q^{-\sigma}\right) \\ &\ll \zeta(\sigma)^{\tau+\varepsilon} \ll (\sigma-1)^{-(\tau+\varepsilon)}\end{aligned}$$

holds uniformly for $1 < \sigma \le 2$.

If now

$$\phi(\sigma) = 1 + \sum_{m=1}^{\infty} g(p^m)p^{-m\sigma}$$

vanishes for $\sigma = 1$, then since the series representing $\phi'(\sigma)$ converges absolutely, $\phi(\sigma) \ll \sigma - 1$ as $\sigma \to 1+$. Thus

$$(\sigma-1)^{-\tau} \ll |G(\sigma)| \ll |\phi(\sigma)| \prod_{\substack{\ell\, prime \\ \ell \ne p}} \left(1 + \sum_{m=1}^{\infty} |g(\ell^m)|\ell^{-m\sigma}\right) \ll (\sigma-1)^{1-(\tau+\varepsilon)},$$

and as $\sigma \to 1+$ a contradiction ensues.

We can now reapply the earlier argument with $g(n)$ in place of $|g(n)|$ and obtain the convergence of the series

$$\sum_q q^{-1}|g(q) - \tau|^\alpha \quad \text{and} \quad \sum_q q^{-1}|g(q) - \tau|^2.$$

As before, the only lower bound that we require is $|G(\sigma)| \gg (\sigma-1)^{-\tau}$, $1 < \sigma \le 2$.

Only the convergence of the series $\sum p^{-1}(g(p) - \tau)$ remains to be assured.

Employing its Euler product representation, as $\sigma \to 1+$

$$G(\sigma) = (1 + o(1))B_1 \exp\left(\sum_{p>p_0} \log\left(1 + \sum_{m=1}^{\infty} g(p^m)p^{-m\sigma}\right)\right)$$

$$= (1 + o(1))B_2 \exp\left(\sum_q g(p)p^{-\sigma}\right),$$

where B_1, B_2 are non-zero constants and p_0 is a suitably large fixed prime. Combining this with the asymptotic estimate $G(\sigma) = (1 + o(1))A\Gamma(\tau)(\sigma - 1)^{-\tau} = (1 + o(1))A\Gamma(\tau)\zeta(\sigma)^{\tau}$, we obtain the existence of

$$\lim_{\sigma \to 1+} \sum_q (g(p) - \tau)p^{-\sigma}.$$

A simple tauberian argument now suffices.

Let $0 < \varepsilon < 1$. We denote the sum function of the series $\sum(g(p) - \tau)p^{-\sigma}$ by $\theta(\sigma)$, set $y = \exp(\varepsilon(\sigma - 1)^{-1})$ and employ the decomposition

$$\theta(\sigma) - \sum_{p \le y}(g(p) - \tau)p^{-1} = \sum_{p>y}(g(p) - \tau)p^{-\sigma} + \sum_{p \le y}(g(p) - \tau)(p^{-\sigma} - p^{-1}).$$

An application of the Cauchy–Schwarz inequality together with the well-known Chebyshev bound $\pi(x) \ll x(\log x)^{-1}$ shows the second of these sums to be

$$\ll \left(\sum_{p>y} |g(p) - \tau|^2 p^{-1}\right)^{1/2} = o(1) \quad \text{as} \quad \sigma \to 1+, \quad y \to \infty.$$

Moreover, $p^{-\sigma} - p^{-1} \ll (\sigma - 1)p^{-1} \log p$ uniformly for $p \le y$. Hence the third sum is similarly

$$\ll \left((\sigma - 1)^2 \sum_{p \le y} (\log p)^2 p^{-1}\right)^{1/2} \left(\sum_p |g(p) - \tau|^2 p^{-1}\right)^{1/2} \ll \varepsilon,$$

the implied constant absolute.

Altogether,

$$\limsup_{\sigma \to 1+} \left|\sum_{p \le y}(g(p) - \tau)p^{-1} - \theta(\sigma)\right| \ll \varepsilon$$

and the desired convergence is obtained.

§ 4.3. Necessity Case, $\tau > 1$

The inequality $1 + ta \le (1 + a)^t$ is not immediately available.

Define the multiplicative function h by $h(p) = g(p)$ if $|g(p)| \leq p^w$, where w is the parameter appearing in the general remarks, $2w\alpha < 1$, and $h = 0$ on all other prime-powers. Then

$$\sum_{n=1}^{\infty} |h(n)|^\alpha n^{-\sigma} = \prod_{|g(p)| \leq p^w} (1 + |g(p)|^\alpha p^{-\sigma}) \leq F(\sigma) \ll (\sigma - 1)^{-\tau^\alpha}$$

is available. Moreover

$$(\sigma - 1)^{-\tau} \ll |G(\sigma)| \ll \exp\left(\sum_p |g(p)| p^{-\sigma}\right)$$

$$\ll \prod_{|g(p)| \leq p^w} (1 + |g(p)| p^{-\sigma}) \ll \sum_{n=1}^{\infty} |h(n)| n^{-\sigma},$$

since $\sum |g(p)| p^{-1}$, $|g(p)| > p^w$, converges.

The argument of the previous section then yields the convergence of the series

$$\sum ||h(p)| - \tau|^\alpha p^{-1} \quad \text{and} \quad \sum ||h(p)| - \tau|^2 p^{-1},$$

hence that of the series

$$\sum |g(p) - \tau|^\alpha p^{-1} \quad \text{and} \quad \sum |g(p) - \tau|^2 p^{-1}$$

taken over those primes p for which $|g(p)| \leq p^w$, with the concomitant non-vanishing of the sums $1 + \sum_{m=1}^{\infty} g(p^m) p^{-m}$.

From our general remarks there is a non-zero constant B_3 for which

$$G(\sigma) = (1 + o(1)) B_3 \exp\left(\sum_p h(p) p^{-\sigma}\right) \quad \text{as} \quad \sigma \to 1+.$$

Combined with the asymptotic estimate $G(\sigma) = (1+o(1)) A\Gamma(\tau)(\sigma-1)^{-\tau}$ this guarantees the existence of

$$\lim_{\sigma \to 1+} \sum (h(p) - \tau) p^{-\sigma}$$

and the convergence of the series $\sum (g(p) - \tau) p^{-1}$, $|g(p)| \leq p^w$. The restriction $|g(p)| \leq p^w$ may then be omitted.

Lemma 4.1. *If* $\alpha > 0$, $|w - 1| \leq 1/2$, *then*

$$|w|^\alpha = 1 + Re(\alpha(w - 1)) + O(|w - 1|^2).$$

Proof of Lemma 4.1. With the principal value of the logarithm

$$|w|^\alpha = \exp(\mathrm{Re}\{\alpha \log(1+w-1)\}) = \exp(\mathrm{Re}(\alpha(w-1)) + O(|w-1|^2))$$
$$= \exp(\mathrm{Re}\{\alpha(w-1)\}) + O(|w-1|^2)$$

and the result is evident.

Replacing w by w/τ, $\tau > 0$,

$$|w|^\alpha - \tau^\alpha = \alpha\tau^{\alpha-1}\mathrm{Re}\,(w-\tau) + O(|w-\tau|^2)$$

uniformly for $|w-\tau| \le \tau/2$. Thus

$$\sum{}' (|g(p)|^\alpha - \tau^\alpha)p^{-\sigma} = \alpha\tau^{\alpha-1}\mathrm{Re}\left\{\sum{}' (g(p)-\tau)p^{-\sigma}\right\}$$
$$+ O\left(\sum{}' |g(p)-\tau|^2 p^{-\sigma}\right),$$

all sums over primes for which $|g(p)-\tau| \le \tau/2$. Combined with what we have established so far

$$\sum_p |h(p)|^\alpha p^{-\sigma} = \tau^\alpha \sum_p p^{-\sigma} + O(1)$$

uniformly for $1 < \sigma \le 2$.

In particular,

$$\prod_p (1 + |h(p)|^\alpha p^{-\sigma}) = \exp\left(\sum_p \{|h(p)|^\alpha p^{-\sigma} + O(|h(p)|^{2\alpha} p^{-2\sigma})\}\right)$$
$$= (1+o(1))B_4(\sigma-1)^{-\tau^\alpha} \quad \text{as} \quad \sigma \to 1+,$$

with $B_4 \ne 0$. As a consequence

$$\prod_{|g(p)|>p^w} \left(1 + \sum_{m=1}^{\infty} |g(p^m)|^\alpha p^{-m\sigma}\right) \ll (\sigma-1)^{\tau^\alpha} F(\sigma) \ll 1, \quad 1 < \sigma \le 2.$$

The product over the primes p with $|g(p)| > p^w$ is uniformly bounded above and the series

$$\sum_{|g(p)|>p^w} \sum_{m=1}^{\infty} |g(p^m)|^\alpha p^{-m}$$

converges.

This achieves the convergence of the series

$$\sum_{|g(p)-\tau|>\tau/2} |g(p)-\tau|^\alpha p^{-1}, \qquad \sum_{|g(p)-\tau|\le\tau/2} |g(p)-\tau|^2 p^{-1}$$

and $\sum(g(p)-\tau)p^{-1}$.

In this case we are left to establish the convergence of the series

$$\sum_{p,m\geq 2}|g(p^m)|^\alpha p^{-m}.$$

For those primes p with $|g(p)|\leq p^w$ we apply the elementary inequality $(1-b)(1+a+b)\geq 1+a/4$, valid for $0\leq b\leq 1/2$, $a\geq 4b^2$. With

$$a=a_p=\sum_{m=2}^{\infty}|g(p^m)|^\alpha p^{-m\sigma},\quad b=|h(p)|^\alpha p^{-\sigma},$$

and provided $p\geq 5$,

$$\exp(-|h(p)|^\alpha p^{-\sigma})(1+|h(p)|^\alpha p^{-\sigma}+a_p)\geq 1+a_p/4$$

The product $\prod(1+a_p/4)$, taken over those primes for which $|g(p)|\leq p^w$ and $a_p\geq 4(|h(p)|^\alpha p^{-\sigma})^2$, is not more than

$$\prod_{|g(p)|\leq p^w}\exp(-|h(p)|^\alpha p^{-\sigma})(1+|h(p)|^\alpha p^{-\sigma}+a_p)$$

$$\leq\exp\left(\sum_p -|h(p)|^\alpha p^{-\sigma}\right)F(\sigma)\ll(\sigma-1)^{\tau^\alpha}(\sigma-1)^{-\tau^\alpha}\ll 1,$$

uniformly for $1<\sigma\leq 2$. As a consequence the series

$$\sum_p\sum_{m=2}^{\infty}|g(p^m)|^\alpha p^{-m\sigma},$$

restricted to the same primes, is bounded uniformly for $1<\sigma\leq 2$.

The corresponding sum over the remaining primes does not exceed

$$4\sum_{|g(p)|\leq p^w}(|g(p)|^\alpha p^{-\sigma})^2\leq 4\sum_p p^{-2(1-\alpha w)}\ll 1,$$

and is likewise uniformly bounded.

The sum

$$\sum_{p,m\geq 2}|g(p^m)|^\alpha p^{-m\sigma}$$

is bounded uniformly for $1<\sigma\leq 2$ and we may let $\sigma\to 1+$.

§ 4.4. Sufficiency

I shall be brief. The argument rests upon the following two results.

Lemma 4.2. *If the real non-negative arithmetic function β satisfies $\beta(1) = 1$, $\beta(qm) \le \beta(q)\beta(m)$ when $(q,m) = 1$ and $y^{-1}\sum_{q\le y}\beta(q)\log q \le \Delta$ uniformly for $1 \le y \le x$, then*

$$\sum_{2\le n\le x}\beta(n) \ll \Delta x \exp\left(\sum_{q\le x}(\beta(q)-1)q^{-1}\right),$$

the implied constant absolute.

Proof of Lemma 4.2 This result is essentially a special case of [5], Lemma 1. See, also, [6], Lemma 2.2.

Lemma 4.3. *With β as in Lemma 4.2 and f a complex-valued additive function*

$$\sum_{n\le x}\beta(n)\left|f(n) - \sum_{q\le x} f(q)\beta(q)q^{-1}\right|^2$$

$$\ll \Delta x \exp\left(\sum_{q\le x}(\beta(q)-1)q^{-1}\right)\sum_{q\le x}|f(q)|^2\beta(q)q^{-1},$$

the implied constant absolute.

Proof of Lemma 4.3. One may proceed along the lines of the proof of the present Theorem 1.1, bearing in mind the argument of [5] and [6], Chapter 2. The term with $n = 1$ may be restored to the first summation since $f(1) = 0$,

$$\left|\sum_{q\le x} f(q)\beta(q)q^{-1}\right|^2 \le \sum_{q\le x}|f(q)|^2\beta(q)q^{-1}\sum_{q\le x}\beta(q)q^{-1},$$

$\beta(q)\log q \le q\Delta$ uniformly for $q \le x$ and

$$\sum_{q\le x}\beta(q)q^{-1} \ll \Delta\sum_{q\le x}(\log q)^{-1} \ll \Delta x(\log x)^{-1}.$$

As for Theorem 1.1, there are versions of Lemma 4.3 involving mean α-powers with $\alpha \ge 1$.

To begin with I shall assume that the function q satisfies $|g(q) - \tau| \le \tau/2$ on all prime-powers q. As before, q_0 will denote the prime of which q is a power.

For $r > 0$, define the multiplicative function $g_r(n)$ by $g_r(q) = g(q)$ if $q_0 \le r$, $g_r(q) = \tau$ otherwise.

In view of the inequality $|e^z - 1| \le |z| \max(e^{Re\, z}, 1)$, valid for all complex z,

$$|g(n) - g_r(n)| \le \left| \sum_{\substack{q \| n \\ q_0 > r}} \log(g(q)\tau^{-1}) \right| \max(|g(n)|, |g_r(n)|),$$

with the principal value of the logarithm.

We note that by Lemma 4.2

$$\sum_{n \le x} \tau^{-\omega(n)} |g(n)|^2 \ll x(\log x)^{-1} \exp\left(\sum_{q \le x} \tau^{-1} |g(q)|^2 \right)$$

$$\ll x(\log x)^{\tau - 1},$$

since $|g(q)|^2 = |g(q) - \tau|^2 + 2\tau \mathrm{Re}(g(q) - \tau) + \tau^2$. Likewise

$$\sum_{n \le x} \tau^{-\omega(n)} |g_r(n)|^2 \ll x(\log x)^{\tau - 1},$$

uniformly for $x \ge 2$, $r \ge 2$.

With the additive function f defined by $f(q) = \log(g(q)\tau^{-1})$ if $q_0 > r$, $f(q) = 0$ otherwise, an application of the Cauchy–Schwarz inequality delivers the bound

$$\sum_{n \le x} |g(n) - g_r(n)| \ll \left(x(\log x)^{\tau - 1} \sum_{n \le x} \tau^{\omega(n)} |f(n)|^2 \right)^{1/2}.$$

Since $\tau f(q) = g(q) - \tau + O(|g(q) - \tau|^2)$, Lemma 4.3 shows that

$$\sum_{n \le x} \tau^{\omega(n)} |f(n) - L|^2 \ll x(\log x)^{\tau - 1} \sum_{\substack{q \le x \\ q_0 > r}} |g(q) - \tau|^2 q^{-1},$$

where

$$L = \sum_{q \le x} \tau f(q) q^{-1} = \sum_{r < p \le x} (g(p) - \tau) p^{-1}$$

$$+ O\left(\sum_{p > r} \sum_{m=2}^{\infty} p^{-m} \right) + O\left(\sum_{q_0 > r} |g(q) - \tau|^2 q^{-1} \right).$$

Hence

$$\limsup_{r \to \infty} \limsup_{x \to \infty} (x(\log x)^{\tau - 1})^{-1} \sum_{n \le x} |g(n) - g_r(n)| = 0.$$

The function $g_r(n)$ is $\tau^{\omega(n)}$ on those integers n not divisible by primes $p \le r$. A classical argument shows that for each r

$$Y(r) = \lim_{x \to \infty} x^{-1} (\log x)^{-(\tau - 1)} \sum_{n \le x} g_r(n)$$

exists and an inspection of Euler products shows that $\lim Y(r)$, $r \to \infty$, exists.

This establishes the mean-value of g subject to our temporary hypothesis on the size of $g(q)$.

The bound

$$\sum_{n \le x} |g(n)|^\alpha \ll x(\log x)^{\tau^\alpha - 1}$$

follows from an application of Lemma 4.2 together with Lemma 4.1.

In the general case we first apply the argument to the multiplicative function g_+ defined by $g_+(q) = g(q)$ if $|g(q) - \tau| \le \tau/2$, and $g_+(q) = \tau$ otherwise. We define the multiplicative function h by Dirichlet convolution, $h * g_+ = g$, and the remaining argument hinges on the absolute convergence of the series $\sum_{m=1}^\infty h(m)m^{-1}$. A similar situation arises in [6], Chapter 11, pp. 103–104, and I suppress the details.

Granted this,

$$\sum_{n \le x} g(n) = \sum_{m \le x} h(m) \sum_{t \le x/m} g_+(t)$$

$$= (1 + o(1))Ax(\log x)^{\tau - 1}, \quad x \to \infty,$$

and an examiniation of the Euler product representation of A shows it not to vanish.

Alternatively, we may define a multiplicative function t by

$$t(p^m) = \begin{cases} 0 \quad \text{if} \quad p \le \tau + 1, \\ g(p^m) \quad \text{if} \quad p > \tau + 1 \quad \text{and} \quad |g(p) - \tau| \le \tau/2, \\ \tau \quad \text{if} \quad p > \tau + 1 \quad \text{and} \quad |g(p) - \tau| > \tau/2. \end{cases}$$

Then $g = t * h$ where the multiplicative function h has associated Euler product

$$\prod_{p \le \tau + 1} \left(\sum_{m=0}^\infty g(p^m) p^{-m\sigma} \right) \prod_{\substack{p > \tau + 1 \\ |g(p) - \tau| > \tau/2}} (1 + (\tau - 1)p^{-\sigma})^{-1} \cdot$$

$$\cdot (1 - p^{-\sigma}) \left(\sum_{m=0}^\infty g(p^m) p^{-m\sigma} \right)$$

and the series $\sum h(n)n^{-1}$ is absolutely convergent. It will suffice to treat the function t.

For $r > \tau + 1$ we define $t_r(p^m) = t(p^m)$ if $m \ge 2$,

$$t_r(p) = \begin{cases} t(p) \quad \text{if} \quad p \le r, \\ \tau \quad \text{if} \quad p > r. \end{cases}$$

We represent a typical integer n as a product uv where $p \mid u$ if and only if $p \| n$. Then

$$\sum_{n \le x} |t_r(n) - t(n)| = \sum_{v \le x} |t(v)| \sum_{u \le x/v} |t_r(u) - t(u)|,$$

where u runs through squarefree integers and v those for which every prime divisor occurs multiply. In particular, $\sum |t(v)| v^{-1}$ converges.

The treatment of $t_r(u)$ $(= \mu(n)^2 t_r(n))$ now follows that for $g_r(n)$.

This completes the proof of Theorem 1.3.

§ 5. Proof of Theorem 1.4

I confine myself to the following remark.

A non-negative arithmetic function h satisfies

$$(\log x)^t \ll \sum_{n \le x} h(n) n^{-1} \ll (\log x)^t$$

for some $t > 0$ and all $x \ge 2$ if and only if the corresponding Dirichlet series $H(\sigma) = \sum_{n=1}^{\infty} h(n) n^{-\sigma}$ satisfies

$$(\sigma - 1)^{-t} \ll H(\sigma) \ll (\sigma - 1)^{-t}$$

uniformly for $1 < \sigma \le 2$.

Assuming only the upper bound on h, an integration by parts shows that

$$H(\sigma) = (\sigma - 1) \int_1^{\infty} y^{-\sigma} \sum_{n \le y} h(n) n^{-1} dy \ll (\sigma - 1)^{-t}, \quad 1 < \sigma \le 2.$$

Conversely, with $\sigma_0 = 1 + (\log x)^{-1}$,

$$\sum_{n \le x} h(n) n^{-1} \le e \sum_{n=1}^{\infty} h(n) n^{-\sigma_0} \ll (\sigma_0 - 1)^{-1} \ll (\log x)^t.$$

Adjoining the lower bound condition on h ensures that $(\sigma - 1)^{-t} \ll H(\sigma)$ with the same uniformity in σ.

Moreover, if both upper and lower bounds are satisfied by $H(\sigma)$, then with $w = \exp(A(\sigma - 1)^{-1})$

$$\sum_{n > w} h(n) n^{-\sigma} \ll (\sigma - 1) \int_w^{\infty} y^{1-\sigma} (\log y)^t dy \ll (\sigma - 1)^{-t} \int_A^{\infty} u^t e^{-u} du,$$

and if A is fixed at a sufficiently large value

$$\sum_{n \le w} h(n) n^{-\sigma} \gg (\sigma - 1)^{-t}, \quad 1 < \sigma \le 2.$$

Specialising σ to $1 + A(\log x)^{-1}$,

$$\sum_{n \le x} h(n) n^{-1} \gg (\log x)^t,$$

the implied constant depending upon A.

This ends the remark.

§ 6. Concluding Remarks

The foregoing is a formal representation, with adjoined details, of the lecture under the same title that I gave during the International Conference: *Functions in Number Theory and Their Probabilistic Aspects*, held in the Research Institute for Mathematical Sciences, Kyoto University, Japan, December 13–17, 2010.

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