

# Limit theorems for random analytic functions and their zeros

*Dedicated to the late Professor Yasunori Okabe*

By

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## Abstract

After we briefly review on determinantal point processes and Gaussian analytic functions, we establish a functional central limit theorem for random analytic functions and the corresponding limit theorem for their zero processes. We also show that the zeros of the complex Wiener integral of the Szegö kernel for the upper half-plane form a determinantal point process on it.

## § 1. Introduction

The Riemann zeta-function is defined as a Dirichlet series or an Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} (1 - p^{-s})^{-1}$$

for  $\text{Re}(s) > 1$ , and it can be meromorphically extended to the whole complex plane with a simple pole at  $s = 1$ . It admits a functional equation

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-s/2} \Gamma(s/2) \zeta(s) = \xi(1-s).$$

The Riemann hypothesis (RH) states that the non-trivial zeros lie on the critical line  $\text{Re}(s) = 1/2$ . It has been of central concern to number theorists and there have been

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many attempts to solve it, among which one of the most fascinating ideas is Hilbert-Pólya's one which suggests that the zeros of Riemann's zeta function might link to the eigenvalues of a certain self-adjoint operator acting on a certain Hilbert space. Although the desired self-adjoint operator is yet to be discovered, their idea has influenced much of the study of zeta functions.

Assuming (RH), we enumerate the nontrivial zeros in the upper half-plane as  $\lambda_j = 1/2 + \sqrt{-1}\gamma_j$  with  $0 < \gamma_1 < \gamma_2 < \dots$ . It is known that

$$N(T) = \#\{j; \gamma_j \leq T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

as  $T \rightarrow \infty$ , which implies that the normalized sequence of imaginary parts  $\{\hat{\gamma}_j = \frac{\gamma_j}{2\pi} \log \frac{\gamma_j}{2\pi}, j = 1, 2, \dots\}$  has unit mean spacing. Montgomery [21] analyzed the Fourier transform of the *empirical* measure of differences of normalized imaginary parts and essentially showed that for any smooth test function  $f$  with  $\text{supp } \hat{f} \subset (-1, 1)$ , as  $T \rightarrow \infty$

$$\frac{1}{T} \sum_{1 \leq i \neq j \leq T} f(\hat{\gamma}_i - \hat{\gamma}_j) \rightarrow \int_{\mathbb{R}} f(x) \rho_2(x) dx,$$

where  $\rho_2(x) = 1 - (\frac{\sin \pi x}{\pi x})^2$  and  $\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi\sqrt{-1}tx} dx$ . From this observation, Montgomery conjectured that the limiting empirical 2-point correlation function coincides with the 2-point correlation function of the limiting point process of eigenvalues of Gaussian unitary ensemble (GUE), also now known as the *determinantal point process* associated with sine kernel  $K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$ . See Section 3 for determinantal point processes and also [7, 30, 27] for details. This conjecture is strongly supported by extensive numerical computation due to Odlyzko [22], and Montgomery's result is extended to the case of 3-point function and  $n$ -point functions by Hejhal [9] and Rudnick-Sarnak [26], respectively. Keating-Snaith [14] used this resemblance between the eigenvalues of random matrices and the zeros of Riemann zeta function to predict the absolute moments of  $\zeta(1/2 + it)$ . Although there is much evidence to believe that there exists a deep connection between these two objects, the full conjecture remains open and the reason why determinantal point process arises from Riemann zeta zeros has not yet been clarified.

A determinantal point process also arises as the zeros of certain *random* analytic function, which was found by Peres-Virág [24]. They showed that when  $\{g_n, n = 0, 1, \dots\}$  are i.i.d. standard complex Gaussian random variables, the zeros of the Gaussian random power series  $X_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} g_n z^n$  form a determinantal point process on the unit disk  $\mathbb{D}$  associated with Bergman kernel  $K(z, w) = \pi^{-1}(1 - z\bar{w})^{-2}$  and the Lebesgue measure  $m(dz)$  on  $\mathbb{D}$ . Krishnapur [16] extended this result to the case of singular points of matrix-valued Gaussian analytic functions i.e., the zeros of  $X_{\mathbb{D}}^{(k)}(z) = \det(\sum_{n=0}^{\infty} G_n z^n)$ , where  $\{G_n, n = 0, 1, \dots\}$  is a sequence of i.i.d. Ginibre

random matrices of size  $k$  whose entries are i.i.d. standard complex Gaussian random variables. The zero set of  $X_{\mathbb{D}}^{(k)}$  is the determinantal point process on  $\mathbb{D}$  associated with  $K^{(k)}(z, w) = \pi^{-1}(1 - z\bar{w})^{-(k+1)}$  and  $\lambda^{(k)}(dz) = k(1 - |z|^2)^{k-1}m(dz)$ . Recently, Ledoan-Merkli-Starr [18] studied a functional central limit theorem for random analytic functions towards Gaussian analytic functions (GAF) and the corresponding limit theorem for their zeros. In the present paper, we provide an extension of their result and some examples of limit theorems. We also show that the Wiener integral of the Szegő kernel for the upper half-plane  $\mathbb{H}$  with respect to the standard complex Brownian motion  $B(t)$

$$X_{\mathbb{H}}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{t - z} dB(t), \quad z \in \mathbb{C} \setminus \mathbb{R}$$

is the counterpart for  $\mathbb{H}$  of the above-mentioned GAF  $X_{\mathbb{D}}(z)$ . Here we should mention that this type of Wiener integrals had been studied by Yasunori Okabe in [23] and subsequent works (cf. [15]) to understand one-dimensional real Gaussian processes as boundary processes of “hyperprocesses” in the framework of Sato’s hyperfunctions. It would also be interesting to study the zero processes of hyperprocesses.

The organization of this paper is as follows. In Section 2, we summarize basic properties of random analytic functions and the associated zero processes. In Section 3, we briefly review on determinantal point processes and Gaussian analytic functions. In Section 4, we discuss a functional central limit theorem for random analytic functions and the corresponding limit theorem for their zeros (Theorems 4.4). In Section 5, we give some examples of Theorem 4.4. In particular, the result obtained in [18] is given in Example 5.4. In Appendix, we show that the zeros of the complex Wiener integral of the Szegő kernel (or the Cauchy kernel) form a determinantal point process on  $\mathbb{H}$  (Theorems 6.2 and 6.3).

## § 2. Random analytic functions and their zero processes

Throughout this paper, for simplicity,  $D \subset \mathbb{C}$  is a connected (open) domain in the complex plane. Let  $Q = Q(D)$  be the set of non-negative integer-valued Radon measures on  $D$ . Here we say that a Borel measure  $\nu$  on  $D$  is a Radon measure if  $\nu(K) < \infty$  for every compact set  $K \subset D$ . An element  $\xi \in Q$  can be expressed as a sum  $\xi = \sum_i m_i \delta_{z_i}$  of delta measures, where the set  $\{z_i\}_i$  has no accumulation points and  $m_i \in \mathbb{N} = \{1, 2, \dots\}$ . It is sometimes convenient to write  $\xi = \sum_i \delta_{z_i}$  with  $z_i$  being repeated  $m_i$  times. For a bounded measurable function  $\varphi$  of compact support, we define  $\langle \xi, \varphi \rangle = \sum_i m_i \varphi(z_i)$  when  $\xi = \sum_i m_i \delta_{z_i} \in Q$ , in particular, we write  $\xi(A)$  for  $\langle \xi, I_A \rangle$  where  $I_A$  is the indicator function of a subset  $A \subset D$ , which stands for the number of points inside  $A$  counted with multiplicity. We equip the space  $Q$  with the  $\sigma$ -field  $\mathcal{B}(Q)$  generated by the functionals  $Q \ni \xi \mapsto \langle \xi, \varphi \rangle \in \mathbb{C}$  for  $\varphi \in C_c(D)$ , the space of

continuous functions on  $D$  with compact support. Here  $\langle \xi, \varphi \rangle$  can also be understood as the pairing of  $\varphi \in C_c(D)$  and  $\xi \in C'_c(D)$  as a positive linear functional on  $C_c(D)$ . We say that  $\xi = \xi(\omega)$  is a *point process* on  $D$  if it is a  $Q$ -valued random variable defined on a probability space  $(\Omega, \mathcal{F}, P)$ .

We introduce the space  $\mathcal{H}(D)$  of complex analytic functions in  $D$  and we equip the metric

$$\rho(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}},$$

which induces the locally uniform convergence of analytic functions. Here  $\|f\|_K = \max_{z \in K} |f(z)|$  is the supremum norm and  $\{K_j, j = 1, 2, \dots\}$  is an exhaustion by compact sets of  $D$ , i.e.,  $K_j, j = 1, 2, \dots$  is an increasing sequence of compact subsets of  $D$  satisfying (i)  $K_j \subset K_{j+1}^\circ, j = 1, 2, \dots$ , (ii) for every compact set  $K \subset D$  there exists  $n$  such that  $K \subset K_n$ , and (iii)  $\bigcup_{j=1}^{\infty} K_j = D$ . It is well-known that  $(\mathcal{H}(D), \rho)$  is a complete separable metric space. The space  $\mathcal{H}(D)$  is equipped with the (topological) Borel  $\sigma$ -field  $\mathcal{B}(\mathcal{H}(D))$  and the totality of probability measures on  $(\mathcal{H}(D), \mathcal{B}(\mathcal{H}(D)))$  is denoted by  $\mathcal{P}(\mathcal{H}(D))$ . By a *random analytic function* on  $D$  we mean an  $\mathcal{H}(D)$ -valued random variable  $X(z) = X(z, \omega)$  on a probability space  $(\Omega, \mathcal{F}, P)$ . The probability law of  $X$  in  $(\mathcal{H}(D), \mathcal{B}(\mathcal{H}(D)))$  is denoted by  $\mu_X \in \mathcal{P}(\mathcal{H}(D))$ . We remark that every  $\mu \in \mathcal{P}(\mathcal{H}(D))$  is uniquely determined by the values of cylinder sets, i.e., the probability law of a random analytic function is uniquely determined by its finite dimensional distributions (cf. [10, 11]).

Throughout this paper, for simplicity, we assume that a random analytic function  $X(z)$  is square-integrable and centered, i.e.,  $E[|X(z)|^2] < \infty$  and  $E[X(z)] = 0$  for each  $z \in D$ . For a random analytic function  $X$ , we define a covariance function by  $S(z, w) = E[X(z)\overline{X(w)}]$ . The following proposition provides us most of the typical and important examples of random analytic functions.

**Proposition 2.1.** *Let  $\{\psi_k\}_k \subset \mathcal{H}(D)$  be a sequence of independent centered random analytic functions defined on the same probability space. Suppose that  $\sum_{k=1}^{\infty} E[|\psi_k(z)|^2]$  is a locally integrable function of  $z$  in  $D$ . Then,  $X(z) = \sum_{k=1}^{\infty} \psi_k(z)$  is convergent in  $\mathcal{H}(D)$  almost surely and thus defines a random analytic function on  $D$ .*

Since the second moments are uniformly locally bounded, it is clear from Kolmogorov's theorem that for each  $z \in D$  the sequence  $\{X_n(z)\}$  converges almost surely. The point is that the almost sure *pointwise* convergence can be strengthened to yield the almost sure *locally uniform* convergence by analyticity. A proof for Proposition 2.1 will be given later in this section. See also Lemma 2.2.3 in [7].

For an analytic function  $f \in \mathcal{H}(D)$ , we denote by  $Z_f$  the set of zeros of  $f$  and define

a non-negative integer-valued Radon measure  $\xi_f$  on  $D$  by

$$\xi_f = \sum_{z \in Z_f} m_z \delta_z,$$

where  $\delta_z$  is the delta measure with unit mass at  $z \in D$  and  $m_z$  is the multiplicity of a zero  $z$ . Since a non-trivial analytic function has only finite number of zeros in every compact set by the identity theorem,  $\xi_f$  turns out to be an element of  $Q$ . When  $f$  is nowhere vanishing in  $D$ ,  $\xi_f$  is understood as the empty configuration  $\emptyset \in Q$ , i.e.,  $\langle \emptyset, \varphi \rangle = 0$  for any  $\varphi \in C_c(D)$ . The following lemma is a restatement of Hurwitz's theorem which shows that the locally uniform convergence of analytic functions implies the vague convergence of the corresponding zeros (cf. [2]).

**Lemma 2.2.** *Suppose that  $f_n$  converges to  $f$  in  $(\mathcal{H}(D), \rho)$  and  $f$  is not identically zero. Then, the sequence  $\{\xi_{f_n}\}$  of the zeros converges to  $\xi_f$  vaguely, i.e., it holds that  $\langle \xi_{f_n}, \varphi \rangle \rightarrow \langle \xi_f, \varphi \rangle$  for any  $\varphi \in C_c(D)$ .*

*Proof.* Let  $K$  be the support of  $\varphi \in C_c(D)$ . Let  $z_1, z_2, \dots, z_L$  be the zeros of  $f$  located on  $K$  with multiplicity  $m_1, m_2, \dots, m_L$ , respectively. For any sufficiently small  $\epsilon > 0$ , we can take a finite open disks  $\{U_{i,\epsilon}, i = 1, 2, \dots, M_\epsilon\}$  of radius less than  $\epsilon$  which covers  $K$  such that  $\{\overline{U_{i,\epsilon}}, i = 1, 2, \dots, L\}$  are disjoint, only  $U_{i,\epsilon}$  contains  $z_i$  for each  $i = 1, 2, \dots, L$  and  $\overline{U_{i,\epsilon}}$  contains no zero for every  $i = L+1, L+2, \dots, M_\epsilon$ . By Hurwitz's theorem (cf.[2]), there exists an  $n_0 = n_0(\epsilon) \in \mathbb{N}$  such that for any  $n \geq n_0$ ,  $\xi_{f_n}(U_{i,\epsilon}) = \xi_f(U_{i,\epsilon}) = m_i$  for  $i = 1, 2, \dots, L$  and  $\xi_{f_n}(U_{i,\epsilon}) = \xi_f(U_{i,\epsilon}) = 0$  for  $i = L+1, L+2, \dots, M_\epsilon$ . Then, for  $n \geq n_0$ , we have

$$\begin{aligned} |\langle \xi_{f_n}, \varphi \rangle - \langle \xi_f, \varphi \rangle| &\leq \sum_{i=1}^L |\langle \xi_{f_n}, \varphi|_{U_{i,\epsilon}} \rangle - \langle \xi_f, \varphi|_{U_{i,\epsilon}} \rangle| \\ &\leq \left( \sum_{i=1}^L m_i \right) \omega_\varphi(\epsilon) \end{aligned}$$

where  $\omega_\varphi(\epsilon) = \sup\{|\varphi(z) - \varphi(w)|; |z - w| < \epsilon\}$  is the modulus of continuity of  $\varphi$ . Hence,

$$\limsup_{n \rightarrow \infty} |\langle \xi_{f_n}, \varphi \rangle - \langle \xi_f, \varphi \rangle| \leq \left( \sum_{i=1}^L m_i \right) \omega_\varphi(\epsilon)$$

Since  $\varphi$  is uniformly continuous on  $K$ , the right-hand side converges to 0 as  $\epsilon \rightarrow 0$ .  $\square$

*Remark.* Every functional  $F_\varphi : \mathcal{H}(D) \rightarrow \mathbb{C}$  defined by  $F_\varphi(f) := \langle \xi_f, \varphi \rangle$  for  $\varphi \in C_c(D)$  is continuous in  $\mathcal{H}(D) \setminus \{0\}$  with respect to the metric  $\rho$ .

If  $X(z, \omega)$  is a random analytic function in  $D$ , it is easy to see that  $\xi_X(\omega) := \xi_{X(\cdot, \omega)}$  defines a point process on  $D$ . We call it a *zero (point) process* of  $X$ .

Now we discuss the relationship between convergence of random analytic functions and that of associated zero processes. The next proposition is an immediate consequence of Lemma 2.2 and the representation theorem due to Skorohod stated below in Theorem 2.4.

**Proposition 2.3.** *Suppose that a sequence of random analytic functions  $\{X_n\}$  converges in law to  $X$ . Then, the zero process  $\xi_{X_n}$  converges in law to  $\xi_X$  provided that  $X \not\equiv 0$  almost surely.*

*Proof.* By Theorem 2.4 below, we can construct random analytic functions  $X_n$ ,  $n = 1, 2, \dots$  and  $X$  on some probability space so that  $\{X_n\}$  converges to  $X (\not\equiv 0)$  in  $(\mathcal{H}(D), \rho)$  almost surely. Then, by Lemma 2.2, the zeros  $\{\xi_{X_n}\}$  converges to  $\xi_X$  vaguely almost surely. This implies the assertion.  $\square$

**Theorem 2.4** (cf. [10]). *Let  $(S, \rho)$  be a complete separable metric space. Suppose that a sequence of probability measures  $\{\mu_n\}_{n=1}^\infty$  on  $(S, \mathcal{B}(S))$  converges weakly to  $\mu$ . Then, on some probability space, one can construct  $S$ -valued random variables  $X_n, n = 1, 2, \dots$  and  $X$  such that (i)  $\mu_n, n = 1, 2, \dots$  and  $\mu$  are the probability law of  $X_n, n = 1, 2, \dots$  and  $X$ , respectively, and (ii)  $X_n$  converges to  $X$  almost surely.*

The next proposition is an analytic process version of a well-known sufficient condition for a sequence of continuous processes to be convergent in law.

**Proposition 2.5.** *Let  $X_n, n = 1, 2, \dots$  be a sequence of random analytic functions. If  $\|X_n\|_K, n = 1, 2, \dots$  is tight for any compact set  $K$ , then  $\mu_{X_n}, n = 1, 2, \dots$  is tight in  $\mathcal{P}(\mathcal{H}(D))$ . Furthermore, if  $\{X_n\}$  converges to  $X$  in the sense of finite dimensional distributions, then  $\{\mu_{X_n}\}$  converges weakly to a limit  $\mu_X$ .*

*Proof.* Let  $\mu_{X_n}, n = 1, 2, \dots$  be the laws of random analytic functions  $X_n, n = 1, 2, \dots$ , respectively. Let  $\{K_j\}$  be an exhaustion of  $D$  by compact sets as before. By tightness of  $\{\|X_n\|_{K_j}, n = 1, 2, \dots\}$  for each  $j$ , for every  $\epsilon > 0$  one can take an increasing sequence of real numbers  $0 < M_1 < M_2 < \dots$  such that  $\sup_n P(\|X_n\|_{K_j} > M_j) \leq 2^{-j}\epsilon$ . We set  $\mathcal{K} = \{h \in \mathcal{H}(D); \|h\|_{K_j} \leq M_j, j = 1, 2, \dots\}$ . Then this is a locally bounded family and hence relatively compact in  $\mathcal{H}(D)$  by Montel's theorem. Moreover, it is easily seen that  $\inf_n \mu_{X_n}(\mathcal{K}) \geq 1 - \epsilon$ . Hence, the sequence  $\{\mu_{X_n}\}_{n=1}^\infty$  is also tight in  $\mathcal{P}(\mathcal{H}(D))$ . The uniqueness of a limit point follows from the convergence in the sense of finite dimensional distributions. Consequently,  $\{\mu_{X_n}\}$  converges to  $\mu_X$  weakly.  $\square$

For complex analytic functions, local integrability implies local boundedness.

**Lemma 2.6.** *For any compact set  $K$  in  $D$  there exists  $\delta > 0$  such that*

$$\|f\|_K^p \leq (\pi\delta^2)^{-1} \int_{\overline{K_\delta}} |f(z)|^p m(dz), \quad f \in \mathcal{H}(D)$$

for any  $p > 0$ , where  $\overline{K_\delta} \subset D$  is the closure of the  $\delta$ -neighborhood of  $K$ .

*Proof.* Take  $\delta > 0$  small enough so that  $\overline{K_\delta}$  is contained in  $D$ . Since the integral  $\int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta$  is a nondecreasing function of  $r$  for every  $p > 0$  (cf. Hardy's convexity theorem [5]), we see that

$$|f(z)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta$$

for  $0 \leq r < \delta$ . Hence, we have

$$\pi\delta^2 |f(z)|^p \leq \int_0^\delta r dr \int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta = \int_{|\zeta - z| \leq \delta} |f(\zeta)|^p m(d\zeta)$$

Therefore, by taking the supremum both sides over  $K$ , we obtain the desired inequality.  $\square$

*Remark.* If  $\sup_n E[|X_n(z)|^p]$  is locally integrable for some  $p > 0$ , then  $\{\mu_{X_n}\}_n$  is tight in  $\mathcal{P}(\mathcal{H}(D))$  since the tightness of  $\{\|X_n\|_K\}_n$  easily follows from Lemma 2.6.

Now we give a proof of Proposition 2.1 as an application of Proposition 2.5.

*Proof of Proposition 2.1.* Let us consider the partial sum  $X_n(z) = \sum_{k=1}^n \psi_k(z)$ . Since  $S^{X_n}(z, z) := \sum_{k=1}^n E[|\psi_k(z)|^2] \nearrow \sum_{k=1}^\infty E[|\psi_k(z)|^2] =: S(z, z)$  as  $n \rightarrow \infty$  and  $S(z, z)$  is locally integrable, the sequence  $\{\mu_{X_n}\}_n$  is tight in  $\mathcal{P}(\mathcal{H}(D))$  by the remark above. Moreover, by Kolmogorov's theorem for sum of independent random variables, any finite dimensional random vector  $(X_n(z_j))_{j=1}^M$  converges a.s., which implies that the limit distribution is uniquely determined. Hence,  $\{\mu_{X_n}\}$  converges weakly to a unique limit, which defines a random analytic function.

Now we recall the Itô-Nisio theorem in [11] which extends Lévy's theorem to sum of Banach space valued independent random variables  $\{\xi_k\}_k$  stating that the almost sure convergence, the convergence in probability and that in law of the sequence of partial sums  $X_n = \sum_{k=1}^n \xi_k$ ,  $n \geq 1$  are equivalent. From this theorem, for each compact set  $K \subset D$ , the  $\{X_n(z), z \in K\}$  is uniformly convergent a.s., and hence  $\{X_n(z), z \in D\}$  is convergent in  $\mathcal{H}(D)$  a.s.  $\square$

*Remark.* Under the condition of Proposition 2.1, the zero process  $\xi_{X_n}$  of the partial sum  $X_n(z) = \sum_{k=1}^n \psi_k(z)$  converges to  $\xi_X$  in law provided that  $X \not\equiv 0$  almost surely.

### § 3. Determinantal point processes and Gaussian analytic functions

First we briefly review the notion of correlation functions. Let  $R$  be a locally compact Hausdorff space with countable basis and  $\xi = \xi(\omega)$  a point process on  $R$ . Here we use the expression  $\xi = \sum_i \delta_{x_i}$  with  $x_i$  being repeated according to its multiplicity instead of  $\xi = \sum_i m_i \delta_{x_i}$ . We fix a non-negative Radon measure  $\lambda$  on  $R$  as a reference measure. If there exists a Radon measure  $\lambda_1$  so that

$$E[\langle \xi, \varphi \rangle] = E\left[\int_R \varphi(x) \xi(dx)\right] = \int_R \varphi(x) \lambda_1(dx)$$

for every  $\varphi \in C_c(R)$ , we say that  $\lambda_1(dx)$  is the first correlation measure. Formally, one can write  $\lambda_1 = E[\xi]$  since the right-hand side can be written as  $\langle \lambda_1, \varphi \rangle$ . Moreover, if  $\lambda_1$  is absolutely continuous with respect to the reference measure  $\lambda$ , the Radon-Nikodym derivative

$$\rho_1(x) := \frac{d\lambda_1}{d\lambda}(x)$$

is called the first correlation function with respect to  $\lambda$ . By definition  $\rho_1(x)$  is the mean density of points at  $x \in R$ . In a similar manner, we define a Radon measure  $\xi_n$  on  $R^n$  from  $\xi$  by

$$\xi_n = \sum_{\substack{x_1, \dots, x_n \in \xi \\ \text{distinct}}} \delta_{x_1, \dots, x_n}.$$

If there exists a Radon measure  $\lambda_n$  on  $R^n$  so that

$$E[\langle \xi_n, \varphi \rangle] = E\left[\int_{R^n} \varphi(x_1, \dots, x_n) \xi_n(dx_1 \dots dx_n)\right] = \int_{R^n} \varphi(x_1, \dots, x_n) \lambda_n(dx_1 \dots dx_n)$$

for every  $\varphi \in C_c(R^n)$ , we say that  $\lambda_n(dx)$  is the  $n$ -th correlation measure. Moreover, if  $\lambda_n$  is absolutely continuous with respect to the product measure  $\lambda^{\otimes n}$ , the Radon-Nikodym derivative

$$\rho_n(x_1, \dots, x_n) := \frac{d\lambda_n}{d\lambda^{\otimes n}}(x_1, \dots, x_n)$$

is called the  $n$ -th correlation function with respect to  $\lambda^{\otimes n}$ .

**Example 3.1** (Poisson point process). A Poisson point process over a space  $R$  is completely determined by a Radon measure  $\nu$  on  $R$  as follows: For any  $n = 1, 2, \dots$  and any disjoint subsets  $A_1, \dots, A_n \in \mathcal{B}(R)$ , (i) random variables  $\xi(A_1), \dots, \xi(A_n)$  are independent and (ii) for any nonnegative integers  $k_1, k_2, \dots, k_n$ ,

$$P(\xi(A_i) = k_i, i = 1, 2, \dots, n) = \prod_{i=1}^n \frac{\nu(A_i)^{k_i}}{k_i!} e^{-\nu(A_i)}.$$



From these two properties, it is easy to see that the first correlation measure is  $\nu$  and if it is absolutely continuous with respect to the base measure  $\lambda$ , then the  $n$ -th correlation function is given by

$$\rho_n(x_1, \dots, x_n) = \prod_{i=1}^n \frac{d\nu}{d\lambda}(x_i).$$

**Example 3.2** (Finite point process). Let  $p_N(x_1, x_2, \dots, x_N)$  be a symmetric probability density function on  $\mathbb{R}^N$  with respect to the Lebesgue measure. By identifying  $(x_1, \dots, x_N) \in \mathbb{R}^N$  with  $\xi = \sum_{i=1}^N \delta_{x_i} \in Q(\mathbb{R})$ ,  $p_N$  induces a probability measure on  $Q(\mathbb{R})$ , which defines a point process on  $\mathbb{R}$ . In this case, it is easy to see that the  $n$ -th correlation function is given by

$$\rho_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} p_N(x_1, \dots, x_N) dx_{n+1} \cdots dx_N.$$

for  $n \leq N$ . For example, the eigenvalues distribution of GUE (Gaussian Unitary Ensemble) of size  $N$  is known to be  $p_N(x_1, \dots, x_N) = Z_N^{-1} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 e^{-\sum_{i=1}^N x_i^2}$ , and in this case, the  $n$ -th correlation is given by

$$\rho_n(x_1, \dots, x_n) = \det(K^{(N)}(x_i, x_j))_{i,j=1}^n,$$

where  $K^{(N)}(x, y) = \sum_{k=0}^{N-1} \varphi_k(x) \varphi_k(y)$  with  $\varphi_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{x^2/2} (-\frac{d}{dx})^k e^{-x^2}$  being the  $k$ -th Hermite function. This point process is a prototype of the class of determinantal point processes. See also Example 3.4.

The class of determinantal point processes is an important one of point processes with negative correlations. It was originally introduced as a model of fermionic particles in physics literature, however, determinantal or fermionic structure was found in many other models in mathematics and physics. Here we recall the definition of determinantal point processes (sometimes for short, DPP).

**Theorem 3.3** (cf. [30, 27]). *Let  $K$  be a self-adjoint integral operator on  $L^2(R, \lambda)$ . Suppose (i)  $K$  is of locally trace class, i.e., the restriction operator  $K_\Lambda$  onto a compact set  $\Lambda$  is of trace class and (ii)  $0 \leq K \leq I$ . Then, there is a unique point process on  $R$  whose Laplace transform is given by*

$$E[\exp(-\langle \xi, \varphi \rangle)] = \text{Det}(I - (1 - e^{-\varphi})K_\Lambda), \quad \forall \varphi \in C_c(R),$$

where  $\text{Det}$  is the so-called Fredholm determinant defined for the class of trace class operators and  $K_\Lambda$  is the restriction of  $K$  on  $\Lambda$ . Moreover, the  $n$ -th correlation function with respect to  $\lambda^{\otimes n}$  is given by

$$\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j)_{i,j=1}^n)$$

for every  $n \in \mathbb{N}$ .

*Remark.* (1) While a Poisson point process is determined by a Radon measure, a determinantal point process is determined by a pair of Radon measure  $\lambda(dx)$  and integral kernel  $K(x, y)$ .

(2) By Theorem 3.3,  $\rho_1(x) = K(x, x)$  and  $\rho_2(x, y) = K(x, x)K(y, y) - |K(x, y)|^2$ . So the negative correlation inequality  $\rho_2(x, y) \leq \rho_1(x)\rho_1(y)$  holds. This property is related to repulsive nature of fermions.

(3) The Laplace transform (functional) determines a point process uniquely. For example, the Laplace transform of Poisson point process with intensity  $\lambda_1$  is given by

$$E[\exp(-\langle \xi, \varphi \rangle)] = \exp\left(-\int_R (1 - e^{-\varphi})\lambda_1(dx)\right), \quad \forall \varphi \in C_c^+(R),$$

where  $C_c^+(R)$  is non-negative continuous functions with compact support. On the other hand, for  $\varphi \in C_c^+(R)$ , from Theorem 3.3, the Laplace transform of a DPP is given by

$$\begin{aligned} \text{Det}(I - (1 - e^{-\varphi})K_\Lambda) &= \exp(\log \text{Det}(I - (1 - e^{-\varphi})K_\Lambda)) \\ &= \exp\left(-\sum_{n=1}^{\infty} \text{Tr}\{[K(1 - e^{-\varphi})]^n\}\right) \end{aligned}$$

Since  $K$  is an integral operator, the first term ( $n = 1$ ) of the right-hand side is written as

$$\text{Tr}\{K(1 - e^{-\varphi})\} = \int_R (1 - e^{-\varphi})K(x, x)\lambda(dx).$$

Since  $\lambda_1(dx) = \rho_1(x)\lambda(dx) = K(x, x)\lambda(dx)$ , the first term in the exponential coincides with that of Poisson point process. If  $\xi_1, \dots, \xi_n$  are independent copies of DPP associated with  $K/n$ , it is easy to see that  $\sum_{i=1}^n \xi_i$  converges weakly to the Poisson point process with intensity  $K(x, x)\lambda(dx)$  as  $n \rightarrow \infty$ .

**Example 3.4.** Gaussian Unitary Ensemble (GUE) is an ensemble of random Hermitian matrices. It has been well investigated by many authors from various points of view. As mentioned in Example 3.2, this is one of the most important example of DPP on  $R = \mathbb{R}^1$ , which is associated with the kernel  $K^{(N)}(x, y) = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y)$  where  $\varphi_k(x)$  is the  $k$ -th Hermite function. The kernel  $K^{(N)}$  defines a rank  $n$  projection operator, which satisfies the conditions in Theorem 3.3. Moreover, under appropriate scaling,  $K^{(N)}(x, y)$  converges to the sine kernel  $K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$ . This also defines a projection operator because it is the Fourier transform of the indicator function of an interval, and thus a DPP. In this case, from Theorem 3.3, the first correlation function  $\rho_1(x) = K(x, x) \equiv 1$  and the second correlation is given by

$$\begin{aligned} \rho_2(x, y) &= K(x, x)K(y, y) - K(x, y)^2 \\ &= 1 - \left(\frac{\sin \pi(x-y)}{\pi(x-y)}\right)^2, \end{aligned}$$

which is nothing but the integrand appearing in the Montgomery conjecture. As mentioned in Introduction, several investigations have shown that the non-trivial zeros of Riemann's zeta function look like a realization of the DPP associated with the sine kernel.

In what follows, we always assume that complex-valued random variables have mean 0.

**Definition 3.5.** A complex-valued random variable of the form  $Z = X + iY$  is called a *complex Gaussian* random variable if  $X$  and  $Y$  are independent real-valued random variables subject to the same Gaussian distribution  $N(0, \sigma^2/2)$  with mean 0 and variance  $\sigma^2/2$ . We also say that  $Z$  has a complex Gaussian distribution  $N_{\mathbb{C}}(0, \sigma^2)$ .

**Definition 3.6** (Gaussian analytic function). A random analytic function  $X$  on  $D$  is a Gaussian analytic function (for short, GAF) if it is also a complex Gaussian process, i.e., any finite linear combination of the form  $\sum_{j=1}^n c_j X(z_j)$  ( $c_j \in \mathbb{C}, z_j \in D$ ) is a complex Gaussian random variable.

*Remark.* The probability law of a complex Gaussian process  $X$  on  $D$  is completely determined by its covariance kernel  $S^X(z, w) := E[X(z)\overline{X(w)}]$  for  $z, w \in D$ . It is nonnegative definite in the sense that  $\sum_{i,j=1}^n S^X(z_i, z_j) \xi_i \bar{\xi}_j \geq 0$  for any  $n \in \mathbb{N}, z_i \in D, \xi_i \in \mathbb{C}$ . Conversely, to each nonnegative definite kernel  $\{S(z, w), z, w \in D\}$ , one can associate a Gaussian process  $\{X(z), z \in D\}$  with covariance kernel  $S$ .

**Example 3.7** (Hyperbolic GAF). Let  $\{\zeta_n, n = 0, 1, \dots\}$  be i.i.d. standard complex Gaussian random variables (see Section 4). We define a one parameter family of GAF by

$$X_L^{hyp}(z) = \sum_{n=0}^{\infty} \sqrt{\frac{(L)_n}{n!}} \zeta_n z^n$$

for  $L > 0$ , where  $(a)_n = a(a+1)\cdots(a+n-1)$  is the Pochhammer symbol. This family is also investigated in physics context as chaotic eigenstates (cf. [19]). It is easy to see that the radius of convergence is one almost surely for every  $L > 0$ , and hence  $X_L^{hyp}(z)$  is considered as a GAF on the unit disk  $\mathbb{D}$ , which is called a hyperbolic GAF with parameter  $L$ . In particular,  $X_1^{hyp}(z)$  is the same as  $X_{\mathbb{D}}(z)$  in the introduction. The covariance kernel of  $X_L^{hyp}(z)$  is given by  $S_L^{hyp}(z, w) = (1 - z\bar{w})^{-L} = \sum_{n=0}^{\infty} \frac{\Gamma(n+L)}{n!\Gamma(L)} (z\bar{w})^n$ . It is remarkable that the hyperbolic GAF  $X_L^{hyp}(z)$  satisfies the transformation rule

$$X_L^{hyp}(z) \stackrel{d}{=} (g'_{\alpha}(z))^{L/2} X_L^{hyp}(g_{\alpha}(z))$$

for the Möbius transformation of the form  $g_{\alpha}(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$  for  $\alpha \in \mathbb{D}$ . Since  $g'_{\alpha}(z)$  is nowhere vanishing in  $\mathbb{D}$ , this implies that the associated zero process is invariant in law under the Möbius transformations.

The analyticity of GAF and the fact that the covariance kernel determines uniquely the probability law of a Gaussian process yield strong consequences on the zeros of GAF. Here we recall some nice properties that GAFs have. See [6, 7, 28, 29] for proofs and more details.

The first correlation function of the zeros of a GAF can easily be computed from its correlation kernel  $S(z, w)$ . The following is sometimes called Edelman-Kostlan formula.

**Theorem 3.8.** *Let  $X(z)$  be a GAF on  $D$  with covariance kernel  $S(z, w)$ . Then, the first correlation function (or the intensity) of its zero process is given by*

$$\rho_1(z) = \frac{1}{\pi} \partial_z \partial_{\bar{z}} \log S(z, z)$$

where  $\partial_z = \frac{1}{2}(\partial_x - \partial_y)$  and  $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + \partial_y)$  with  $z = x + iy$ .

*Remark.* When  $S(z, z) = 0$ ,  $X$  has a deterministic (non-random) zero at  $z$ . The first correlation measure includes an atom at  $z$ , and hence  $\rho_1(z)$  does not exist at such  $z$ .

**Example 3.9.** The first correlation function for the zeros of the hyperbolic GAF  $X_L^{hyp}(z)$  is given by

$$\rho_1(z) = \frac{L}{\pi} (1 - z\bar{z})^{-2}.$$

from Theorem 3.8 since  $S_L^{hyp}(z, w) = (1 - z\bar{w})^{-L}$ . This implies that the zeros accumulate towards the boundary of  $\mathbb{D}$ .

The next theorem which is sometimes called Calabi's rigidity shows that the first correlation measure essentially characterizes the law of the zero process of a GAF.

**Theorem 3.10.** *Let  $X$  and  $Y$  be two GAFs on  $D$ . If the first correlation measures of zero processes  $\xi_X$  and  $\xi_Y$  coincide, then there exists a non-vanishing deterministic analytic function  $h$  such that  $Y \stackrel{d}{=} hX$ . In particular,  $\xi_X \stackrel{d}{=} \xi_Y$ .*

Since correlation functions determine point processes completely, one should have the formula for higher correlation functions from the first correlation function. However, the explicit formula for describing each  $n$ -th correlation function in terms of the first correlation is not known.

Here we recall the formula for correlation functions, which is a special case of the so-called Kac-Rice formulas [7, 8, 12, 25]. It is essentially obtained by the change of variables.

**Theorem 3.11.** *Let  $X(z)$  be a GAF on  $D$  with covariance kernel  $S(z, w)$ . The  $n$ -th correlation function of the zeros of  $X(z)$  is given by the formula*

$$\rho_n(z_1, z_2, \dots, z_n) = \frac{E[|X'(z_1)X'(z_2) \cdots X'(z_n)|^2 | X(z_1) = X(z_2) = \cdots = X(z_n) = 0]}{\det(\pi(S(z_i, z_j))_{i,j=1}^n)}$$

for distinct  $z_1, z_2, \dots, z_n$  whenever  $\det(S(z_i, z_j))_{i,j=1}^n > 0$ .

Although we can compute every correlation function from this formula in principle, it is too complicated for large  $n$  to determine what the corresponding point process is. Peres and Virág found that it could be carried out for hyperbolic GAF for  $L = 1$ , namely,  $X_1^{hyp}(z) = \sum_{n=0}^{\infty} \zeta_n z^n$  or we write  $X_{\mathbb{D}}(z)$  in the Introduction.

**Theorem 3.12** ([24]). *The zero process of hyperbolic GAF  $X_1^{hyp}(z)$  (or  $X_{\mathbb{D}}(z)$ ) is the determinantal point process on the unit disk  $\mathbb{D}$  associated with the Bergman kernel*

$$(3.1) \quad K_{\mathbb{D}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}.$$

In particular, the  $n$ -th correlation function is given by the determinantal form

$$\rho_n(z_1, \dots, z_n) = \pi^{-n} \det \left( \frac{1}{(1 - z_i \bar{z}_j)^2} \right)_{i,j=1}^n$$

Once we realize a point process is determinantal, we can compute many quantities that we would like to know.

Next we give another example of a determinantal point process which arises as the zeros of a GAF.

**Example 3.13** (Zeros of complex Wiener integral of Szegő kernel). We consider a Gaussian analytic function defined on the upper half-plane  $\mathbb{H}$  as the Wiener integral of Szegő kernel

$$X_{\mathbb{H}}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{t - z} dB(t) = \int_{\mathbb{R}} S_{\mathbb{H}}(z, t) dB(t), \quad z \in \mathbb{H},$$

where  $S_{\mathbb{H}}(z, w) = \frac{1}{2\pi i(\bar{w} - z)}$  and  $B(t)$  is a standard complex Brownian motion. Such a Wiener integral can be defined for  $f \in L^2(\mathbb{R})$  in the  $L^2$ -sense for each  $z \in \mathbb{H}$ . From Proposition 2.1, we can define it as a random analytic function as in the Appendix. It is known that

$$E \left[ \int_{\mathbb{R}} f(t) dB(t) \overline{\int_{\mathbb{R}} g(t) dB(t)} \right] = \int_{\mathbb{R}} f(t) \overline{g(t)} dt.$$

By the reproducing kernel property of the Szegő kernel, i.e.,  $\int_{\mathbb{R}} S_{\mathbb{H}}(z, t) S_{\mathbb{H}}(t, w) dt = S_{\mathbb{H}}(z, w)$ , we see that the covariance kernel of  $X_{\mathbb{H}}(z)$  is equal to  $S_{\mathbb{H}}(z, w)$  itself. The zero process of  $X_{\mathbb{H}}(z)$  is a determinantal point process associated with the kernel

$$K_{\mathbb{H}}(z, w) = 4\pi S_{\mathbb{H}}(z, w)^2 = \frac{-1}{\pi(\bar{w} - z)^2}.$$

A proof of this result and its slight extension will be provided in the appendix. The first correlation function (or the intensity) is given by

$$\rho_1(z) = K_{\mathbb{H}}(z, z) = \frac{1}{4\pi(\text{Im} z)^2},$$

which, of course, can also be computed by using Theorem 3.8. Also we can see that the zero process  $\xi_{X_{\mathbb{H}}}$  is invariant under  $SL_2(\mathbb{R})$ -action in the sense that  $\xi_{X_{\mathbb{H}}} \stackrel{d}{=} \xi_{X_{\mathbb{H}} \circ T}$  for any  $T \in SL_2(\mathbb{R})$ .

For DPPs and more details, we refer the readers to [7, 30, 27].

#### § 4. Central limit theorem for random analytic functions

We denote by  $\mathcal{C}_v$  the set of square integrable, complex-valued random variables with mean 0 such that the real part and the imaginary part are mutually orthogonal and have the same finite variance  $v \geq 0$ . We write  $\mathcal{C}$  for  $\bigcup_{v \geq 0} \mathcal{C}_v$ . It is obvious that if  $\zeta \in \mathcal{C}$  then  $E[(\operatorname{Re}\zeta)^2] = E[(\operatorname{Im}\zeta)^2] = \frac{1}{2}E[|\zeta|^2]$  and  $E[\zeta^2] = 0$ . If  $\zeta$  is a complex Gaussian random variable whose distribution is  $N_{\mathbb{C}}(0, \sigma^2)$ , then  $\zeta \in \mathcal{C}_{\sigma^2}$ .

**Lemma 4.1.** *Let  $\{\zeta_k\}_k \subset \mathcal{C}_1$  be independent complex-valued random variables with unit variance and set  $Y = \sum_k \theta_k \zeta_k$  for  $\theta_k \in \mathbb{C}$  with  $\sum_k |\theta_k|^2 < \infty$ . Then,  $Y \in \mathcal{C}$ .*

*Proof.* It is easy to see that (i)  $\theta Z \in \mathcal{C}$  for any  $\theta \in \mathbb{C}$  and  $Z \in \mathcal{C}$ , (ii)  $Z_1 + Z_2 \in \mathcal{C}$  if  $Z_1, Z_2 \in \mathcal{C}$  are independent and (iii) the class  $\mathcal{C}$  is closed under the  $L^2$ -convergence, which imply the assertion.  $\square$

*Remark.* When  $\{\theta_k\}_k$  are independent random variables that are independent of  $\{\zeta_k\}_k$ , the same conclusion of Lemma 4.1 holds under the condition  $\sum_k E[|\theta_k|^2] < \infty$ .

We recall a complex version of the central limit theorem under the Lindeberg condition.

**Proposition 4.2.** *Let  $\{Z_{n,k}\} \subset \mathcal{C}$  be an array of complex random variables. Suppose  $\{Z_{n,k}\}$  are independent for fixed  $n$  and satisfy the following two conditions: (i)  $\lim_{n \rightarrow \infty} \sum_k E[|Z_{n,k}|^2] = \sigma^2$  and (ii)  $\lim_{n \rightarrow \infty} \sum_k E[|Z_{n,k}|^2; |Z_{n,k}| > \epsilon] = 0$  for any  $\epsilon > 0$ . Then,  $\{\sum_k Z_{n,k}\}_n$  converges to  $N_{\mathbb{C}}(0, \sigma^2)$  in law as  $n \rightarrow \infty$ .*

*Proof.* The central limit theorem under the Lindeberg condition for an array of real random variables  $\{X_{n,k}\}$  with mean 0 is as follows (cf. [4]): Suppose  $\{X_{n,k}\}_k$  are independent for fixed  $n$  and satisfy the following two conditions: (j)  $\lim_{n \rightarrow \infty} \sum_k E[|X_{n,k}|^2] = \sigma^2$  and (jj)  $\lim_{n \rightarrow \infty} \sum_k E[|X_{n,k}|^2; |X_{n,k}| > \epsilon] = 0$  for any  $\epsilon > 0$ . Then,  $\{\sum_k X_{n,k}\}$  converges to  $N(0, \sigma^2)$  in law.

It suffices to show that for every  $\lambda, \mu \in \mathbb{R}$ ,  $\{\lambda \operatorname{Re} Z_{n,k} + \mu \operatorname{Im} Z_{n,k}\}_n$  satisfies (j) with  $(\lambda^2 + \mu^2)\sigma^2/2$  and (jj) for any  $\epsilon > 0$ . A little consideration shows that this is the case.  $\square$

**Corollary 4.3.** *Let  $\{\zeta_k\}_k \subset \mathcal{C}_1$  be i.i.d. complex random variables and  $\{f_{n,k}\}_{n,k}$  be an array of complex random variables independent of  $\{\zeta_k\}_k$ . Suppose that (1)  $\{f_{n,k}\}_k$  for fixed  $n$  are independent, (2)  $\lim_{n \rightarrow \infty} \sum_k E[|f_{n,k}|^2] = \sigma^2$  and (3)  $\lim_{n \rightarrow \infty} \sum_k E[|f_{n,k}|^{2+\delta}] = 0$  for some  $\delta > 0$ . Then, a sequence of complex random variables  $X_n = \sum_k f_{n,k} \zeta_k, n = 1, 2, \dots$  converges to  $N_{\mathbb{C}}(0, \sigma^2)$  in law.*

*Proof.* The condition (i) in Proposition 4.2 is obvious from (2). It suffices to check the condition (ii) in Proposition 4.2 by putting  $Z_{n,k} = f_{n,k} \zeta_k$ . Let  $F(t) = E[|\zeta_1|^2; |\zeta_1| > t]$  for  $t \geq 0$ . Since  $\{f_{n,k}\}_{n,k}$  is independent of  $\{\zeta_k\}_k$ , we have

$$\begin{aligned} R_{n,\epsilon} &:= \sum_k E[|f_{n,k} \zeta_k|^2; |f_{n,k} \zeta_k| > \epsilon] \\ &= \sum_k E[|f_{n,k}|^2 F(\frac{\epsilon}{|f_{n,k}|}); |f_{n,k}| > 0] \\ &= \sum_k E[|f_{n,k}|^2 F(\frac{\epsilon}{|f_{n,k}|}); |f_{n,k}| > \eta] + \sum_k E[|f_{n,k}|^2 F(\frac{\epsilon}{|f_{n,k}|}); 0 < |f_{n,k}| \leq \eta] \\ &\leq F(0) \eta^{-\delta} \sum_k E[|f_{n,k}|^{2+\delta}] + F(\frac{\epsilon}{\eta}) \sum_k E[|f_{n,k}|^2] \end{aligned}$$

for any  $\eta > 0$ . Then,

$$\limsup_{n \rightarrow \infty} R_{n,\epsilon} \leq \sigma^2 F(\frac{\epsilon}{\eta}).$$

Since  $\eta > 0$  is arbitrary, it holds that  $R_{n,\epsilon} \rightarrow 0$  as  $n \rightarrow \infty$  for every  $\epsilon > 0$ .  $\square$

**Theorem 4.4.** *Let  $\{\zeta_k\}_k \subset \mathcal{C}_1$  be i.i.d. complex random variables and  $\{\psi_{n,k}(z)\}$  an independent array of random analytic functions on  $D$  independent of  $\{\zeta_k\}_k$  such that  $\sum_k E[|\psi_{n,k}(z)|^2] < \infty$  for every  $z \in D$ . We consider a sequence  $\{X_n(z)\}$  of random analytic functions on  $D$  of the form*

$$X_n(z) = \sum_k \zeta_k \psi_{n,k}(z), \quad z \in D$$

with finite covariance kernel  $S_n(z, w) = \sum_k E[\psi_{n,k}(z) \overline{\psi_{n,k}(w)}]$ . Suppose that

(A1) The covariance kernel  $S_n(z, w)$  converges to  $S(z, w)$  for every  $z, w \in D$ .

(A2) There exists a locally integrable function  $g(z)$  such that  $\sup_n S_n(z, z) \leq g(z)$ .

(A3) There exists a positive constant  $\delta > 0$  such that  $\lim_{n \rightarrow \infty} \sum_k E[|\psi_{n,k}(z)|^{2+\delta}] = 0$  for every  $z \in D$ .

Then,  $\{X_n\}$  converges in law to the Gaussian analytic function  $X$  with covariance kernel  $S(z, w)$ . In particular, the sequence  $\{\xi_{X_n}\}$  of the zero processes converges in law to  $\xi_X$  provided that  $X \not\equiv 0$  almost surely.

*Proof of Theorem 4.4.* We apply Corollary 4.3 to the random variable

$$Y_n(\lambda) = \sum_{j=1}^M \lambda_j X_n(z_j) = \sum_k \left( \sum_{j=1}^M \lambda_j \psi_{n,k}(z_j) \right) \zeta_k =: \sum_k f_{n,k} \zeta_k$$

for  $\lambda := (\lambda_1, \dots, \lambda_M) \in \mathbb{C}^M$  and distinct points  $z_1, \dots, z_M \in D$ . For (2) in Corollary 4.3, as  $n \rightarrow \infty$

$$\sum_k E[|f_{n,k}|^2] = \sum_{j,l=1}^M \lambda_j \bar{\lambda}_l S_n(z_j, z_l) \rightarrow \sum_{j,l=1}^M \lambda_j \bar{\lambda}_l S(z_j, z_l)$$

from (A1). For (3) in Corollary 4.3, for  $p > 2$ , by Hölder's inequality

$$\sum_k E[|f_{n,k}|^p] = \sum_k E\left[\left|\sum_{j=1}^M \lambda_j \psi_{n,k}(z_j)\right|^p\right] \leq C_{p,\lambda} \sum_{j=1}^M \sum_k E[|\psi_{n,k}(z_j)|^p] \rightarrow 0$$

from (A3). Then,  $\{X_n\}$  converges to the Gaussian process with covariance kernel  $S(z, w)$  in the sense of finite dimensional distributions.

Let us take  $\delta > 0$  so small that the closure of the  $\delta$ -neighborhood of  $K$  is contained in  $D$ . From Lemma 2.6 with  $p = 2$ , we see that

$$\pi \delta^2 E[\|X_n\|_K^2] \leq \int_{\overline{K_\delta}} E[|X_n(z)|^2] m(dz) = \int_{\overline{K_\delta}} S_n(z, z) m(dz).$$

By (A2), it holds that  $\sup_n E[\|X_n\|_K^2] < \infty$ . Therefore, by Proposition 2.5, the sequence  $\{\|X_n\|_K\}_n$  is tight. Consequently, a sequence of random analytic functions  $\{X_n\}$  converges in law to the Gaussian analytic function  $X$  with covariance kernel  $S(z, w)$ . The last part of the theorem follows from Proposition 2.3.  $\square$

*Remark.* Theorem 4.4 also holds when  $\zeta_n$  is identically 1 for every  $n$ .

## § 5. Examples

In this section, we give some examples to which Theorem 4.4 is applied. We always assume that  $\{\zeta_k\}_k \subset \mathcal{C}_1$  are i.i.d. complex random variables (but not necessarily complex standard normal).

**Example 5.1.** Let us consider the random analytic function

$$X(z) = \sum_{k \in \mathbb{Z}} \zeta_k S_{\mathbb{H}}(z, k) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} \frac{\zeta_k}{k - z}$$



on the upper-half plane  $\mathbb{H}$ , where  $S_{\mathbb{H}}(z, w) = \frac{1}{2\pi i(\bar{w}-z)}$  is the Szegő kernel for the upper half-plane. It is easy to see that

$$S^X(z, w) := E[X(z)\overline{X(w)}] = S_{\mathbb{H}}(z, w) \frac{\cot \pi \bar{w} - \cot \pi z}{2i} = S_{\mathbb{H}}(z, w) \frac{\sin \pi(z - \bar{w})}{2i \sin \pi z \cdot \sin \pi \bar{w}}.$$

We notice that if  $\zeta_k$  is complex standard normal, by Theorem 3.8, we have

$$\rho_1^X(z) = \rho_{1, \mathbb{H}}(z) - \frac{\pi}{\sinh^2 2\pi y} \sim \frac{\pi}{3} - \frac{4\pi^3}{15} y^2 + O(y^4), \quad y = \operatorname{Im} z \rightarrow 0.$$

Here  $\rho_{1, \mathbb{H}}(z)$  in the right-hand side is the same as in Example 3.13. It implies that the zeros of  $X(z)$  does not accumulate on the real line.

By Theorem 4.4 we can show that the scaled random analytic function  $X_n(z) = \sqrt{n}X(nz)$  converges to the GAF  $X_{\mathbb{H}}(z)$  with covariance kernel  $S_{\mathbb{H}}(z, w)$ . In particular, the zero process of  $X_n(z)$ , or equivalently the zero process of  $X(z)$  scaled by  $1/n$  converges in law to the determinantal point process with kernel  $K_{\mathbb{H}}(z, w) = \frac{-1}{\pi(\bar{w}-z)^2}$ .

*Proof.* Since  $S^{X_n}(z, w) = nS^X(nz, nw)$  and  $nS_{\mathbb{H}}(nz, nw) = S_{\mathbb{H}}(z, w)$ ,

$$\begin{aligned} |S^{X_n}(z, w) - S_{\mathbb{H}}(z, w)| &= |S_{\mathbb{H}}(z, w)| \cdot \left| \frac{\sin \pi n(z - \bar{w})}{2i \sin \pi n z \sin \pi n \bar{w}} - 1 \right| \\ &= |S_{\mathbb{H}}(z, w)| \cdot \left| \frac{e^{2i\pi n z} + e^{-2i\pi n \bar{w}} - 2e^{2i\pi n(z - \bar{w})}}{(1 - e^{2i\pi n z})(1 - e^{-2i\pi n \bar{w}})} \right| \\ &\leq |S_{\mathbb{H}}(z, w)| \frac{e^{-2\pi n a} + e^{-2\pi n b} + 2e^{-2\pi n(a+b)}}{(1 - e^{-2\pi n a})(1 - e^{-2\pi n b})}, \end{aligned}$$

where  $a = \operatorname{Im} z$  and  $b = \operatorname{Im} w$ . Hence,  $S^{X_n}(z, w)$  converges to  $S_{\mathbb{H}}(z, w)$  uniformly on  $K \times K$  for any compact set  $K \subset \mathbb{H}$ . In particular, the assumptions (A1) and (A2) in Theorem 4.4 hold. For the assumption (A3), for  $\delta > 0$ , we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\sqrt{n} S_{\mathbb{H}}(nz, k)|^{2+\delta} &= \sum_{k \in \mathbb{Z}} \left| \frac{\sqrt{n}}{2\pi(k - nz)} \right|^{2+\delta} \\ &= \frac{1}{(2\pi\sqrt{n})^{2+\delta}} \sum_{k \in \mathbb{Z}} \left| \frac{1}{\frac{k}{n} - z} \right|^{2+\delta} \\ &= O\left(\frac{1}{n^{\delta/2}}\right). \end{aligned}$$

Therefore, we obtain the result.  $\square$

**Example 5.2.** This is a randomized version of Example 5.1. Let us consider the random analytic function

$$Y(z) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} \frac{\zeta_k}{t_k - z}$$

on the upper-half plane  $\mathbb{H}$ , where  $t_k = k + \eta_k$  with  $\{\eta_k\}_{k \in \mathbb{Z}}$  being i.i.d. real random variables. Since  $S^Y(z, w) = E[S^X(z - \eta_0, w - \eta_0)]$  where  $S^X(z, w)$  is the one in Example 5.1, in almost the same way as before, we can show that  $Y_n(z) = \sqrt{n}Y(nz)$  converges to the GAF  $X_{\mathbb{H}}(z)$  with covariance kernel  $S_{\mathbb{H}}(z, w)$ . In particular, the zero process of  $Y_n(z)$  converges in law to the determinantal point process with kernel  $K_{\mathbb{H}}(z, w) = \frac{-1}{\pi(\bar{w} - z)^2}$ .

**Example 5.3.** We consider the Szegő kernel  $S_{\mathbb{D}}(z, w) = (1 - z\bar{w})^{-1}$  for the unit disk  $\mathbb{D}$ . Now we define

$$X_n(z) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \zeta_k S_{\mathbb{D}}(z, e^{i\theta_k}) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \frac{\zeta_k}{1 - ze^{-i\theta_k}}$$

where  $\{\zeta_k\}_{k=0}^{\infty} \subset \mathcal{C}_1$  are i.i.d. random variables, and either (i)  $\theta_k = \frac{2\pi k}{n}, k = 0, 1, \dots, n-1$  or (ii)  $\{\theta_k\}$  are i.i.d. uniform random variables on  $[0, 2\pi]$ . It is easy to see that  $S^{X_n}(z, w) \rightarrow S_{\mathbb{D}}(z, w)$  as  $n \rightarrow \infty$ . Moreover,

$$\sum_{k=0}^{n-1} E \left[ \left| \frac{1}{\sqrt{n}} \frac{1}{1 - ze^{-i\theta_k}} \right|^{2+\delta} \right] \leq n^{-\delta/2} \left( \frac{1}{1 - |z|} \right)^{2+\delta} \rightarrow 0.$$

Then,  $\{X_n\}$  converges to the hyperbolic GAF  $X_1^{hyp}(z)$ , and the corresponding zero process  $\{\xi_{X_n}\}$  converges in law to the determinantal point process associated with the Bergman kernel  $K_{\mathbb{D}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$ .

Theorem 4.4 is an extension of the following result obtained by Ledoan-Merkli-Starr [18].

**Example 5.4** ([18]). For fixed  $L > 0$ , we consider a random analytic function defined by

$$X(z) = \sum_{n=0}^{\infty} \sqrt{\frac{(L)_n}{n!}} \zeta_n z^n,$$

where  $(a)_n = a(a+1)\cdots(a+n-1)$  is the Pochhammer symbol. In particular, when  $\{\zeta_n\}_{n=0}^{\infty}$  are i.i.d. standard complex normal random variables, we write  $X_L^{hyp}(z)$  as in Example 3.7. The convergence radius of  $X(z)$  is 1 almost surely. The covariance kernel  $S^X(z, w)$  of  $X(z)$  is equal to  $S_{\mathbb{D}, L}(z, w) = (1 - z\bar{w})^{-L}$  which is the same as that of  $X_L^{hyp}(z)$ . Now we consider a Möbius transformation

$$g_{\alpha}(z) = \frac{z - \alpha}{1 - z\bar{\alpha}}, \quad (|\alpha| < 1).$$

It is easily seen that one can take  $(g'_{\alpha}(z))^{1/2}$  as a nowhere vanishing analytic function on the unit disk. Then, the random analytic function

$$X_{\alpha}(z) := g'_{\alpha}(z)^{L/2} X(g_{\alpha}(z))$$

converges to  $X_L^{hyp}(z)$  in law as  $|\alpha| \rightarrow 1$ .

*Proof.* While a Fourier-analytic technique is used in [18] to show the condition (A3), here we use an asymptotic behavior of the Gauss hypergeometric functions. It is easy to verify that

$$S_{\mathbb{D},L}(g_\alpha(z), g_\alpha(w)) = g'_\alpha(z)^{-L/2} \overline{g'_\alpha(w)^{-L/2}} S_{\mathbb{D},L}(z, w).$$

The conditions (A1) and (A2) in Theorem 4.4 is satisfied since  $S^{X_\alpha}(z, w) = S_{\mathbb{D},L}(z, w)$  for all  $\alpha \in \mathbb{D}$ . We show the condition (A3) for  $\delta = 2$ . Let  $\psi_k^{(\alpha)}(z) = \sqrt{\frac{(L)_k}{k!}} (g'_\alpha(z))^{L/2} g_\alpha(z)^k$ . Then,

$$\begin{aligned} (5.1) \quad \sum_{k=0}^{\infty} |\psi_k^{(\alpha)}(z)|^4 &= \sum_{k=0}^{\infty} \frac{(L)_k^2}{(k!)^2} |g'_\alpha(z)|^{2L} |g_\alpha(z)|^{4k} \\ &= |g'_\alpha(z)|^{2L} {}_2F_1(L, L; 1; |g_\alpha(z)|^4). \end{aligned}$$

Here  ${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$  is the Gauss hypergeometric function and

$${}_2F_1(L, L; 1; x) \sim C_L \times \begin{cases} (1-x)^{-(2L-1)}, & L > 1/2 \\ -\log(1-x), & L = 1/2 \\ 1, & 0 < L < 1/2 \end{cases}$$

as  $x \uparrow 1$  (cf. [17]). Since  $1 - |g_\alpha(z)|^2 = \frac{(1-|z|^2)(1-|\alpha|^2)}{|1-z\bar{\alpha}|^2} = (1-|z|^2)|g'_\alpha(z)|$ , the right-hand side of (5.1) converges to 0 as  $|\alpha| \rightarrow 1$  for every  $z \in \mathbb{D}$  and  $L > 0$ .  $\square$

**Example 5.5.** Let us consider the random Dirichlet series defined by

$$X(z) = \sum_p \frac{\Theta_p}{p^{1/2-iz}}, \quad z \in \mathbb{H},$$

where  $\{\Theta_p\}_p$  are i.i.d. uniform random variables on  $\{z \in \mathbb{C}; |z| = 1\}$  and the sum is taken over all primes. This defines a random analytic function on  $\mathbb{H}$  by Proposition 2.1. It is the first order approximation of  $\sum_p -\log(1 - \frac{\Theta_p}{p^{\sigma-iz}})$ , which is the limiting random analytic function (rotated by 90 degrees) appearing in the Bohr-Jessen theorem for the empirical distribution of  $\log \zeta(z + it), t \in \mathbb{R}$  (cf. [20], [31]). Then,  $X_\epsilon(z) := \frac{1}{\sqrt{\log(1/\epsilon)}} X(\epsilon z)$  for  $\epsilon > 0$  converges in law to the constant function  $Y(z) \equiv \zeta$  where  $\zeta \sim N_{\mathbb{C}}(0, 1)$  as  $\epsilon \rightarrow 0$ . Hence, the zero process  $\xi_{X_\epsilon}$  converges in law to the empty configuration.

*Proof.* For  $z, w \in \mathbb{H}$ , the covariance kernel of  $X_\epsilon$  is given by

$$S_\epsilon(z, w) = E[X_\epsilon(z) \overline{X_\epsilon(w)}] = \frac{1}{\log(1/\epsilon)} \sum_p \frac{1}{p^{1-i\epsilon(z-\bar{w})}}$$

For  $f \in C_b^1(0, \infty)$  under suitable condition, integration by parts yields

$$\begin{aligned}
 (5.2) \quad \sum_p \frac{f(\log p)}{p} &= \int_2^\infty \frac{f(\log t)}{t \log t} dt - \int_2^\infty (f'(\log t) - f(\log t)) \frac{R(t)}{t^2} dt \\
 &= \int_{\log 2}^\infty f(s) \frac{ds}{s} - \int_{\log 2}^\infty (f'(s) - f(s)) \frac{R(e^s)}{e^s} ds \\
 &=: (I) - (II),
 \end{aligned}$$

where  $R(x) = \pi(x) - \text{li}(x)$  for  $x \geq 2$ , where  $\pi(x)$  is the number of primes up to  $x$  and  $\text{li}(x) = \int_2^x \frac{dt}{\log t}$ . By the prime number theorem with error bound, there exist positive constants  $c_1$  and  $M$  such that

$$(5.3) \quad \left| \frac{R(x)}{x} \right| \leq M e^{-c_1 \sqrt{\log x}}$$

for  $x \geq 2$  (cf. Theorem 1.1 [31], [20]). Now if we take  $f(s) = e^{i\epsilon(z-\bar{w})s}$  with  $z - \bar{w} \in \mathbb{H}$ , the left-hand side of (5.2) is equal to  $\log(1/\epsilon) S_\epsilon(z, w)$ . From the estimates (5.3) and  $|f'(s) - f(s)| \leq \epsilon|z - \bar{w}| + 1$ ,  $(II) = O(1)$  as  $\epsilon \rightarrow 0$ . By carrying out a contour integral for  $(I)$ , we can easily see that

$$(I) = \int_{\log 2}^\infty e^{i\epsilon(z-\bar{w})s} \frac{ds}{s} = \log(1/\epsilon) + O(1) \quad \text{as } \epsilon \rightarrow 0$$

for  $z, w \in \mathbb{H}$  uniformly on compact subsets of  $\mathbb{H} \times \mathbb{H}$ . Hence, the conditions (A1) and (A2) follow. In particular,  $S_\epsilon(z, w) \rightarrow 1$  for every  $z, w \in \mathbb{H}$ . For the condition (A3) in Theorem 4.4,

$$\sum_p \left| \frac{1}{\sqrt{\log(1/\epsilon)}} \frac{1}{p^{1/2-i\epsilon z}} \right|^4 \leq \frac{1}{(\log(1/\epsilon))^2} \sum_p \frac{1}{p^{2+4\epsilon \text{Im} z}} \rightarrow 0$$

as  $\epsilon \rightarrow 0$ .

Consequently,  $X_\epsilon(z)$  converges in law to the GAF with covariance kernel  $S(z, w) \equiv 1$ , which is the constant function  $\zeta \sim N_{\mathbb{C}}(0, 1)$ .  $\square$

## § 6. Appendix

Let  $D$  be a domain in the complex plane  $\mathbb{C}$  and  $\{X(z), z \in D\}$  be a centered Gaussian analytic function with covariance kernel  $S(z, w)$ , i.e.,

$$E[X(z)] = 0, \quad E[X(z) \overline{X(w)}] = S(z, w).$$

The reproducing kernel Hilbert space corresponding to  $S$  is denoted by  $H_S$ . It is known that the covariance kernel for the GAF  $\{X(z), z \in D\}$  given that  $X(\alpha) = 0$  for  $\alpha \in D$  is given by

$$S^\alpha(z, w) = S(z, w) - \frac{S(z, \alpha) S(\alpha, w)}{S(\alpha, \alpha)}$$

whenever  $S(\alpha, \alpha) > 0$ . The space  $H_{S^\alpha}$  is a subspace of functions in  $H_S$  that vanish at  $\alpha$ . We inductively define  $S^{\alpha_1, \dots, \alpha_n}$  by

$$S^{\alpha_1, \dots, \alpha_n}(z, w) := (S^{\alpha_1, \dots, \alpha_{n-1}}(z, w))^{\alpha_n}$$

for  $\alpha_1, \alpha_2, \dots, \alpha_n \in D$  with  $\det(S(\alpha_j, \alpha_k))_{j,k=1}^n > 0$ . It is easy to see that it does not depend on the order of  $\alpha_1, \dots, \alpha_n$ . Then, the covariance kernel for the conditional GAF  $\{X(z), z \in D\}$  given that  $X(\alpha_1) = X(\alpha_2) = \dots = X(\alpha_n) = 0$  is equal to  $S^{\alpha_1, \dots, \alpha_n}$ . Also, we see that the covariance kernel for  $X'(z)$  is given by  $\partial_z \partial_{\bar{w}} S(z, w)$ . From these observations together with Theorem 3.11, we have the following theorem. This is a slightly different expression from Corollary 3.4.2 in [7].

**Proposition 6.1.** *The  $n$ -th correlation function of the zeros of GAF with covariance kernel  $S(z, w)$  is given by the formula*

$$\rho_n(z_1, z_2, \dots, z_n) = \frac{\text{per}(\partial_z \partial_{\bar{w}} S^{z_1, z_2, \dots, z_n}(z_j, z_k))_{j,k=1}^n}{\det(\pi S(z_j, z_k))_{j,k=1}^n}$$

for distinct  $z_1, z_2, \dots, z_n \in D$  with  $\det(S(z_j, z_k))_{j,k=1}^n > 0$ , where  $\text{per } A$  is the permanent of an  $n$  by  $n$  matrix  $A = (a_{jk})_{j,k=1}^n$  defined by

$$\text{per } A = \sum_{\sigma \in \mathcal{S}_n} \prod_{j=1}^n a_{j\sigma(j)},$$

where  $\mathcal{S}_n$  is the symmetric group of order  $n$ .

*Proof.* From the observations above, the joint density of the conditional Gaussian vector  $(X'(z_1), X'(z_2), \dots, X'(z_n))$  given that  $X(z_1) = X(z_2) = \dots = X(z_n) = 0$  is the complex Gaussian with covariance  $(\partial_z \partial_{\bar{w}} S^{z_1, \dots, z_n}(z_j, z_k))_{j,k=1}^n$ . Moreover, the second absolute moment of the product  $X_1 X_2 \dots X_n$  of complex Gaussian random variables is equal to the permanent of the covariance matrix of the Gaussian vector  $(X_1, X_2, \dots, X_n)$ . Therefore, we obtain the desired expression from the formula in Theorem 3.11.  $\square$

We consider the Szegő kernel for the upper half-plane  $\mathbb{H}$

$$S_{\mathbb{H}}(z, w) = \frac{1}{2\pi i} \frac{1}{\bar{w} - z}.$$

The corresponding reproducing kernel Hilbert space is the Hardy space  $H^2(\mathbb{H})$  on  $\mathbb{H}$ . The kernel can be expanded as

$$S_{\mathbb{H}}(z, w) = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)}$$

by an orthonormal basis of  $H^2(\mathbb{H})$

$$e_n(z) = \frac{1}{\sqrt{\pi}} \frac{1}{1-iz} \left( \frac{1+iz}{1-iz} \right)^n, \quad n \geq 0.$$

We define a Gaussian analytic function by the Wiener integral

$$X_{\mathbb{H}}(z) = \int_{-\infty}^{\infty} S_{\mathbb{H}}(z, t) dB(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{t-z} dB(t), \quad z \in \mathbb{H}$$

on the upper half-plane  $\mathbb{H}$  (and also the lower half-plane  $\mathbb{H}_-$ ), where  $B(t)$  is a complex Brownian motion. It can be expanded by the orthonormal basis  $\{e_n(z), n \geq 0\}$  as

$$X_{\mathbb{H}}(z) = \sum_{n=0}^{\infty} \xi_n e_n(z),$$

where  $\xi_n = \int_{\mathbb{R}} \overline{e_n(t)} dB(t)$ ,  $n \geq 0$  are i.i.d. standard complex Gaussian random variables. By Proposition 2.1, we can define  $X_{\mathbb{H}}(z)$  as a random analytic function. It also follows from Remark 6.2 in [23] together with [1] which states that one can take an analytic version of  $X_{\mathbb{H}}(z)$  if it has the strong derivative in the  $L^2$ -sense with respect to  $z \in \mathbb{C} \setminus \mathbb{H}$ .

**Theorem 6.2.** *The zeros of GAF  $X_{\mathbb{H}}(z)$  is a determinantal point process on the upper half-plane  $\mathbb{H}$  associated with the Bergman kernel*

$$(6.1) \quad K_{\mathbb{H}}(z, w) = 4\pi S_{\mathbb{H}}(z, w)^2 = \frac{-1}{\pi(\overline{w} - z)^2}$$

for  $\mathbb{H}$  acting as a projection operator on  $L^2(\mathbb{H}, m(dz))$ .

This theorem is the counterpart for  $\mathbb{H}$  of Theorem 3.12. Indeed, if we write

$$\tilde{S}_{\mathbb{D}}(z, w) = \frac{1}{2\pi} S_{\mathbb{D}}(z, w) = \frac{1}{2\pi(1-z\overline{w})} = \sum_{n=0}^{\infty} \varphi_n(z) \overline{\varphi_n(w)}$$

with  $\{\varphi_n(z) = \frac{z^n}{\sqrt{2\pi}}\}_{n=0}^{\infty}$  being CONS of  $H^2(\mathbb{D})$ , it is easy to check that

$$S_{\mathbb{H}}(z, w) = T_i(z)^{1/2} \overline{T_i(w)^{1/2}} \tilde{S}_{\mathbb{D}}(T_i(z), T_i(w)),$$

where  $T_{\beta}(z) = \frac{z-\beta}{z-\overline{\beta}}$  for  $\beta \in \mathbb{H}$ ; in particular, when  $\beta = i (= \sqrt{-1})$ ,  $T_i(z)$  is the Cayley transform which maps  $\mathbb{H}$  conformally to the unit disk  $\mathbb{D}$ .

Here we give a slight extension of this theorem.

**Theorem 6.3.** *Let  $X_v(z)$  be a GAF whose covariance kernel is given by*

$$S_v(z, w) := \frac{1}{2\pi} \int_0^{\infty} e^{i(z-\overline{w})\lambda} e^{-v\lambda} d\lambda = \frac{1}{2\pi} \frac{1}{v - i(z - \overline{w})}, \quad z, w \in \mathbb{H}$$

for  $v \geq 0$ . Then, the zeros of  $X_v(z)$  is a determinantal point process on  $\mathbb{H}$  associated with kernel

$$K_v(z, w) = 4\pi S_v(z, w)^2 = \frac{1}{\pi(v - i(z - \bar{w}))^2}$$

Since  $S_v(z, w) = S_{\mathbb{H}}(z + iv/2, w + iv/2)$ , this is the covariance kernel of  $X_{\mathbb{H}}(\cdot + iv/2)$ . Therefore, the zeros of  $X_v(z)$  is equal to those of  $X_{\mathbb{H}}(z)$  in the restricted domain  $\text{Im} z > v/2$ . The theorem above is the consequence from Theorem 6.2 (if we admit) and the fact that a determinantal point process restricted in a subdomain is again determinantal. Here we give a direct proof for Theorem 6.3 along the line of the proof in [24] for the sake of readers' convenience,

**Lemma 6.4.** For  $z_1, z_2, \dots, z_n, z, w \in \mathbb{H}$ ,

$$S_v^{z_1, \dots, z_n}(z, w) = S_v(z, w) \gamma_n(z) \overline{\gamma_n(w)}.$$

where  $\gamma_n(z) = \prod_{k=1}^n h_{z_k}(z)$  and  $h_a(z) = 2\pi(z - a)S_v(z, a)$  for  $a \in \mathbb{H}$ .

*Proof.* We notice that (i)  $S_v^a(z, w) = S_v(z, w)h_a(z)\overline{h_a(w)}$  for every  $a \in \mathbb{H}$  and (ii) if  $L(z, w) = Q(z, w)g(z)\overline{g(w)}$  for some  $Q$  and  $g$ , then  $L^a(z, w) = Q^a(z, w)g(z)\overline{g(w)}$ . From (i) and (ii), we can show the equality by induction.  $\square$

Here we recall two determinant identities. The first one is often called Cauchy's determinant identity and the second one Borchardt's identity. Proofs can be found in [7].

**Proposition 6.5.** Let  $p_j, q_j, j = 1, 2, \dots, n$  be complex numbers such that  $p_j \neq q_k$  for any  $j$  and  $k$ . Then,

$$\det \left( \frac{1}{p_j - q_k} \right)_{j,k=1}^n = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{1 \leq j < k \leq n} (p_j - p_k)(q_j - q_k)}{\prod_{1 \leq j, k \leq n} (p_j - q_k)}$$

and

$$\text{per} \left( \frac{1}{p_j - q_k} \right)_{j,k=1}^n \det \left( \frac{1}{p_j - q_k} \right)_{j,k=1}^n = \det \left( \frac{1}{(p_j - q_k)^2} \right)_{j,k=1}^n.$$

*Proof of Theorem 6.3.* Since  $h_a(a) = 0$ ,  $h'_a(a) = 2\pi S_v(a, a)$  and  $S_v(z, w) = S_0(z + iv/2, w + iv/2)$ , by Cauchy's determinant formula, it is easily seen that

$$\prod_{j=1}^n |\gamma'_n(z_j)|^2 = \{\det(2\pi S_v(z_j, z_k))_{j,k=1}^n\}^2.$$

We observe that

$$\partial_z \partial_{\bar{w}} S_v^{z_1, \dots, z_n}(z_j, z_k) = S_v(z_j, z_k) \gamma'_n(z_j) \overline{\gamma'_n(z_k)}$$

since  $\gamma_n(z_j) = 0$  for any  $j = 1, 2, \dots, n$ . Then, by Borchardt's identity, we have

$$\begin{aligned} \text{per}(\partial_z \partial_{\bar{w}} S_v^{z_1, \dots, z_n}(z_j, z_k)) &= \text{per}(S_v(z_j, z_k))_{j,k=1}^n \prod_{j=1}^n |\gamma'_n(z_j)|^2 \\ &= \det(K_v(z_j, z_k))_{j,k=1}^n \det(\pi S_v(z_j, z_k))_{j,k=1}^n \end{aligned}$$

Hence we obtain

$$\rho_n(z_1, z_2, \dots, z_n) = \det(K_v(z_j, z_k))_{j,k=1}^n$$

from Proposition 6.1. □

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