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An extension of Voronin's functional independence for a general Dirichlet series

By

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Abstract

We give a hybrid joint denseness result for values of an axiomatically defined general Dirichlet series $F(s)$ and its derivatives. A typical example of $F(s)$ is the Lerch zeta-function $L(\lambda,\alpha,s)=\sum_{n=0}^{\infty} e^{2\pi i n\lambda}(n+\alpha)^{-s}$ with α transcendental. Further, from this result we deduce two independence properties of those functions. One of them is stronger than the functional independence in the sense of Voronin for those functions.

§ 1. Introduction

The study of differential independence for Dirichlet series has a long history. At the International Congress of Mathematicians in 1900, Hilbert stated that the Riemann zeta-function $\zeta(s)$ and its derivatives are algebraically independent over the rational functions $\mathbb{C}(s)$. His proof is based on the functional equation of $\zeta(s)$ and the similar independence property of the Gamma-function $\Gamma(s)$ proved by Hölder in 1886.

Much later, Voronin obtained another proof of Hilbert's result and a stronger result, from the viewpoint of the value-distribution of $\zeta(s)$. In fact, Voronin proved that for any $\sigma \in (\frac{1}{2}, 1]$ and any non-negative integer K, the set

$$\left\{ \left(\zeta(\sigma+it), \zeta^{(1)}(\sigma+it), \ldots, \zeta^{(K)}(\sigma+it)\right) \in \mathbb{C}^{K+1} \;\middle|\; t \in \mathbb{R} \right\}$$

is dense in $\mathbb{C}^{K+1}$ (see [16]), and from this he deduced the following functional independence for $\zeta(s)$ and its derivatives (see [17] and [7, p. 254]).

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Theorem A. *Let K and J be non-negative integers. Let $G_0, \dots, G_J : \mathbb{C}^{K+1} \to \mathbb{C}$ be continuous functions. If*

$$\sum_{j=0}^{J} s^j G_j\bigl(\zeta(s), \zeta^{(1)}(s), \dots, \zeta^{(K)}(s)\bigr) = 0$$

holds identically for s, then we have $G_j \equiv 0$ for all $0 \le j \le J$.

Similar and stronger results for L-functions have been established. See e.g. [18], [15], [10] and [11]. In the proofs of these results, the fact that the set $\{\log p \,|\, p \text{ is prime}\}$ is linearly independent over the rationals $\mathbb{Q}$ is crucial.

For $\lambda \in \mathbb{R}$ and $0 < \alpha \le 1$, the Lerch zeta-function $L(\lambda, \alpha, s)$ is defined by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n \lambda}}{(n+\alpha)^s}.$$

This series has an analytic continuation to $\mathbb{C}$ (except for a simple pole at $s = 1$ when $\lambda \in \mathbb{Z}$). In the sequel we assume that α is transcendental. Then the numbers $\{\log(n+\alpha) | n = 0, 1, \dots\}$ are linearly independent over $\mathbb{Q}$. As an analog of Theorem A, Garunkštis and Laurinčikas [8] [5, p. 137] obtained the following result for $L(\lambda, \alpha, s)$.

Theorem B. *The function $L(\lambda, \alpha, s)$ and its derivatives are functionally independent in the sense of Voronin. That is, we have the following: Let K and J be non-negative integers. Let $G_0, \dots, G_J : \mathbb{C}^{K+1} \to \mathbb{C}$ be continuous functions. If*

$$\sum_{j=0}^{J} s^j G_j\bigl(L(\lambda, \alpha, s), L^{(1)}(\lambda, \alpha, s), \dots, L^{(K)}(\lambda, \alpha, s)\bigr) = 0$$

holds identically for s, then we have $G_j \equiv 0$ for all $0 \le j \le J$.

As a related result, we have the next theorem, which is due to Amou and Katsurada [1, Theorem 1, Corollay 2]. Let $\mathcal{D}_s$ denote the set of Dirichlet polynomials $\{\sum_{n=1}^{N} a_n n^{-s} \,|\, N \in \mathbb{N}, a_n \in \mathbb{C}\}$.

Theorem C. *The function $L(\lambda, \alpha, s)$ and its derivatives are algebraically independent over $\mathcal{D}_s$.*

In this paper we show two independence results (Theorems 1 and 2) for a general Dirichlet series $F(s)$ mentioned below and its derivatives, from the viewpoint of their value-distribution (Theorem 3). A typical example of $F(s)$ is the Lerch zeta-function $L(\lambda, \alpha, s)$, where α is transcendental as above, and our Theorems 1 and 2 with $F(s) = L(\lambda, \alpha, s)$ are stronger than Theorems B and C, respectively.

We now give a general Dirichlet series which is treated in the present paper. Let

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n}{e^{\lambda_n s}}$$

be a general Dirichlet series satisfying the following conditions (I), (II) and (III):

(I) $0\le\lambda_1<\lambda_2<\cdots$, $\lim_{n\to\infty}\lambda_n=\infty$, and $a_n\in\mathbb{C}$ for all $n=1,2,\ldots$.

(II) There exists a real number $\sigma_0>0$ such that $F(s)$ converges absolutely in the half-plane $\operatorname{Re}s>\sigma_0$ and such that

$$\sum_{n\le X}\frac{|a_n|}{e^{\lambda_n\sigma_0}}\longrightarrow\infty\quad\text{as } X\to\infty.$$

Further, the series

$$\sum_{n=1}^{\infty}\frac{|a_n|^2}{e^{\lambda_n s}}$$

converges for some real number $s=\sigma_1$ with $\sigma_0<\sigma_1<2\sigma_0$.

(III) The numbers $\{\lambda_n \mid n=1,2,\ldots\}$ are linearly independent over the rationals $\mathbb{Q}$.

To state our results, we introduce the following. For a real number c, let $\mathcal{D}_{s,c}$ denote the set of all general Dirichlet series $D(s)$ such that each $D(s)$ converges absolutely at some complex number $s=s_0$ with $\operatorname{Re}s_0<c$, where s_0 may depend on $D(s)$. If $c_1>c_2$ then

$$\mathcal{D}_{s,c_1}\supset\mathcal{D}_{s,c_2}.$$

For any real number c we have

$$\mathcal{D}_{s,c}\supset\mathcal{D}_{s,-\infty}\supset\mathcal{D}_s,$$

where $\mathcal{D}_{s,-\infty}$ denotes the set of general Dirichlet polynomials $\{\sum_{n=1}^{N}a_ne^{-\lambda_n s}\mid N\in\mathbb{N},a_n\in\mathbb{C},\lambda_n\in\mathbb{R}\}$ and $\mathcal{D}_s$ is as before.

The next theorem shows that the Dirichlet series $F(s)$ and its derivatives are functionally independent in the sense of Voronin and further that they have a stronger independence property. Actually, in this theorem, the case $D_0(s)\equiv1,\ldots,D_J(s)\equiv1$ is the functional independence in the sense of Voronin for those functions.

Theorem 1. *Let $F(s)$ be as above. Let K and J be non-negative integers. Let $G_0,\ldots,G_J:\mathbb{C}^{K+1}\to\mathbb{C}$ be continuous functions, and $D_0(s),\ldots,D_J(s)\in\mathcal{D}_{s,\sigma_0}$. If*

$$\sum_{j=0}^{J}s^jD_j(s)\,G_j\bigl(F(s),F^{(1)}(s),\ldots,F^{(K)}(s)\bigr)=0$$

holds identically for $\mathrm{Re}\, s > \sigma_0$*, then we have*

$$G_j \equiv 0 \quad or \quad D_j(s) \equiv 0$$

for all $0 \le j \le J$*.*

Let $\mathbb{N}_0$ denote the set of non-negative integers. In this paper, the symbol $\sum'$ will denote a finite sum.

Theorem 2. *Let* $F(s)$ *be as above. Let* K *be a non-negative integer. Let*

$$P(s, X_0, \ldots, X_K) = \sum_{a, a_0, \ldots, a_K \in \mathbb{N}_0}{}' D(s; a, a_0, \ldots, a_K) s^a X_0^{a_0} \cdots X_K^{a_K}, \tag{1.1}$$

be a polynomial in $(K+2)$*-variables* $s, X_0, \ldots, X_K$ *whose coefficients* $D(s; a, a_0, \ldots, a_K)$ *are general Dirichlet series in* $\mathcal{D}_{s,\sigma_0}$*. If*

$$P\big(s, F(s), \ldots, F^{(K)}(s)\big) = 0$$

holds identically for $\mathrm{Re}\, s > \sigma_0$*, then* P *is the zero polynomial.*

Theorems 1 and 2 are obtained from the next theorem, which is a "hybrid" joint denseness result on values of $F(s)$ and its derivatives. For related hybrid type results, see e.g. [6] and [13]. As usual, for $x \in \mathbb{R}$ let $\|x\| := \min_{n \in \mathbb{Z}} |x - n|$. Let *meas* denote the Lebesgue measure on $\mathbb{R}$.

Theorem 3. *Let* $F(s)$ *be as above, and let* K *be a non-negative integer. Let* $z_k \in \mathbb{C}$ $(0 \le k \le K)$*,* $t_0 \in \mathbb{R}$*,* $\varepsilon > 0$ *and* $\delta > 0$*. Let* $\alpha_1, \ldots, \alpha_N$ *be real numbers linearly independent over* $\mathbb{Q}$*, and* $\theta_1, \ldots, \theta_N \in \mathbb{R}$*. Then there exists a real number* $\sigma_0' > \sigma_0$ *such that, for every* σ *with* $\sigma_0 < \sigma \le \sigma_0'$*, the set of real numbers* t *satisfying*

$$\left|F^{(k)}(\sigma + it_0 + it) - z_k\right| < \varepsilon \quad \text{for any } 0 \le k \le K$$

and

$$\|\alpha_j t - \theta_j\| < \delta \quad \text{for any } 1 \le j \le N$$

has a positive lower density, that is,

$$\liminf_{T \to \infty} \frac{1}{T} meas\Big(\Big\{t \in [0,T] \;\Big|\; \Big|F^{(k)}(\sigma + it_0 + it) - z_k\Big| < \varepsilon \text{ for any } 0 \le k \le K \text{ and } \|\alpha_j t - \theta_j\| < \delta \text{ for any } 1 \le j \le N\Big\}\Big) > 0.$$

§ 2. Preliminary results

Let $\mathbb{T}$ denote the unit circle $\{s \in \mathbb{C} \mid |s| = 1\}$.

Lemma 4. *As in condition (III), let* $\{\lambda_n \mid n = 1, 2, \dots\}$ *be real numbers linearly independent over* $\mathbb{Q}$. *As in Theorem 3, let* $\alpha_1, \dots, \alpha_N$ *be real numbers linearly independent over* $\mathbb{Q}$, *and* $\theta_1, \dots, \theta_N \in \mathbb{R}$. *Then there exist a finite set* $B = B(\alpha_1, \dots, \alpha_N) \subset \mathbb{N}$ *and real numbers* θ_n^* $(n \in B)$ *such that for any finite set* $A \subset \mathbb{N} \setminus B$, *any real numbers* ϕ_n $(n \in A)$ *and any* $\delta > 0$, *the set of real numbers* t *satisfying the inequalities*

$$\max_{1\le j\le N} \|\alpha_j t - \theta_j\| < \delta, \quad \max_{n\in B} \left\| -\frac{\lambda_n}{2\pi} t - \theta_n^* \right\| < \delta \quad \text{and} \quad \max_{n\in A} \left\| -\frac{\lambda_n}{2\pi} t - \phi_n \right\| < \delta$$

has a positive lower density.

Proof. Lemma 2.6 of [12] gives this lemma, since the linear independence of $\{\lambda_n \mid n = 1, 2, \dots\}$ over $\mathbb{Q}$ implies the linear independence of $\{-\frac{\lambda_n}{2\pi} \mid n = 1, 2, \dots\}$ over $\mathbb{Q}$. □

Lemma 5. *Let* $F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ *be a general Dirichlet series satisfying conditions (I) and (II). Let* K *be any non-negative integer, and* $t_0 \in \mathbb{R}$. *For each* $n \in \mathbb{N}$, *let* $\boldsymbol{F}_n$ *be the element of* $\mathbb{C}^{K+1}$ *given by*

$$\boldsymbol{F}_n := \left(\frac{a_n}{e^{\lambda_n(\sigma_0+it_0)}}, \frac{(-\lambda_n)\, a_n}{e^{\lambda_n(\sigma_0+it_0)}}, \frac{(-\lambda_n)^2\, a_n}{e^{\lambda_n(\sigma_0+it_0)}}, \dots, \frac{(-\lambda_n)^K a_n}{e^{\lambda_n(\sigma_0+it_0)}} \right).$$

Let y *be any positive real number. Then the set*

$$\left\{ \sum_{y\le n\le \nu} c_n \boldsymbol{F}_n \;\middle|\; \nu \ge y,\ c_n \in \mathbb{T} \text{ for every } n \in \mathbb{N} \text{ with } y \le n \le \nu \right\}$$

is dense in $\mathbb{C}^{K+1}$.

Proof. Lemma 4 of [9] gives this lemma, since if $c \in \mathbb{T}$ then $ce^{-\lambda_n i t_0} \in \mathbb{T}$. □

Lemma 6. *Let* $F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ *be a general Dirichlet series satisfying conditions (I) and (II). Let* K *be any non-negative integer, and* $t_0 \in \mathbb{R}$. *Then there exist a real number* σ_2 *with* $0 < \sigma_2 < \sigma_0$ *and a sequence* $\{\varepsilon_n \in \mathbb{T} \mid n = 1, 2, \dots\}$ *such that for every* $0 \le k \le K$ *the series*

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n (-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_2+it_0)}}$$

converges.

Proof. This is obtained from [9, Lemma 5]. □

From Lemmas 5 and 6 we obtain the next proposition.

Proposition 7. *Let $F(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ be a general Dirichlet series satisfying conditions (I) and (II). Let B be a finite subset of $\mathbb{N}$, and $\theta_n^* \in \mathbb{R}$ for $n \in B$. Let K be a non-negative integer and $t_0 \in \mathbb{R}$. Let $z_k \in \mathbb{C}$ $(0 \le k \le K)$ and $\varepsilon > 0$ be arbitrary. Then there exist a sequence $\{b_n \in \mathbb{T} \mid n \in \mathbb{N} \setminus B\}$, a large real number $X_0 > 0$ and a real number $\sigma_0' > \sigma_0$ such that if σ satisfies $\sigma_0 \le \sigma \le \sigma_0'$ then for all $X > X_0$ and $0 \le k \le K$ we have*

$$\left| z_k - \sum_{n \in \mathbb{N}\setminus B, n \le X} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{n \in B, n \le X} \frac{e^{2\pi i \theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} \right| < \varepsilon.$$

Proof. Let σ_2 and $\{\varepsilon_n \in \mathbb{T} | n = 1, 2, \dots\}$ be as in Lemma 6. Then by a well-known property of Dirichlet series (see [4, p. 28, Corollary 1.3]), the series

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(s+it_0)}}$$

converges uniformly on compacta (in particular, on the segment $[\sigma_0, \sigma_0+1]$) in the half-plane $\operatorname{Re} s > \sigma_2$. Thus we can take a large number $y > 0$ such that

$$y > \sup_{n \in B} n \tag{2.1}$$

and such that

$$\sup_{\sigma_0 \le \sigma \le \sigma_0+1} \left| \sum_{n \ge y_1} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} \right| < \frac{\varepsilon}{4} \tag{2.2}$$

for every $y_1 \ge y$ and $0 \le k \le K$.

By Lemma 5, there exist a real number $\nu \ge y$ and numbers $\{c_n \in \mathbb{T} \mid y \le n \le \nu\}$ such that

(2.3)
$$\left| \left(z_k - \sum_{n \in \mathbb{N}\setminus B, n<y} \frac{(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}} - \sum_{n \in B, n<y} \frac{e^{2\pi i \theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}} \right) - \sum_{y \le n \le \nu} \frac{c_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}} \right| < \frac{\varepsilon}{4}$$

for every $0 \le k \le K$.

For each $n \in \mathbb{N} \setminus B$, we put

$$b_n := \begin{cases} 1 & \text{if } 1 \le n < y\ , \\ c_n & \text{if } y \le n \le \nu, \\ \varepsilon_n & \text{if } n > \nu. \end{cases} \tag{2.4}$$

By continuity, there exists a real number $\sigma_0 < \sigma_0' < \sigma_0 + 1$ such that if σ satisfies $\sigma_0 \le \sigma \le \sigma_0'$, then

$$\begin{aligned}\Bigg|\Bigg(\sum_{n\in\mathbb{N}\setminus B, n\le\nu} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}} + \sum_{n\in B, n<y} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}}\Bigg) \\ - \Bigg(\sum_{n\in\mathbb{N}\setminus B, n\le\nu} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} + \sum_{n\in B, n<y} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg)\Bigg| < \frac{\varepsilon}{4}\end{aligned} \tag{2.5}$$

for every $0 \le k \le K$.

Let $X_0 := \nu$ and let σ be a real number with $\sigma_0 \le \sigma \le \sigma_0'$. Let X be any real number greater than X_0. By the triangle inequality, (2.5), (2.3), (2.1) and (2.4), we have

$$\begin{aligned}&\Bigg|z_k - \sum_{n\in\mathbb{N}\setminus B, n\le\nu} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{n\in B, n<y} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| \\ &< \Bigg|z_k - \sum_{n\in\mathbb{N}\setminus B, n\le\nu} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}} - \sum_{n\in B, n<y} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma_0+it_0)}}\Bigg| + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.\end{aligned} \tag{2.6}$$

By (2.2) we have

$$\begin{aligned}\Bigg|\sum_{\nu<n\le X} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| &= \Bigg|\sum_{n>\nu} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{n>X} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| \\ &\le \Bigg|\sum_{n>\nu} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| + \Bigg|\sum_{n>X} \frac{\varepsilon_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| < \frac{\varepsilon}{2}.\end{aligned} \tag{2.7}$$

Thus, from (2.1), (2.6), (2.4) and (2.7) we conclude

$$\begin{aligned}&\Bigg|z_k - \sum_{n\in\mathbb{N}\setminus B, n\le X} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{n\in B, n\le X} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| \\ =&\Bigg|z_k - \sum_{n\in\mathbb{N}\setminus B, n\le\nu} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{n\in B, n<y} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{\nu<n\le X} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| \\ \le&\Bigg|z_k - \sum_{n\in\mathbb{N}\setminus B, n\le\nu} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} - \sum_{n\in B, n<y} \frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| + \Bigg|\sum_{\nu<n\le X} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg| \\ <& \ \varepsilon.\end{aligned}$$

This completes the proof. □

§ 3. Proof of Theorem 3

Let B and θ_n^* $(n \in B)$ be as in Lemma 4. According to Proposition 7, there exist a sequence $\{b_n \in \mathbb{T} \mid n \in \mathbb{N} \setminus B\}$, a large real number $X_0 > 0$ and a real number $\sigma_0' > \sigma_0$

such that if σ satisfies $\sigma_0 \le \sigma \le \sigma_0'$, then for all $X \ge X_0$ and $0 \le k \le K$ we have

$$\left| z_k - \left(\sum_{n \in \mathbb{N} \setminus B, n \le X} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma + it_0)}} + \sum_{n \in B, n \le X} \frac{e^{2\pi i \theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} \right) \right| < \frac{\varepsilon}{3}. \tag{3.1}$$

In the following, we fix a number σ with $\sigma_0 < \sigma \le \sigma_0'$. By condition (II), the series $F^{(k)}(s)$ converges absolutely in the half-plane $\operatorname{Re} s > \sigma_0$ for every $0 \le k \le K$. Therefore, if X_1 is a large positive real number satisfying

$$\lambda_n > 1 \text{ for all } n > X_1 \quad \text{and} \quad \sum_{n > X_1} \frac{\lambda_n^K |a_n|}{e^{\lambda_n \sigma}} < \frac{\varepsilon}{3},$$

then, for all $0 \le k \le K$, $X \ge X_1$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} \left| F^{(k)}(\sigma + it_0 + it) - \sum_{n \le X} \frac{(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0+it)}} \right| &\le \sum_{n > X} \left| \frac{(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0+it)}} \right| \\ \le \sum_{n>X} \frac{\lambda_n^k |a_n|}{e^{\lambda_n \sigma}} \le \sum_{n>X_1} \frac{\lambda_n^k |a_n|}{e^{\lambda_n \sigma}} &\le \sum_{n>X_1} \frac{\lambda_n^K |a_n|}{e^{\lambda_n \sigma}} < \frac{\varepsilon}{3}. \end{aligned} \tag{3.2}$$

We fix a large number X_2 satisfying

$$X_2 > \max\{X_0, X_1\} \quad \text{and} \quad X_2 > n \quad \text{for all } n \in B.$$

For each $b_n \in \mathbb{T}$ in (3.1), we write $b_n = e^{2\pi i \phi_n}$ with $0 \le \phi_n < 1$. Then, by continuity, there exists a small number δ_1 with $0 < \delta_1 < \delta$ such that if $t \in \mathbb{R}$ satisfies

$$\left\| -\frac{\lambda_n}{2\pi} t - \phi_n \right\| < \delta_1 \qquad \text{for any } n \in \mathbb{N} \setminus B \text{ with } n \le X_2 \tag{3.3}$$

and

$$\left\| -\frac{\lambda_n}{2\pi} t - \theta_n^* \right\| < \delta_1 \qquad \text{for any } n \in B, \tag{3.4}$$

then for any $0 \le k \le K$ we have

$$\left| \left(\sum_{n \in \mathbb{N} \setminus B, n \le X_2} \frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} + \sum_{n \in B} \frac{e^{2\pi i \theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}} \right) - \sum_{n \le X_2} \frac{(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0+it)}} \right| < \frac{\varepsilon}{3}.$$

This, (3.1) and (3.2) imply that if $t \in \mathbb{R}$ satisfies (3.3) and (3.4), then for any $0 \le k \le K$ we have

$$\begin{aligned} &\left| F^{(k)}(\sigma + it_0 + it) - z_k \right| \\ &= \left| F^{(k)}(\sigma + it_0 + it) - \sum_{n \le X_2} \frac{(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0+it)}} + \sum_{n \le X_2} \frac{(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0+it)}} \right. \end{aligned} \tag{3.5}$$

$$-\Bigg(\sum_{n\in\mathbb{N}\setminus B, n\le X_2}\frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}+\sum_{n\in B}\frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg)$$
$$+\Bigg(\sum_{n\in\mathbb{N}\setminus B, n\le X_2}\frac{b_n(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}+\sum_{n\in B}\frac{e^{2\pi i\theta_n^*}(-\lambda_n)^k a_n}{e^{\lambda_n(\sigma+it_0)}}\Bigg)-z_k\Bigg|$$
$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Lemma 4 gives

$$\liminf_{T\to\infty}\frac{1}{T}meas\Bigg(\Bigg\{t\in[0,T]\ \Bigg|\ \Bigg\|-\frac{\lambda_n}{2\pi}t-\phi_n\Bigg\|<\delta_1\quad\text{for any } n\in\mathbb{N}\setminus B\text{ with } n\le X_2,$$
$$\Bigg\|-\frac{\lambda_n}{2\pi}t-\theta_n^*\Bigg\|<\delta_1\quad\text{for any } n\in B$$
$$\text{and } \|\alpha_j t-\theta_j\|<\delta_1\quad\text{for any } 1\le j\le N\Bigg\}\Bigg)>0.$$

Using this and (3.5) and recalling $\delta_1<\delta$, we conclude

$$\liminf_{T\to\infty}\frac{1}{T}meas\Bigg(\Bigg\{t\in[0,T]\ \Bigg|\ \Big|F^{(k)}(\sigma+it_0+it)-z_k\Big|<\varepsilon\quad\text{for any } 0\le k\le K$$
$$\text{and } \|\alpha_j t-\theta_j\|<\delta\quad\text{for any } 1\le j\le N\Bigg\}\Bigg)>0.$$

This completes the proof of Theorem 3.

§ 4. Proof of Theorem 1

We can prove Theorem 1 by using an argument in [10]. For the sake of completeness of the present paper, we will give a detailed proof. Let $F(s), K, J, G_j, D_j(s)$ be as in Theorem 1. Assume that for some $0\le j\le J$ we have $G_j\not\equiv 0$ and $D_j(s)\not\equiv 0$. In order to obtain Theorem 1, it suffices to show that there exists a complex number s_* with $\operatorname{Re}s_*>\sigma_0$ such that

$$\sum_{j=0}^{J}s_*^j\,D_j(s_*)\,G_j\big(F(s_*),F^{(1)}(s_*)\ldots,F^{(K)}(s_*)\big)\ne 0. \tag{4.1}$$

Let $J_0:=\max\{0\le j\le J\mid G_j\not\equiv 0\text{ and }D_j(s)\not\equiv 0\}$. By the definition of $\mathcal{D}_{s,\sigma_0}$ the series $D_{J_0}(s)$ is holomorphic for $\operatorname{Re}s>\sigma_1$ with some real number $\sigma_1<\sigma_0$, and we have $D_{J_0}(s)\not\equiv 0$. Hence, by a fundamental property of a holomorphic function (see [14, Theorem 10.18]), we have

$$c_0:=|D_{J_0}(\sigma_0+it_0)|\ne 0 \tag{4.2}$$

for some real number t_0. We write

$$D_{J_0}(s)=\sum_{m=1}^{\infty}\frac{b_m}{e^{\nu_m s}}$$

and set

$$\varepsilon := \frac{c_0}{100}.$$

Since by the definition of $\mathcal{D}_{s,\sigma_0}$ the series $D_{J_0}(s)$ converges absolutely at $s = \sigma_0$, we have a large positive integer M such that

$$\sum_{m>M} \frac{|b_m|}{e^{\nu_m \sigma_0}} < \varepsilon. \tag{4.3}$$

By definition, there exists a small number $\delta_0 = \delta_0(\varepsilon, \{b_m\}, \{\nu_m\}, M) > 0$ such that if $t \in \mathbb{R}$ satisfies

$$\left\| -\frac{\nu_m}{2\pi} t \right\| < \delta_0 \quad \text{for every integer } 1 \le m \le M \tag{4.4}$$

then

$$\left| \sum_{m \le M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} - \sum_{m \le M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0)}} \right| < \varepsilon.$$

Hence it follows from the triangle inequality, (4.2) and (4.3) that, for any $t \in \mathbb{R}$ satisfying (4.4), we have

$$\begin{aligned} |D_{J_0}(\sigma_0 + it_0 + it)| &= \left| \sum_{m \le M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} + \sum_{m > M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} \right| \\ &\ge \left| \sum_{m \le M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} \right| - \left| \sum_{m > M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} \right| \\ &\ge \left| \sum_{m \le M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0)}} \right| - \varepsilon - \left| \sum_{m > M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} \right| \\ &\ge \left| D_{J_0}(\sigma_0 + it_0) - \sum_{m > M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0)}} \right| - \varepsilon - \sum_{m > M} \left| \frac{b_m}{e^{\nu_m(\sigma_0 + it_0 + it)}} \right| \\ &\ge |D_{J_0}(\sigma_0 + it_0)| - \left| \sum_{m > M} \frac{b_m}{e^{\nu_m(\sigma_0 + it_0)}} \right| - \varepsilon - \sum_{m > M} \frac{|b_m|}{e^{\nu_m \sigma_0}} \\ &\ge |D_{J_0}(\sigma_0 + it_0)| - \sum_{m > M} \frac{|b_m|}{e^{\nu_m \sigma_0}} - \varepsilon - \sum_{m > M} \frac{|b_m|}{e^{\nu_m \sigma_0}} \\ &\ge c_0 - 3\varepsilon > \frac{c_0}{2}. \end{aligned} \tag{4.5}$$

Let $\{\alpha_1, \dots, \alpha_N\}$ be a basis of the vector space over $\mathbb{Q}$ generated by the numbers ν_m $(1 \le m \le M)$. Let N_0 be an integer such that, for each $1 \le m \le M$,

$$\nu_m = \sum_{j=1}^{N} n_{m,j} \frac{\alpha_j}{N_0}, \tag{4.6}$$

where $n_{m,j} \in \mathbb{Z}$. Then, since we have the inequalities

$$\|\theta_1 + \theta_2\| \le \|\theta_1\| + \|\theta_2\| \qquad (\theta_1, \theta_2 \in \mathbb{R})$$

and

$$\|n\theta\| \le |n| \, \|\theta\| \qquad (\theta \in \mathbb{R}, n \in \mathbb{Z})$$

(see e.g. [2, p. ix]), there exists a small number $\delta_1 = \delta_1(\delta_0, N, \{n_{m,j}\}) > 0$ such that if t satisfies

$$\left\| \frac{\alpha_j}{2\pi N_0} t \right\| < \delta_1 \quad \text{for every integer } 1 \le j \le N \tag{4.7}$$

then t satisfies (4.4). This fact and (4.5) imply that, for any $t \in \mathbb{R}$ satisfying (4.7), we have

$$|D_{J_0}(\sigma_0 + it_0 + it)| > \frac{c_0}{2}. \tag{4.8}$$

By recalling that $D_{J_0}(s)$ converges absolutely at $s = \sigma_0$ and taking a large number M_0 with $\sum_{m>M_0} |b_m| \, e^{-\nu_m \sigma_0} < \frac{\varepsilon}{4}$, we find that, for any $\sigma \ge \sigma_0$ and $\tau \in \mathbb{R}$,

$$\begin{aligned}
|D_{J_0}(\sigma_0 + i\tau) - D_{J_0}(\sigma + i\tau)| &\le \left| D_{J_0}(\sigma_0 + i\tau) - \sum_{m \le M_0} \frac{b_m}{e^{\nu_m(\sigma_0 + i\tau)}} \right| \\
&\quad + \left| \sum_{m \le M_0} \frac{b_m}{e^{\nu_m(\sigma + i\tau)}} - D_{J_0}(\sigma + i\tau) \right| + \left| \sum_{m \le M_0} \frac{b_m}{e^{\nu_m(\sigma_0 + i\tau)}} - \sum_{m \le M_0} \frac{b_m}{e^{\nu_m(\sigma + i\tau)}} \right| \\
&\le \sum_{m > M_0} \left| \frac{b_m}{e^{\nu_m(\sigma_0 + i\tau)}} \right| + \sum_{m > M_0} \left| \frac{b_m}{e^{\nu_m(\sigma + i\tau)}} \right| + \sum_{m \le M_0} \left| \frac{b_m}{e^{\nu_m(\sigma_0 + i\tau)}} \left(1 - \frac{1}{e^{\nu_m(\sigma - \sigma_0)}} \right) \right| \\
&\le \sum_{m > M_0} \frac{|b_m|}{e^{\nu_m \sigma_0}} + \sum_{m > M_0} \frac{|b_m|}{e^{\nu_m \sigma}} + \sum_{m \le M_0} \frac{|b_m|}{e^{\nu_m \sigma_0}} \left(1 - \frac{1}{e^{\nu_m(\sigma - \sigma_0)}} \right) \\
&\le \frac{\varepsilon}{2} + \sum_{m \le M_0} \frac{|b_m|}{e^{\nu_m}} \left(1 - \frac{1}{e^{\nu_m(\sigma - \sigma_0)}} \right).
\end{aligned}$$

Thus there exists a real number $\sigma_0'' = \sigma_0''(\varepsilon, M_0, \{b_m\}, \{\nu_m\}) > \sigma_0$ such that

$$|D_{J_0}(\sigma_0 + i\tau) - D_{J_0}(\sigma + i\tau)| < \varepsilon \tag{4.9}$$

uniformly for $\sigma_0 \le \sigma \le \sigma_0''$ and $\tau \in \mathbb{R}$.

Since $G_{J_0} = G_{J_0}(z_0, \ldots, z_K)$ is a continuous function and $G_{J_0} \not\equiv 0$, there exist a constant $c_1 > 0$ and a bounded open set $U \subset \mathbb{C}^{K+1}$ such that

$$|G_{J_0}(z_0, \ldots, z_K)| > c_1 \qquad \text{for all } (z_0, \ldots, z_K) \in U. \tag{4.10}$$

By Theorem 3, there exist a real number σ and a sequence of real numbers $\{t_n \,|\, n = 1, 2, \ldots\}$ satisfying

$$\sigma_0 < \sigma < \min\{\sigma_0'', \sigma_0 + 1\}, \tag{4.11}$$

$$\lim_{n\to\infty} t_n = \infty, \tag{4.12}$$

$$\boldsymbol{w}_n := \big(F(\sigma + it_0 + it_n), \ldots, F^{(K)}(\sigma + it_0 + it_n)\big) \in U \quad \text{for any } n, \tag{4.13}$$

and

$$\left\| \frac{\alpha_j}{2\pi N_0} t_n \right\| < \delta_1 \quad \text{for any } 1 \le j \le N \text{ and } n. \tag{4.14}$$

We write

$$s_n := \sigma + it_0 + it_n.$$

Using (4.14) and (4.8), we have

$$|D_{J_0}(\sigma_0 + it_0 + it_n)| > \frac{c_0}{2} \qquad \text{for any } n,$$

which, together with (4.9) and (4.11), gives

$$|D_{J_0}(s_n)| > |D_{J_0}(\sigma_0 + it_0 + it_n)| - \varepsilon > \frac{c_0}{2} - \varepsilon > \frac{c_0}{4} \qquad \text{for any } n. \tag{4.15}$$

Combining (4.15), (4.13) and (4.10), we have

$$|D_{J_0}(s_n)\, G_{J_0}(\boldsymbol{w}_n)| \ge \frac{c_0 c_1}{4} \qquad \text{for any } n. \tag{4.16}$$

Now, in the case $J_0 \ge 1$, we shall deduce (4.1). If $D(s) = \sum_{m=1}^{\infty} c_m e^{-\mu_m s} \in \mathcal{D}_{s,\sigma_0}$, then we have

$$|D(\sigma + i\tau)| \le \sum_{m=1}^{\infty} \left| \frac{c_m}{e^{\mu_m(\sigma + i\tau)}} \right| = \sum_{m=1}^{\infty} \frac{|c_m|}{e^{\mu_m \sigma}} \le \sum_{m=1}^{\infty} \frac{|c_m|}{e^{\mu_m \sigma_0}} \tag{4.17}$$

uniformly for $\sigma \ge \sigma_0$ and $\tau \in \mathbb{R}$. Hence there exists a constant $C_0 > 0$ such that

$$|D_j(s_n)| < C_0 \qquad \text{for any } 0 \le j \le J \text{ and } n. \tag{4.18}$$

Since all G_j are bounded on the bounded open set U, by (4.13) there exists a constant $C_1 > 0$ such that

$$|G_j(\boldsymbol{w}_n)| < C_1 \qquad \text{for any } 0 \le j \le J \text{ and } n. \tag{4.19}$$

By (4.11) and (4.12), we have

$$\lim_{n\to\infty} |s_n| = \infty.$$

Consequently, since $J_0 \ge 1$ and $\frac{c_0 c_1}{4} > 0$, it follows from (4.16), (4.18), and (4.19) that

$$\begin{aligned}
&\left|\sum_{j=0}^{J} s_n^j D_j(s_n) G_j(\boldsymbol{w}_n)\right| \\
&= \left| s_n^{J_0} D_{J_0}(s_n)\, G_{J_0}(\boldsymbol{w}_n) + s_n^{J_0-1} D_{J_0-1}(s_n)\, G_{J_0-1}(\boldsymbol{w}_n) + \cdots + D_0(s_n)\, G_0(\boldsymbol{w}_n)\right| \\
&\ge \left| s_n^{J_0} D_{J_0}(s_n)\, G_{J_0}(\boldsymbol{w}_n)\right| - \left| s_n^{J_0-1} D_{J_0-1}(s_n)\, G_{J_0-1}(\boldsymbol{w}_n)\right| - \cdots - \left| D_0(s_n)\, G_0(\boldsymbol{w}_n)\right| \\
&\ge |s_n|^{J_0} \left(\frac{c_0 c_1}{4} - \frac{C_0 C_1}{|s_n|} - \cdots - \frac{C_0 C_1}{|s_n|^{J_0}} \right) \\
&\longrightarrow \infty \qquad \text{as } n \to \infty.
\end{aligned}$$

Thus (4.1) holds in the present case.

In the case $J_0 = 0$, (4.1) is obtained from (4.16), since $\frac{c_0 c_1}{4} > 0$. We have completed the proof of Theorem 1.

§ 5. Proof of Theorem 2

We follow an argument in [10] again. For the sake of completeness of the present paper, we will give a detailed proof. Let $F(s)$ and K be as in Theorem 2. Let $P(s, X_0, \ldots, X_K)$ be as in (1.1). Assume that $P(s, X_0, \ldots, X_K)$ is not the zero polynomial. In order to obtain Theorem 2, it suffices to show that there exists a complex number s_* with $\operatorname{Re} s_* > \sigma_0$ such that

$$P\big(s_*, F(s_*), \ldots, F^{(K)}(s_*)\big) \neq 0. \tag{5.1}$$

We order the terms of $P(s, X_0, \ldots, X_K)$ lexicographically with

$$s > X_0 > \cdots > X_K,$$

and let $(d, d_0, \cdots, d_K)$ denote the multidegree of $P(s, X_0, \ldots, X_K)$ (see e.g. [3]). The symbol $\sum'$ will denote a finite sum.

First we shall consider
Case 1: $d \neq 0$ and $d_k \neq 0$ for some $0 \leq k \leq K$.
Let $\kappa := \min\{0 \leq k \leq K \mid d_k \neq 0\}$. Then

$$
\begin{aligned}
P(s, X_0, \ldots, X_K) &= D(s; d, 0, \ldots, 0, d_\kappa, \ldots, d_K) s^d X_\kappa^{d_\kappa} \cdots X_K^{d_K} \\
&+ \sideset{}{'}\sum_{\substack{a_K \in \mathbb{N}_0, \\ a_K < d_K}} D(s; d, 0, \ldots, 0, d_\kappa, \ldots, d_{K-1}, a_K) s^d X_\kappa^{d_\kappa} \cdots X_{K-1}^{d_{K-1}} X_K^{a_K} \\
&+ \cdots \\
&+ \sideset{}{'}\sum_{\substack{a_\kappa, \ldots, a_K \in \mathbb{N}_0, \\ a_\kappa < d_\kappa}} D(s; d, 0, \ldots, 0, a_\kappa, \ldots, a_K) s^d X_\kappa^{a_\kappa} \cdots X_K^{a_K} \\
&+ \sideset{}{'}\sum_{\substack{a, a_0, \ldots, a_K \in \mathbb{N}_0, \\ a < d}} D(s; a, a_0, \ldots, a_K) s^a X_0^{a_0} \cdots X_K^{a_K},
\end{aligned}
\tag{5.2}
$$

where

$$D_0(s) := D(s; d, 0, \ldots, 0, d_\kappa, \ldots, d_K) \not\equiv 0. \tag{5.3}$$

As in (4.2), by (5.3) we have

$$c := |D_0(\sigma_0 + it_0)| \neq 0$$

for some real number t_0. We write

$$D_0(s) = \sum_{m=1}^{\infty} \frac{b_m}{e^{\nu_m s}}$$

and set

$$\varepsilon := \frac{c}{100}.$$

Then, let $M, \{\alpha_1, \ldots, \alpha_N\}$ and N_0 be as in the proof of Theorem 1 (see (4.3) and (4.6)). As in (4.8), there exists a small number $\delta_1 > 0$ such that if t satisfies

$$\left\| \frac{\alpha_j}{2\pi N_0} t \right\| < \delta_1 \qquad \text{for every integer } 1 \leq j \leq N$$

then

$$|D_0(\sigma_0 + it_0 + it)| > \frac{c}{2}. \tag{5.4}$$

As in (4.9), there exists a real number $\sigma_0'' > \sigma_0$ such that

$$|D_0(\sigma_0 + i\tau) - D_0(\sigma + i\tau)| < \varepsilon \tag{5.5}$$

uniformly for $\sigma_0 \le \sigma \le \sigma_0''$ and $\tau \in \mathbb{R}$.

For a large positive integer n, we set

$$z_{0,n} := n, \quad z_{1,n} := \log n, \quad z_{2,n} := \log\log n, \quad \dots, \quad z_{K,n} := \log\cdots\log n. \tag{5.6}$$

Let n_0 be a large positive integer with $z_{K,n_0} > 10$. According to Theorem 3, for each integer $n \ge n_0$ there exist real numbers σ_n and t_n such that

$$\sigma_0 < \sigma_n < \min\{\sigma_0'', \sigma_0 + 1\}, \tag{5.7}$$

$$t_n > e^n, \tag{5.8}$$

$$\left|F^{(k)}(\sigma_n + it_0 + it_n) - z_{k,n}\right| < \frac{1}{100} \quad \text{for any } 0 \le k \le K, \tag{5.9}$$

and

$$\left\|\frac{\alpha_j}{2\pi N_0} t_n\right\| < \delta_1 \quad \text{for any } 1 \le j \le N. \tag{5.10}$$

We write

$$s_n := \sigma_n + it_0 + it_n.$$

Since (5.10) and (5.4) give

$$|D_0(\sigma_0 + it_0 + it_n)| > \frac{c}{2} \qquad \text{for any } n \ge n_0,$$

it follows from (5.5) and (5.7) that

$$|D_0(s_n)| \ge |D_0(\sigma_0 + it_0 + it_n)| - \varepsilon > \frac{c}{2} - \varepsilon > \frac{c}{4} \qquad \text{for any } n \ge n_0. \tag{5.11}$$

By (4.17), there exists a positive constant C_0 such that

$$\sup_{\substack{a, a_0, \dots, a_K \text{ in } (1.1) \\ n \ge n_0}} |D(s_n; a, a_0, \dots, a_K)| < C_0.$$

This, (5.2) and (5.11) imply that, for any $n \ge n_0$,

$$\begin{aligned} &\left|P\bigl(s_n, F(s_n), \dots, F^{(K)}(s_n)\bigr)\right| \\ &\ge |D(s_n; d, 0, \dots, 0, d_\kappa, \dots, d_K)|\, |s_n|^d \left|F^{(\kappa)}(s_n)\right|^{d_\kappa} \cdots \left|F^{(K)}(s_n)\right|^{d_K} \\ &\quad - \sideset{}{'}\sum_{\substack{a_K \in \mathbb{N}_0, \\ a_K < d_K}} |D(s_n; d, 0, \dots, 0, d_\kappa, \dots, d_{K-1}, a_K)|\, |s_n|^d \left|F^{(\kappa)}(s_n)\right|^{d_\kappa} \cdots \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\times \left|F^{(K-1)}(s_n)\right|^{d_{K-1}} \left|F^{(K)}(s_n)\right|^{a_K} \\
&- \quad \cdots \\
&- \sideset{}{'}\sum_{\substack{a_\kappa,\dots,a_K\in\mathbb{N}_0,\\ a_\kappa<d_\kappa}} |D(s_n;d,0,\dots,0,a_\kappa,\dots,a_K)|\,|s_n|^d \left|F^{(\kappa)}(s_n)\right|^{a_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{a_K} \\
&- \sideset{}{'}\sum_{\substack{a,a_0,\dots,a_K\in\mathbb{N}_0,\\ a<d}} |D(s_n;a,a_0,\dots,a_K)|\,|s_n|^a\,|F(s_n)|^{a_0}\cdots\left|F^{(K)}(s_n)\right|^{a_K} \\
&\ge \frac{c}{4}\,|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{d_K} \\
&- \sideset{}{'}\sum_{\substack{a_K\in\mathbb{N}_0,\\ a_K<d_K}} C_0\,|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K-1)}(s_n)\right|^{d_{K-1}}\left|F^{(K)}(s_n)\right|^{a_K} \\
&- \quad \cdots \\
&- \sideset{}{'}\sum_{\substack{a_\kappa,\dots,a_K\in\mathbb{N}_0,\\ a_\kappa<d_\kappa}} C_0\,|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{a_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{a_K} \\
&- \sideset{}{'}\sum_{\substack{a,a_0,\dots,a_K\in\mathbb{N}_0,\\ a<d}} C_0\,|s_n|^a\,|F(s_n)|^{a_0}\cdots\left|F^{(K)}(s_n)\right|^{a_K} \\
&= |s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{d_K} \\
&\quad\Bigg(\frac{c}{4} - \sideset{}{'}\sum_{\substack{a_K\in\mathbb{N}_0,\\ a_K<d_K}} C_0\frac{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K-1)}(s_n)\right|^{d_{K-1}}\left|F^{(K)}(s_n)\right|^{a_K}}{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{d_K}} \\
&\qquad - \cdots - \sideset{}{'}\sum_{\substack{a_\kappa,\dots,a_K\in\mathbb{N}_0,\\ a_\kappa<d_\kappa}} C_0\frac{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{a_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{a_K}}{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{d_K}} \\
&\qquad\qquad - \sideset{}{'}\sum_{\substack{a,a_0,\dots,a_K\in\mathbb{N}_0,\\ a<d}} C_0\frac{|s_n|^a\,|F(s_n)|^{a_0}\cdots\left|F^{(K)}(s_n)\right|^{a_K}}{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{d_K}}\Bigg).
\end{aligned}
$$

For every $m \in \mathbb{N}$ we have

$$\frac{(\log x)^m}{x} \longrightarrow 0 \qquad \text{as } x \to \infty.$$

This fact, (5.6), (5.7), (5.8) and (5.9) imply that the numbers

$$\frac{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K-1)}(s_n)\right|^{d_{K-1}}\left|F^{(K)}(s_n)\right|^{a_K}}{|s_n|^d\left|F^{(\kappa)}(s_n)\right|^{d_\kappa}\cdots\left|F^{(K)}(s_n)\right|^{d_K}} \quad (a_K \in \mathbb{N}_0, a_K < d_K),$$

$$\ldots, \quad \frac{|s_n|^d \left|F^{(\kappa)}(s_n)\right|^{a_\kappa} \cdots \left|F^{(K)}(s_n)\right|^{a_K}}{|s_n|^d \left|F^{(\kappa)}(s_n)\right|^{d_\kappa} \cdots \left|F^{(K)}(s_n)\right|^{d_K}} \quad (a_\kappa, \ldots, a_K \in \mathbb{N}_0, a_\kappa < d_\kappa),$$

$$\frac{|s_n|^a \left|F(s_n)\right|^{a_0} \cdots \left|F^{(K)}(s_n)\right|^{a_K}}{|s_n|^d \left|F^{(\kappa)}(s_n)\right|^{d_\kappa} \cdots \left|F^{(K)}(s_n)\right|^{d_K}} \quad (a, a_0, \ldots, a_K \in \mathbb{N}_0, a < d)$$

go to 0 as $n \to \infty$, and that

$$|s_n|^d \left|F^{(\kappa)}(s_n)\right|^{d_\kappa} \cdots \left|F^{(K)}(s_n)\right|^{d_K} \longrightarrow \infty \qquad \text{as } n \to \infty.$$

Therefore, we have

$$\left|P\big(s_n, F(s_n), \ldots, F^{(K)}(s_n)\big)\right| \longrightarrow \infty \qquad \text{as } n \to \infty.$$

Thus (5.1) holds in Case 1.

Similarly we can verify (5.1) in the following remaining cases:
Case 2 : $d \neq 0$ and $d_k = 0$ for all $0 \leq k \leq K$,
Case 3 : $d = 0$ and $d_k \neq 0$ for some $0 \leq k \leq K$,
Case 4 : $d = 0$ and $d_k = 0$ for all $0 \leq k \leq K$.
We have completed the proof of Theorem 2.

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References

[1] Amou, M. and Katsurada, M., Differential transcendence of a class of generalized Dirichlet series, *Illinois. J. Math.*, **45** (2001), 939–948.
[2] Cassels, J., *An Introduction to Diophantine Approximation*, Cambridge Tracts in Mathematics and Mathematical Physics. No. 45. Reprinted, 1965.
[3] Cox, D., Little, J. and O'Shea, D., *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer-Verlag, 1992
[4] Laurinčikas, A., *Limit Theorems for the Riemann Zeta-Function*, Kluwer, 1996.
[5] Garunkštis, R. and Laurinčikas, A., On one Hilbert's problem for the Lerch zeta-function, *Publ. Inst. Math.*, **65** (1999), 63–68.
[6] Kaczorowski, J. and Kulas, M., On the non-trivial zeros off the critical line for L-functions from the extended Selberg class, *Monatsh. Math.*, **150** (2007), 217–232.
[7] Karatsuba, A. and Voronin, S., *The Riemann Zeta-Function*, Walter de Gruyter, 1992.
[8] Laurinčikas, A. and Garunkštis, R., *The Lerch zeta-function*, Kluwer Academic Publishers, 2002.
[9] Nagoshi, H., Values of general Dirichlet series and simultaneous Diophantine approximation, *Lith. Math. J.*, **50** (2010), 227–238.
[10] Nagoshi, H., Independence of L-functions and their derivatives, *in preparation.*

[11] Nagoshi, H., Difference-differential independence of the Riemann-zeta function, *in preparation.*

[12] Pańkowski, Ł., Some remarks on the generalized strong recurrence for L-functions, *New directions in value-distribution theory of zeta and L-functions*, pp. 305–315, Ber. Math., Shaker Verlag, 2009.

[13] Pańkowski, Ł., Hybrid joint universality theorem for Dirichlet L-functions, *Acta Arith.*, **141** (2010), 59–72.

[14] Rudin, W., *Real and Complex Analysis*, Third Edition, McGraw-Hill, 1987.

[15] Steuding, J., *Value-Distribution of L-functions*, Lecture Notes in Mathematics **1877**, Springer-Verlag, 2007.

[16] Voronin, S., On the distribution of nonzero values of the Riemann zeta-function, *Trudy Mat. Inst. Steklov.*, **128** (1972), 131–150; Proc. Steklov Inst. Math. **128** (1972), 153–175.

[17] Voronin, S., On differential independence of ζ-functions, *Sov. Math., Dokl.*, **14** (1973), 607–609.

[18] Voronin, S., The functional independence of Dirichlet L-functions, *Acta Arith.*, **27** (1975), 493–503. (Russian)