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On algebraic solutions to Painlevé VI

By

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Abstract

We announce some results which might bring a new insight into the classification of algebraic solutions to the sixth Painlevé equation. They consist of the rationality of parameters, trigonometric Diophantine conditions, and what the author calls the Tetrahedral Theorem regarding the absence of algebraic solutions in certain situations. The method is based on fruitful interactions between the moduli theoretical formulation of Painlevé VI and dynamics on character varieties via the Riemann-Hilbert correspondence.

§ 1. Introduction

All algebraic solutions to the Gauss hypergeometric equation were classified by H.A. Schwarz [29] in 1873. After him this classifiaction has been known as Schwarz's list. On the other hand the sixth Painlevé equation is known as a nonlinear generalization of the Gauss equation. So we are naturally led to the problem of classifying all glgebraic solutions to Painlevé VI. This problem is still open (as of this writing) and there is a vast literature on this theme including [1, 2, 4, 5, 7, 10, 11, 12, 13, 14, 20, 21, 24, 31]. The attempt at solving this problem could be entitled *Towards a nonlinear Schwarz's list* as P. Boalch employs these words as the title of his survey [6], in which the current states of the subject are nicely presented. The aim of this article is to announce some results which might bring a new insight into this subject.

The above-mentioned problem for Painlevé VI is closely related to a problem from topology, that is, to classifying all finite orbits of the mapping class group action on certain character varieties, where the Painlevé-equation side and the character-variety side are connected by the so-called Riemann-Hilbert correspondence. Our philosophy

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is that working on both sides together, going back and forth between them, should be more fruitful than working on only one side of them. The mixture of methods from both sides should go much farther than either side could go by itself. The main results of this article are the rationality of parameters (§5), trigonometric Diophantine conditions (§6), and what the author calls the Tetrahedral Theorem (§10) which is concerned with the absence of algebraic solutions in certain situations.

The contents of this article are based on the following talks by the author: (1) a series of talks at IRMAR, l'Université de Rennes, March, 2008. The author thanks S. Cantat and F. Loray for stimulating discussions; (2) a talk at the Conference on Exact WKB Analysis and Microlocal Analysis in RIMS, Kyoto, May, 2008. This article is a contribution to its Proceedings; (3) a talk at the International Conference "From Painlevé to Okamoto" in The University of Tokyo, June, 2008. A full account of this announcement will be given in [18].

Note. After this article had been submitted to the Editor, a preprint [22] giving a complete classification of algebraic solutions was posted in the e-Print ArXiv by O. Lisovyy and Y. Tykhyy. Their approach is quite straightforward. First, they use trigonometric Diophantine conditions to show that all monodromy data that can lead to finite orbits necessarily belong to an explicitly defined finite set (with two exceptions), and then all possibilities are checked by computers. Hearing of their work, the author puts this note here instead of revising the Introduction to a large extent.

§ 2. Dynamics on Character Varieties

Let X be a real orientable closed surface with a finite number of punctures. By definition a relative $SL_2(\mathbb{C})$-character variety of X is the moduli space of Jordan equivalence classes of representations into $SL_2(\mathbb{C})$ of the fundamental group $\pi_1(X)$ with prescribed local representations around the punctures. Hereafter a relative $SL_2(\mathbb{C})$-character variety is simply referred to as a character variety. It is acted on by the mapping class group of X in a natural manner.

In this article we are interested in the basic case where X is the quadruply-punctured sphere. In this case the character varieties are realized as the four-parameter family of complex affine cubic surfaces $\mathcal{S}(\theta) = \{ x = (x_1, x_2, x_3) \in \mathbb{C}^3_x \,:\, f(x, \theta) = 0 \}$ parametrized by $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4_\theta$, where $f(x, \theta)$ is defined by

$$f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4.$$

The surface $\mathcal{S}(\theta)$ is a $(2, 2, 2)$-surface, that is, the defininig function $f(x, \theta)$ is a quadratic polynomial in each variable x_i $(i = 1, 2, 3)$. Thus the line through a point $x \in \mathcal{S}(\theta)$ parallel to the x_i-axis passes through a unique second point $x' = \sigma_i(x) \in \mathcal{S}(\theta)$. This

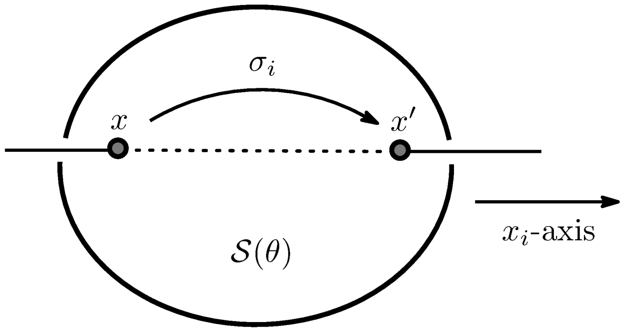

Figure 1. Involutions on the $(2,2,2)$-surface $\mathcal{S}(\theta)$

induces an involutive automorphism $\sigma_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta)$ for each $i = 1, 2, 3$ (see Figure 1). Let G be the group generated by these three involutions. Then it turns out that the generators have no other relations than the trivial ones $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$. Namely,

$$G := \langle \sigma_1, \sigma_2, \sigma_3 \rangle = \langle \sigma_1, \sigma_2, \sigma_3 \,|\, \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \rangle \curvearrowright \mathcal{S}(\theta).$$

Each element $\sigma \in G$ can be written in a unique way as a word $\sigma = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}$ in the alphabet $\{\sigma_1, \sigma_2, \sigma_3\}$ such that the consecutive indices i_ν and $i_{\nu+1}$ are all distinct. Let $G(2)$ denote the subgroup of all even words in G. It is an index-two normal subgroup of G. In the present case the mapping class group action is realized as the group action $G(2) \curvearrowright \mathcal{S}(\theta)$. Now we are interested in the following problem.

Problem 1. Classify all finite orbits of the action $G(2) \curvearrowright \mathcal{S}(\theta)$.

Let $V = \{\theta \in \Theta \,:\, \Delta(\theta) = 0\}$ be the discriminant locus of the family of cubics $\mathcal{S}(\theta)$ parametrized by $\theta \in \Theta$, where $\Delta(\theta)$ is the discriminant of $f(x, \theta)$ as a polynomial of x. For any $\theta \in V$ the surface $\mathcal{S}(\theta)$ has at most four simple singulatities. Let

$$\varphi : \widetilde{\mathcal{S}}(\theta) \to \mathcal{S}(\theta) \tag{2.1}$$

be an algebraic minimal desingularization. Then the action $G \curvearrowright \mathcal{S}(\theta)$ lifts to the smooth surface $\widetilde{\mathcal{S}}(\theta)$ in a unique way and Problem 1 is refined into the following problem.

Problem 2. Classify all finite orbits of the lifted action $G(2) \curvearrowright \widetilde{\mathcal{S}}(\theta)$.

It is easy to see that the singular points of $\mathcal{S}(\theta)$ are exactly the fixed points of the action $G(2) \curvearrowright \mathcal{S}(\theta)$ so that the exceptional set $\mathcal{E}(\theta) \subset \widetilde{\mathcal{S}}(\theta)$ is invariant under the lifted action $G(2) \curvearrowright \widetilde{\mathcal{S}}(\theta)$. Problem 2 is finer than Problem 1 to the extent that Problem 2 demands to classify finite orbits on the exceptional set $\mathcal{E}(\theta)$. But this extra task is not so heavy as will be explained in §4. So one can safely say that the two problems are approximately the same.

§ 3. The Sixth Painlevé Equation

The sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is a Hamiltonian system with a complex time variable $z \in Z := \mathbb{P}^1 - \{0, 1, \infty\}$ and unknown functions $q = q(z)$ and $p = p(z)$,

$$\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q},$$

depending on complex parameters κ in the 4-dimensional affine space

$$\mathcal{K} := \{\, \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5_\kappa \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \,\},$$

where the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z-1)H(\kappa) = (q_0 q_z q_1)p^2 - \{\kappa_1 q_1 q_z + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_z\} p + \kappa_0(\kappa_0 + \kappa_4) q_z$$

with $q_\nu := q - \nu$ for $\nu \in \{0, z, 1\}$. It is known that $\mathrm{P_{VI}}(\kappa)$ has the Painlevé property in Z, that is, any meromorphic solution germ to $\mathrm{P_{VI}}(\kappa)$ at a base point $z \in Z$ admits a global analytic continuation along any path in Z emanating from z as a meromorphic function. In fact, this property is a natural consequence of our solution to the Riemann-Hilbert problem based on a moduli theoretical formulation of the sixth Painlevé equation (see [15, 16]). For the Painlevé equation we are interested in the following problem.

Problem 3. Classify all algebraic solutions to $\mathrm{P_{VI}}(\kappa)$.

For the current states of the problem we refer to the nice survey article [6]. We also consider a closely related problem (Problem 4 below, which turns out to be an equivalent problem). Fix a base point $z \in Z$ and let $\mathcal{M}_z(\kappa)$ be the set of all meromorphic solution germs to $\mathrm{P_{VI}}(\kappa)$ at the point z. Thanks to the Painlevé property, any germ $Q \in \mathcal{M}_z(\kappa)$ can be continued analytically along any loop $\gamma \in \pi_1(Z, z)$ into a second germ $\gamma_* Q \in \mathcal{M}_z(\kappa)$. This defines an automorphism $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ and hence a group action $\pi_1(Z, z) \curvearrowright \mathcal{M}_z(\kappa)$, called the nonlinear monodromy action.

Problem 4. Classify all finite orbits of the action $\pi_1(Z, z) \curvearrowright \mathcal{M}_z(\kappa)$.

Since any algebraic solution to $\mathrm{P_{VI}}(\kappa)$ has only finitely many local branches at the base point z which are permuted by the $\pi_1(Z, z)$-action, there is the natural inclusion:

$$\begin{aligned} &\{\, \text{germs at } z \text{ of algebraic solutions to } \mathrm{P_{VI}}(\kappa) \,\} \\ &\qquad \hookrightarrow \{\, \text{finite } \pi_1(Z, z)\text{-orbits on } \mathcal{M}_z(\kappa) \,\} \end{aligned} \tag{3.1}$$

One may be worried about the difference of the two sets. In fact there is no difference.

Theorem 3.1 ([17]). *The inclusion* (3.1) *is surjective and hence Problems* 3 *and* 4 *are equivalent.*

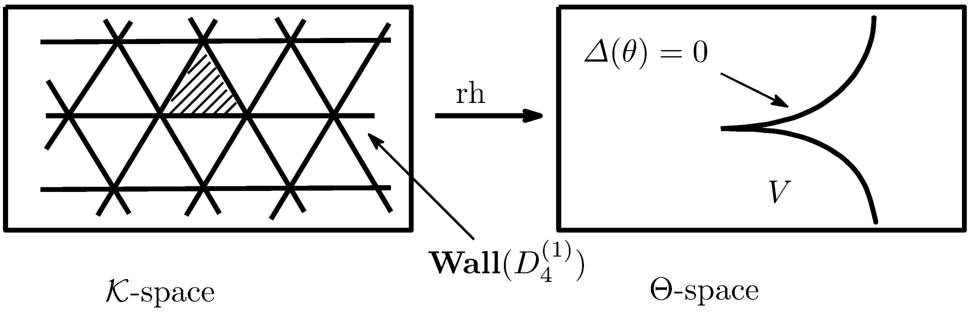

Figure 2. The Riemann-Hilbert correspondence in the parameter level

There is a small gap in an argument of [17], which is to be filled in [18].

§ 4. Riemann-Hilbert correspondence

In order to connect Problem 2 with Problem 3 (or equivalently with Problem 4), we review the Riemann-Hilbert correspondence [15, 16, 17]. It exists in the parameter level and in the moduli level.

Firstly, the parameter space $\mathcal{K}$ is acted on by the affine Weyl group $W(D_4^{(1)})$ of type $D_4^{(1)}$ and the Riemann-Hilbert correspondence in the parameter level is a holomorphic map $\mathrm{rh} : \mathcal{K} \to \Theta$ that is a branched $W(D_4^{(1)})$-covering ramifying along $\mathbf{Wall}(D_4^{(1)})$ and mapping it onto the discriminant locus $V \subset \Theta$ of the family of cubic surfaces, where $\mathbf{Wall}(D_4^{(1)})$ is the union of all reflecting hyperplanes for the reflection group $W(D_4^{(1)})$ (see Figure 2). Secondly, developing a suitable moduli theory [15, 16] allows us to realize the set $\mathcal{M}_z(\kappa)$ as the moduli space of (certain) stable parabolic connections and thereby to provide it with the structure of a smooth quasi-projective rational surface. The Riemann-Hilbert correspondence (in the moduli level),

$$\mathrm{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \mathcal{S}(\theta), \quad Q \mapsto \rho, \qquad \text{with} \quad \theta = \mathrm{rh}(\kappa), \tag{4.1}$$

is defined to be the holomorphic map sending each connection Q to its monodromy representation ρ up to Jordan equivalence. A basic fact for the map (4.1) is the following.

Theorem 4.1 ([15, 16])**.** *The Riemann-Hilbert correspondence* (4.1) *is a proper surjection that yields an analytic minimal resolution of simple singularities.*

By the minimality of the resolution, the Riemann-Hilbert correspondence (4.1) uniquely lifts to a biholomorphism $\widetilde{\mathrm{RH}}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \widetilde{\mathcal{S}}(\theta)$ such that the diagram

$$\begin{tikzcd} \mathcal{M}_z(\kappa) \arrow[r, "\widetilde{\mathrm{RH}}_{z,\kappa}"] \arrow[d, equal] & \widetilde{\mathcal{S}}(\theta) \arrow[d, "\varphi"] \\ \mathcal{M}_z(\kappa) \arrow[r, "\mathrm{RH}_{z,\kappa}"] & \mathcal{S}(\theta) \end{tikzcd}$$

is commutative. The lifted Riemann-Hilbert correspondence $\widetilde{\mathrm{RH}}_{z,\kappa}$ gives a (strict) conjugacy between the nonlinear monodromy action $\pi_1(Z,z) \curvearrowright \mathcal{M}_z(\kappa)$ and the mapping class group action $G(2) \curvearrowright \widetilde{\mathcal{S}}(\theta)$. In these circumstances the exceptional set $\mathcal{E}_z(\kappa) \subset \mathcal{M}_z(\kappa)$ of the resolution (4.1) just corresponds to the exceptional set $\mathcal{E}(\theta) \subset \widetilde{\mathcal{S}}(\theta)$ of the resolution (2.1). We remark that $\mathcal{E}_z(\kappa)$ parametrizes the so-called Riccati solutions to $\mathrm{P_{VI}}(\kappa)$, namely, those solutions which can be written in terms of the Riccati equations associated with Gauss hypergeometric equations (see [15]).

The lifted Riemann-Hilbert correspondence and Theorem 3.1 yield the diagram:

$$\begin{array}{ccc} \{\text{ germs at } z \text{ of algebraic solutions to } \mathrm{P_{VI}}(\kappa)\,\} & = & \{\text{ finite } \pi_1(Z,z)\text{-orbits on } \mathcal{M}_z(\kappa)\,\} \\ & & \text{bijection} \updownarrow \widetilde{\mathrm{RH}}_{z,\kappa} \\ & & \{\text{ finite } G(2)\text{-orbits on } \widetilde{\mathcal{S}}(\theta)\,\} \end{array}$$

In summary, Problem 1 is almost equivalent to Problem 2, while Problems 2, 3 and 4 are all equivalent. The difference of Problem 2 from Problem 1 amounts to classifying all Riccati algebraic solutions to $\mathrm{P_{VI}}(\kappa)$, which in turn can be reduced to classifying all algebraic solutions to the Gauss hypergeometric equation, the classical problem settled by H.A. Schwarz [29].

§ 5. Rationality of Parameters

An algebraic solution to $\mathrm{P_{VI}}(\kappa)$ is said to be of degree d if it has exactly d local branches (germs) at a base point $z \in Z$. On the other hand a finite $G(2)$-orbit in $\widetilde{\mathcal{S}}(\theta)$ is said to be of degree d if it has exactly d elements. Note that these two concepts of degree are consistent under the lifted Riemann-Hilbert correspondence $\widetilde{\mathrm{RH}}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \widetilde{\mathcal{S}}(\theta)$ with $\theta = \mathrm{rh}(\kappa)$. Naturally one may guess that those parameters $\kappa \in \mathcal{K}$ for which $\mathrm{P_{VI}}(\kappa)$ admits at least one algebraic solutions of degree $d \geq d_0$ should have a very "sparse" distribution, for some (perhaps reasonably large) integer d_0. Actually, with the choice of $d_0 = 7$, this guess is true in the following sense.

Theorem 5.1. *We have the following rationality conditions.*

(1) *If* $\mathrm{P_{VI}}(\kappa)$ *admits an algebraic solution of degree* $d \geq 7$*, then* κ_0*,* κ_1*,* κ_2*,* κ_3 *and* κ_4 *must be rational numbers.*

(2) *If* $\mathrm{P_{VI}}(\kappa)$ *admits an algebraic solution of degree* $d \geq 1$ *without univalent local branches at any fixed singular point* $z = 0,\ 1,\ \infty$*, then* $d\kappa_0$*,* $d\kappa_1$*,* $d\kappa_2$*,* $d\kappa_3$ *and* $d\kappa_4$ *must be integers.*

Since $\mathbb{Q} \subset \mathbb{R}$, assertion (1) of Theorem 5.1 allows us to concentrate our attention on the real part $\mathcal{K}_{\mathbb{R}}$ of the complex affine space $\mathcal{K}$, as far as algebraic solutions of degree $d \geq 7$ are concerned.

Example 5.2. In order to illustrate assertion (2) of Theorem 5.1, we look at the "Klein solution" constructed by Boalch [4] based on the Klein complex reflection group of order 336 in $SL_3(\mathbb{C})$,

$$\begin{cases} z = \dfrac{(7s^2-7s+4)^2}{s^3(4s^2-7s+7)^2}, \\[2ex] q = \dfrac{(s+1)(7s^2-7s+4)}{2s(s^2-s+1)(4s^2-7s+7)}, \\[2ex] p = -\dfrac{2s(s+1)(s-2)(2s-1)(s^2-s+1)(4s^2-7s+7)}{21(s-1)^2(4s^2-s+4)(7s^2-7s+4)}, \end{cases}$$

for which $d = 7$ and $\kappa = (1/7, 1/7, 1/7, 1/7, 2/7)$. This solution has ramification indices 3, 2, 2 (a partition of $d = 7$) at each of the three fixed singular points $z = 0, 1, \infty$. Namely, it has one local branch of valency 3 and two local branches of valency 2 (and hence no univalent local branch) at each fixed singular point. Observe that $d\kappa_i$ $(i = 0, 1, 2, 3, 4)$ are integers.

Two remarks are in order regarding Theorem 5.1.

Remark 5.3. For $i = 1$, 2, item (i) corresponds to assertion (i) of Theorem 5.1.

(1) One may ask why condition $d \geq 7$ is imposed and how the assertion is derived. A brief answer to these questions will be given in §7 (especially in Lemma 7.1 and the discussions thereafter). One may also ask what happens if $d \leq 6$. It is known that there exist three exceptional classes of non-Riccati algebraic solutions to $\mathrm{P_{VI}}(\kappa)$ for which κ depends continuously on some complex parameters. All of them are simple solutions of degree $d \leq 4$. Except for these solutions, it seems that assertion (1) remains true for all non-Riccati algebraic solutions of degree $d \leq 6$, although a further check is needed to swear its truth (see also Remark 6.2).

(2) Assertion (2) is not necessarily true if the solution under consideration has a univalent local branch at a fixed singular point. This can be seen by the "generic" icosahedral solution of Boalch [5],

$$\begin{cases} z = \dfrac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2}, \\[2ex] q = \dfrac{3s(3s-4)(s^2+1)(3s^2-2s+4)}{2(2s-1)^2(9s^2+4)}, \\[2ex] p = -\dfrac{(2s-1)^2(9s^2+4)(9s^2+3s+10)}{90s(3s-4)(s^2+1)(3s^2-3s+2)(3s^2+2s+2)}, \end{cases}$$

for which $d = 12$ and $\kappa = (1/5, 11/60, 17/60, 7/60, 1/60)$. This solution has ramification indices (partitions of $d = 12$): 5, 3, 2, 2 at $z = 0, \infty$; and 3, 3, 2, 2, 1, 1 at $z = 1$. So it has two univalent local branches at $z = 1$. Observe that $d\kappa_i$ ($i = 0, 1, 2, 3, 4$) are *not* integers. We remark that assertion (2) is valid for any $d \geq 1$ (not only for $d \geq 7$).

§ 6. Trigonometric Diophantine Conditions

The rationality result in Theorem 5.1 is stated in the Painlevé-equation side. Switching to the character-variety side, we present another result showing that the coordinates of any finite orbit of degree $d \geq 7$ are tied down by very tight conditions, namely, by certain trigonometric Diophantine conditions. In this section we work on $\mathcal{S}(\theta)$ downstairs rather than $\widetilde{\mathcal{S}}(\theta)$ upstairs so that the degree means the number of points in the finite $G(2)$-orbit on $\mathcal{S}(\theta)$ under consideration.

Theorem 6.1. *Given any $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{C}^4$, let $\mathcal{O} \subset \mathcal{S}(\theta)$ be a (possibly infinite) $G(2)$-orbit of degree $d \geq 7$?D Then the orbit $\mathcal{O}$ is finite if and only if*

$$\mathcal{O} \subset \mathcal{S}(\theta) \cap (2\cos\pi\mathbb{Q})^3. \tag{6.1}$$

If this is the case then θ_1, θ_2, θ_3 and θ_4 must be real cyclotomic integers such that

$$-8 < \theta_1, \theta_2, \theta_3 < 8, \qquad -28 < \theta_4 < 28.$$

As a corollary, if $\mathcal{O} \subset \mathcal{S}(\theta)$ is a finite orbit of degree $d \geq 7$, then θ must be real and the orbit $\mathcal{O}$ must lie in the real part $\mathcal{S}(\theta)_{\mathbb{R}}$ of the complex surface $\mathcal{S}(\theta)$. Thus it is also important to investigate the real dynamics on the real cubic surface $\mathcal{S}(\theta)_{\mathbb{R}}$ with $\theta \in \mathbb{R}^4$.

Remark 6.2. All finite orbits of degree $d \leq 4$ has been classified by Cantat and Loray [9]. During the author's visit to Rennes in March 2008, having heard of the author's results for degree $d \geq 7$, F. Loray carried out computer experiments to determine all finite orbits of degrees 5 and 6. These orbits correspond to some algebraic solutions by Theorem 3.1 and actually it seems that they correspond to already known algebraic solutions (a further careful check is needed).

Remark 6.3. It follows from (6.1) in Theorem 6.1 that enumerating all finite orbits of degree $d \geq 7$ on our character varieties can be embedded into the problem of solving the trigonometric Diophantine equation

$$\sum_{k=1}^{8} \cos\pi\xi_k = 0, \qquad \xi = (\xi_1, \dots, \xi_8) \in \mathbb{Q}^8. \tag{6.2}$$

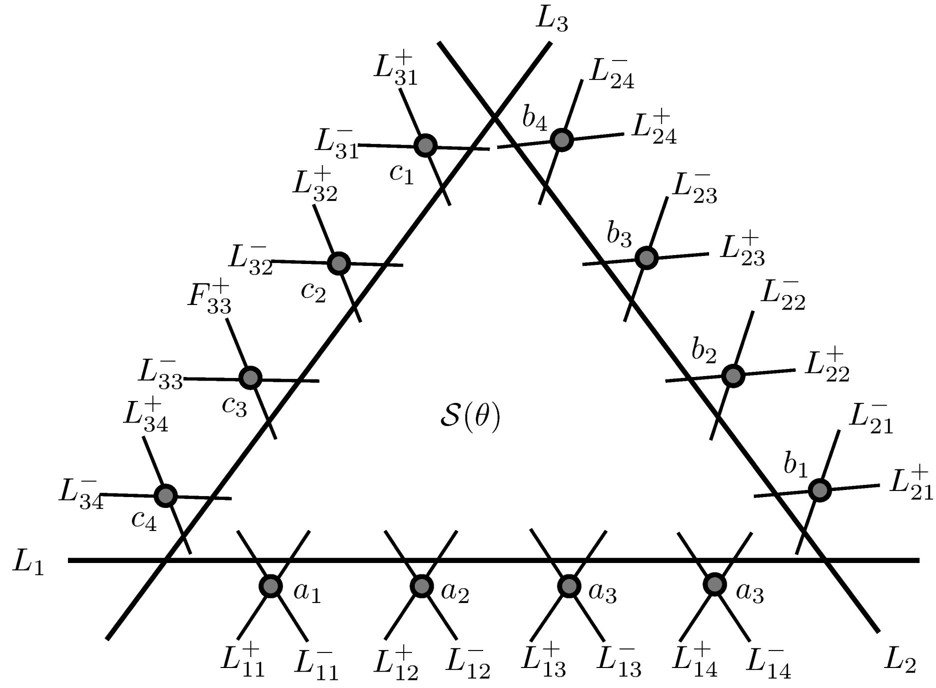

Figure 3. The 27 lines viewed from the tritangent lines at infinity

Similar but more tractable trigonometric Diophantine equations have appeared in many places (see e.g. [27, 28] and the references therein). Although it is harder than those, equation (6.2) still seems to be a tractable problem in computer-assisted mathematics. However, even if one succeeds in enumerating all solutions to equation (6.2), there remains the extra job of identifying which solutions are relevant to our original problem. In any case the author prefers more insightful geometric approaches.

The proof of Theorem 6.1 relies largely on the direct manipulations of the dynamics on the character variety, but it also depends heavily on Theorem 5.1, which in turn is obtained by the combination of some main discussions on the Painlevé-equation side and some auxiliary discussions on the character-variety side. Behind this complicated circle of ideas, there exists the geometry of cubic surfaces, especially the configuration of lines on a cubic surface. In the next section we give a brief account of this, leaving a full explanation in [18].

§ 7. Lines on a Cubic Surface

Compactify the affine cubic surface $\mathcal{S}(\theta)$ by the standard embedding $\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}}(\theta) \subset \mathbb{P}^3$. Then $\overline{\mathcal{S}}(\theta)$ is obtained from $\mathcal{S}(\theta)$ by adding the tritangent lines at infinity, $L = L_1 \cup L_2 \cup L_3$, as in Figure 3. For simplicity we assume that $\theta = \mathrm{rh}(\kappa)$ with $\kappa \in \mathcal{K} - \mathbf{Wall}(D_4^{(1)})$. Then the projective cubic surface $\overline{\mathcal{S}}(\theta)$ is smooth and it contains twenty-seven lines, whose configuration is depicted in Figure 3. The lines at infinity, L_1, L_2, L_3, are three among them. The remaining twenty-four lines are divided into three groups, each consisting of eight lines, according to the three lines at infinity. Namely, for each $i = 1, 2, 3$, the line L_i meets exactly eight lines, say, L_{ij}^{ε} as in Figure 3, where $j = 1, 2, 3, 4$ and $\varepsilon = \pm$. This group of eight lines are divided into four intersecting pairs $\{L_{ij}^{+}, L_{ij}^{-}\}_{j=1}^{4}$. Any other pair from the same group has no intersections.

Assume that a finite $G(2)$-orbit $\mathcal{O} \subset \mathcal{S}(\theta)$ be given. To it we can associate an "ON/OFF" data $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \{0,1\}^{12}$ as follows. As for $\boldsymbol{a} = (a_1, a_2, a_3, a_4) \in \{0,1\}^4$, put

$$a_j := \begin{cases} 1 \text{ (ON)}, & \text{if } \mathcal{O} \text{ passes through the intersection point } L_{1j}^{+} \cap L_{1j}^{-}, \\ 0 \text{ (OFF)}, & \text{otherwise}, \end{cases}$$

for $j = 1, 2, 3, 4$. In a similar manner we can define $\boldsymbol{b} = (b_1, b_2, b_3, b_4) \in \{0,1\}^4$ and $\boldsymbol{c} = (c_1, c_2, c_3, c_4) \in \{0,1\}^4$ by replacing $L_{1j}^{\pm}$ with $L_{2j}^{\pm}$ and $L_{3j}^{\pm}$ repectively. Then certain arguments that are too involved to be included here lead to the matrix

$$M(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) := \begin{pmatrix} d_1 & a_3 - a_4 & c_1 - c_2 & b_1 - b_2 \\ a_3 - a_4 & d_2 & b_3 - b_4 & c_3 - c_4 \\ c_1 - c_2 & b_3 - b_4 & d_3 & a_1 - a_2 \\ b_1 - b_2 & c_3 - c_4 & a_1 - a_2 & d_4 \end{pmatrix},$$

where d_i $(i = 1, 2, 3, 4)$ are nonnegative integers defined by

$$\begin{cases} d_1 := a_3 + a_4 + b_1 + b_2 + c_1 + c_2, \\ d_2 := a_3 + a_4 + b_3 + b_4 + c_3 + c_4, \\ d_3 := a_1 + a_2 + b_3 + b_4 + c_1 + c_2, \\ d_4 := a_1 + a_2 + b_1 + b_2 + c_3 + c_4. \end{cases}$$

It turns out that the column vector $\boldsymbol{\kappa} = {}^t(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ must satisfy a linear equation

$$[dI_4 - M(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})]\, \boldsymbol{\kappa} = \text{a certain integer vector}, \tag{7.1}$$

where d is the order of the orbit $\mathcal{O}$ and I_4 is the identity matrix of rank 4.

Lemma 7.1. *For any $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \{0,1\}^{12}$, $M(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ has no eigenvalues ≥ 7.*

This is verified by a computer check exhausting all $2^{12} = 4096$ possibilities for the data $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. It is also observed that actually some of $0, 1, 2, 3, 4, 5, 6$ are eigenvalues of the matrix $M(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. The author is indebted to A. Maruyama and T. Uehara for the job of these verifications. We are now able to give the following.

Sketch of the proof of Theorem 5.1. If $d \geq 7$ then Lemma 7.1 implies that $dI_4 - M(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is invertible in rational numbers since it is an integer matrix, so that

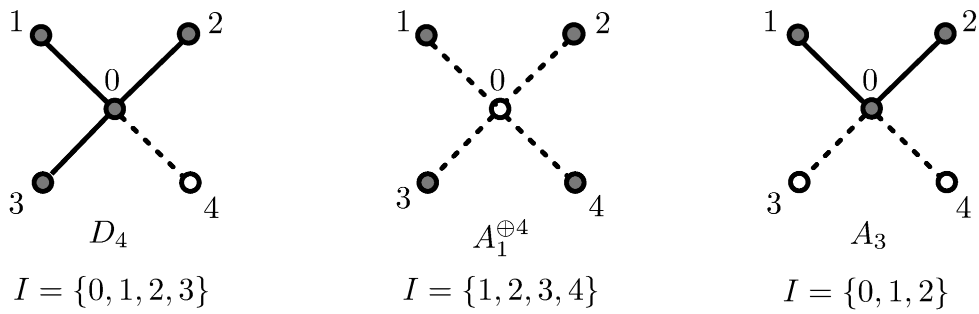

Figure 4. Some $D_4^{(1)}$-strata and their abstract Dynkin types

equation (7.1) can be settled to conclude that $\boldsymbol{\kappa}$ is a vector with rational entries. This proves assertion (1). Let us proceed to assertion (2). Put $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. It is shown in [17] that for each $i = 1, 2, 3$ the line L_i at infinity is attached to the fixed singular point z_i and the univalent solution germs at z_i are in one-to-one correspondence with those intersection points $L_{ij}^+ \cap L_{ij}^-$, $j \in \{1, 2, 3, 4\}$, which lie in the affine part $\mathcal{S}(\theta)$ of $\overline{\mathcal{S}}(\theta)$. Thus if the algebraic solution under consideration has no univalent local branches at any fixed singular point, then we must have $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ and $M(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = O$. Then equation (7.1) implies that $d\boldsymbol{\kappa}$ must be an integer vector, from which assertion (2) readily follows. Note that this argument is valid for an arbitrary integer $d \geq 1$. □

This section ends with three remarks. Firstly, even if $d \leq 6$ some useful information about $\boldsymbol{\kappa}$ can be extracted from equation (7.1). Secondly, if $\kappa \in \mathbf{Wall}(D_4^{(1)})$ then the line configuration is degenerate and the situation becomes more complicated than the case $\kappa \in \mathcal{K} - \mathbf{Wall}(D_4^{(1)})$ discussed above, but basically a similar argument is feasible. Finally we refer to the original paper [18] for the most important thing: why and how equation (7.1) occurs.

§ 8. Stratifications of Parameters

We define a stratification of $\mathcal{K}$ in terms the proper subdiagrams of the Dynkin diagram $D_4^{(1)}$. To this end we index the nodes of the Dynkin diagram $D_4^{(1)}$ by the numbers $0, 1, 2, 3, 4$, where 0 represents the central node (see Figure 4). Let $\mathcal{I}$ be the set of all proper subsets of $\{0, 1, 2, 3, 4\}$ including the empty set $\emptyset$. For each $I \in \mathcal{I}$, put

$$\begin{cases} \overline{\mathcal{K}}_I = \text{the } W(D_4^{(1)})\text{-translates of the subset } \{\, \kappa \in \mathcal{K} \,:\, \kappa_i = 0 \ (i \in I) \,\}, \\ \mathcal{K}_I = \overline{\mathcal{K}}_I - \displaystyle\bigcup_{|J| = |I|+1} \overline{\mathcal{K}}_J, \\ D_I = \text{the Dynkin subdiagram of } D_4^{(1)} \text{ that has nodes } \bullet \text{ exactly in } I. \end{cases} \tag{8.1}$$

It turns out that for any pair $(I, I') \in \mathcal{I} \times \mathcal{I}$, either $\mathcal{K}_I = \mathcal{K}_{I'}$ or $\mathcal{K}_I \cap \mathcal{K}_{I'} = \emptyset$ holds so that the partition $\{\mathcal{K}_I\}_{I \in \mathcal{I}}$ defines a stratification of $\mathcal{K}$, called the $D_4^{(1)}$-*stratification*.

$$\begin{array}{ccccccccc} \emptyset & \longrightarrow & A_1 & \longrightarrow & A_1^{\oplus 2} & \longrightarrow & A_1^{\oplus 3} & \longrightarrow & A_1^{\oplus 4} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A_2 & \longrightarrow & A_3 & \longrightarrow & D_4 & & \end{array}$$

Figure 5. Adjacency relations among the $F_4^{(1)}$-strata

For $I = \emptyset$ one has the big open stratum $\mathcal{K}_\emptyset = \mathcal{K} - \mathbf{Wall}(D_4^{(1)})$ and other examples of strata are given in Figure 4.

The automorphism group of Dynkin diagram $D_4^{(1)}$ is the symmetric group S_4 of degree 4 acting by permuting the nodes 1, 2, 3, 4 while fixing the central node 0. The group $W(D_4^{(1)})$ extended by S_4 is the affine Weyl group $W(F_4^{(1)})$ of type $F_4^{(1)}$. A coaser stratification of $\mathcal{K}$ can be defined in the same way as in the case of $D_4^{(1)}$-stratification by replacing the group $W(D_4^{(1)})$ with $W(F_4^{(1)})$ in (8.1). It is called the $F_4^{(1)}$-*stratification*. Note that the $F_4^{(1)}$-stratification encodes only the *abstract* Dynkin type of the subdiagram D_I, while the $D_4^{(1)}$-stratification encodes not only the abstract Dynkin type of D_I but also the inclusion patern $D_I \hookrightarrow D_4^{(1)}$, a kind of marking. Thus the $F_4^{(1)}$-strata can be indexed by the abstract Dynkin subdiagrams of $D_4^{(1)}$. The adjacency relations among them are given in Figure 5, where $* \to **$ indicates that the stratum $**$ is in the closure of $*$.

§ 9. On Various Strata

Theorems 5.1 and 6.1 are results that can be stated without refering to the stratification. Besides them, there are such results that differ stratum by stratum. A factor that might cause such a difference is the topology (or perhaps the shape) of the real character variety $\mathcal{S}(\theta)_{\mathbb{R}}$ (see [3]). On one hand the topology changes as the stratum varies and on the other hand the dynamics of the mapping class group action on $\mathcal{S}(\theta)_{\mathbb{R}}$ is a *priori* defined by the space $\mathcal{S}(\theta)_{\mathbb{R}}$ itself, so that the topology or the shape of the space should have a strong influence on the dynamics.

We focus our attention on the $F_4^{(1)}$-strata of positive codimensions. A careful inspection shows that it is natural to divide those strata into two sequences (Figure 5):

$$\text{(S1)} \quad A_1^{\oplus 2} \to A_1^{\oplus 3} \to A_1^{\oplus 4}, \qquad \text{(S2)} \quad A_1 \to A_2 \to A_3 \to D_4.$$

In this section we are concerned with the strata belonging to the former sequence (S1).

Example 9.1 (Stratum of type $A_1^{\oplus 4}$). This is the locus where the classically well-known Picard solutions exist (see [25])?D The corresponding character variety $\mathcal{S}(\theta)$

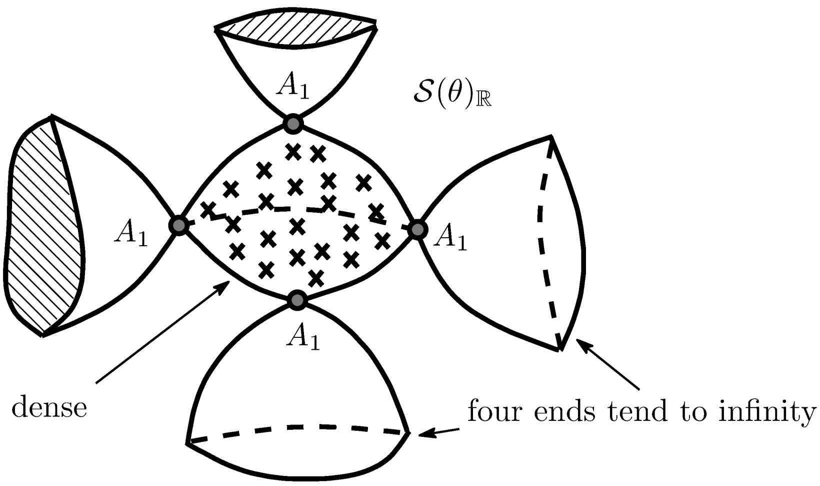

Figure 6. Real Cayley cubic $\mathcal{S}(\theta)_{\mathbb{R}}$ with four A_1-singularities

is the Cayley cubic, with parameters $\theta = (0, 0, 0, -4)$. The Picard solutions can be settled by quadrature in terms of the Legendre family of elliptic curves. However the way in which they are integrated is irreducible in the sense of Nishioka [26] and Umemura [30], but reducible in the sense of Casale [8] and Malgrange [23] (see Cantat and Loray [9])?D This world is amenable to torus structures in two ways. Firstly an elliptic curve is a (real) torus and secondly the Cayley cubic enjoys a (complex) orbifold torus structure?C $\mathcal{S}(\theta) \cong (\mathbb{C}^\times)^2/(\text{an involution})$, where the four A_1-singularities (all real) just come from the four fixed points of the involution. On this stratum there are countably many algebraic solutions, which correspond to the finite-order points of elliptic curves?D The finite orbits on the Cayley cubic are dense in the unique bounded connected component of the real Cayley cubic $\mathcal{S}(\theta)_{\mathbb{R}}$ with the four singular points removed (see Figure 6).

Example 9.2 (Stratum of type $A_1^{\oplus 3}$)**.** This is the locus discussed by Dubrovin and Mazzocco [12], although they made use of a different parametrization of the character variety. On this stratum they showed that there are exactly five algebraic solutions up to some equivalence.

Example 9.3 (Stratum of type $A_1^{\oplus 2}$)**.** This stratum is not well understood yet. We content ourselves with giving an example, the orbit in Figure 7. It is a finite G-orbit (also a $G(2)$-orbit) of degree 6 with parameters $\theta = (2\sqrt{2}, 2\sqrt{2}, 3, 4) \in \Theta$, which is the rh-image of $\kappa = (1/4, 0, 0, 1/12, 5/12) \in \mathcal{K}$, certainly a point of type $A_1^{\oplus 2}$.

§ 10. Tetrahedral Theorem

The strata belonging to the sequence (S2) admit a unified treatment.

$$\begin{tikzcd}
 & & \sigma_1,\ \sigma_2 & & \\
 & & \circlearrowleft & & \\
 & & (\sqrt{2},\sqrt{2},0) \arrow[d, leftrightarrow, "\sigma_3"'] & & \\
(0,\sqrt{2},2) \arrow[r, leftrightarrow, "\sigma_3"] & (0,\sqrt{2},1) \arrow[r, leftrightarrow, "\sigma_1"] & (\sqrt{2},\sqrt{2},1) \arrow[r, leftrightarrow, "\sigma_2"] & (\sqrt{2},0,1) \arrow[r, leftrightarrow, "\sigma_3"] & (\sqrt{2},0,2) \\
\circlearrowleft & \circlearrowleft & & \circlearrowleft & \circlearrowleft \\
\sigma_1,\ \sigma_2 & \sigma_2 & & \sigma_1 & \sigma_1,\ \sigma_2
\end{tikzcd}$$

Figure 7. A finite orbit of degree 6 on the stratum of type $A_1^{\oplus 2}$

$D_4^{(1)}$-strata along sequence (S2)	skeletons of tetrahedron
one stratum of abstract type A_1	one 3-cell
$\downarrow$	$\downarrow$
four strata of abstract type A_2	four faces
$\downarrow$	$\downarrow$
six strata of abstract type A_3	six edges
$\downarrow$	$\downarrow$
four strata of abstract type D_4	four vertices

Table 1. A parallelism in adjacency relations

Theorem 10.1. *On any $F_4^{(1)}$-stratum belonging to the sequence* (S2)*, there is no non-Riccati algebraic solutions of degree $d \geq 7$ without univalent local branches at any fixed singular point.*

This theorem can be used to classify all algebraic solutions on the strata belonging to the sequence (S2). We may refer to Theorem 10.1 as the *Tetrahedral Theorem* for the following reasons.

Remark 10.2 (Parallelism)**.** There is a parallelism as in Table 1 between the adjacency relations for the $D_4^{(1)}$-strata along the sequence (S2) and those for the skeletons of the (regular) tetrahedron. This parallelism is not by chance. Behind it there exists an interesting story starting with the algebraic geometry of Painlevé VI and ending up with some elementary geometry of a regular tetrahedron of edge length $\sqrt{2}$. Indeed, in the course of establishing Theorem 10.1 we come across the regular tetrahedron in Figure 8 (right), which lies in the 3-dimensional space with coordinates $(m_0/d, m_1/d, m_\infty/d)$,

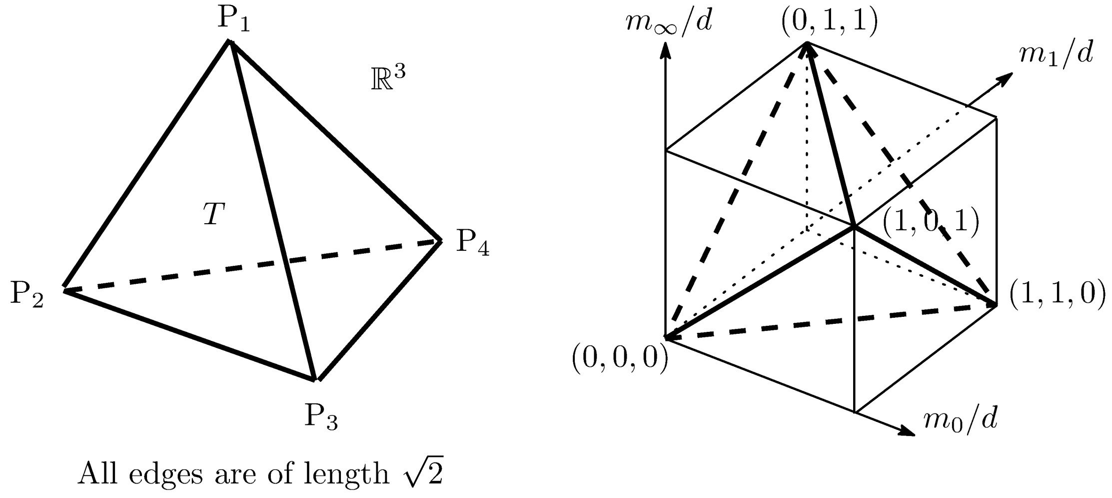

Figure 8. Tetrahedron for the Tetrahedral Theorem

where d is the degree of the algebraic solution under consideration and (m_0, m_1, m_∞) is a triplet of positive integers encoding certain information of how the algebraic solution branches at the fixed singular points $z = 0, 1, \infty$. A full account can be found in [18].

We explain what kind of elementary geometry comes up. Let $T = \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4 \subset \mathbb{R}^3$ be a regular tetrahedron with edge length $\sqrt{2}$ as in Figure 8 (left); $C = \mathrm{QP}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4 \subset \mathbb{R}^4$ the cone over the base T with side lengths $\overline{\mathrm{QP}}_i = r_i$ for $i = 1, 2, 3, 4$, as in Figure 9; and let R be the orthogonal projection of the vertex Q down to the 3-space $\mathbb{R}^3$ that contains the tetrahedron T. Moreover let $\overrightarrow{\mathrm{R}}$ and $\overrightarrow{\mathrm{P}_i}$ denote the position vectors of the points R and P_i respectively. Write

$$\overrightarrow{\mathrm{R}} = \alpha_1 \overrightarrow{\mathrm{P}_1} + \alpha_2 \overrightarrow{\mathrm{P}_2} + \alpha_3 \overrightarrow{\mathrm{P}_3} + \alpha_4 \overrightarrow{\mathrm{P}_4},$$

in terms of the barycentric coordinates $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. In the Painlevé situation, T is the tetrahedron of Figure 8 (right) and the vertices P_i $(i = 1, 2, 3, 4)$ are just those of the latter tetrahedron. A basic fact we need is the following lemma.

Lemma 10.3. *If the side lengths r_i $(i = 1, 2, 3, 4)$ are chosen as*

$$\begin{cases} r_1^2 = (\kappa_1 - 1)^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2, \\ r_2^2 = \kappa_1^2 + (\kappa_2 - 1)^2 + \kappa_3^2 + \kappa_4^2, \\ r_3^2 = \kappa_1^2 + \kappa_2^2 + (\kappa_3 - 1)^2 + \kappa_4^2, \\ r_4^2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2 + (\kappa_4 - 1)^2, \end{cases} \tag{10.1}$$

with $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathcal{K}_{\mathbb{R}}$, then

$$\overline{\mathrm{QR}}^2 = \kappa_0^2, \qquad \alpha_i = \kappa_i + \frac{\kappa_0}{2} \qquad (i = 1, 2, 3, 4). \tag{10.2}$$

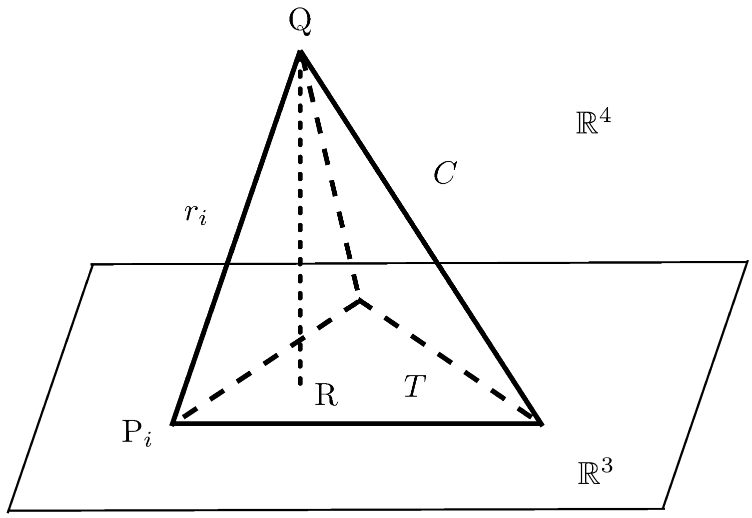

Figure 9. 4-dimensional cone C over the tetrahedron T

It is difficult to explain in short words why the choice (10.1) is natural in our situation and we refer to [18] for a detailed explanation. Anyway, in the course of establishing Theorem 10.1 we encounter a sort of *territory problem*, where the territory of the vertex P_i is the 3-dimensional open ball $B_i := B(\mathrm{P}_i, r_i)$ of radius r_i with center at the point P_i. To explain what this problem is all about, we begin by stating a key observation in the following lemma.

Lemma 10.4. *If* $\mathrm{P_{VI}}(\kappa)$ *with* $\kappa \in \mathcal{K}_{\mathbb{R}}$ *admits a non-Riccati algebraic solution without univalent local branches at any fixed singular point, then the balls* B_i $(i = 1, 2, 3, 4)$ *must have at least one points in common.*

As the contraposition of this lemma, if the four balls have no points in common then there is no algebraic solution with the prescribed property. Now a natural question is when they have points in common and when not. Let us restrict our attention to the case where the point R lies in the interior of T, that is, where the barycentric coordinates $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfy the inequalities

$$\alpha_i > 0 \qquad (i = 1, 2, 3, 4). \tag{10.3}$$

In this case, if the balls B_i $(i = 1, 2, 3, 4)$ have at least one points in common, then R must be such a point in common. With this fact we are able to give the following.

Sketch of the proof of Theorem 10.1. The proof is by contradiction. Assume that $\mathrm{P_{VI}}(\kappa)$ has a non-Riccati algebraic solution of degree $d \geq 7$ without univalent local branches at any fixed singular point. Then we must have $\kappa \in \mathcal{K}_{\mathbb{R}}$ from Theorem 5.1.

After applying a suitable Bäcklund transformation we may assume that κ lies in the (closed) fundamental $W(D_4^{(1)})$-alcove $\{\kappa \in \mathcal{K}_{\mathbb{R}} : \kappa_i \ge 0 \ (i = 0,1,2,3,4)\}$. Now assume that κ lies on the stratum of type A_1. Then there is a unique index $i_0 \in \{0,1,2,3,4\}$ such that $\kappa_{i_0} = 0$ and $\kappa_i > 0$ for the remaining indices i. After applying a further Bäcklund transformation we may assume that $i_0 = 0$, namely, that $\kappa_0 = 0$ and $\kappa_i > 0$ for $i = 1,2,3,4$. So it follows from formula (10.2) that

$$\overline{\mathrm{QR}} = 0, \qquad \alpha_i = \kappa_i > 0 \qquad (i = 1,2,3,4). \tag{10.4}$$

The former condition in (10.4) means that $\mathrm{R} = \mathrm{Q}$ and hence $\overline{\mathrm{RP}}_i = \overline{\mathrm{QP}}_i = r_i$ so that R is a point of the boundary sphere ∂B_i. Since B_i is an open ball, R does not belong to B_i. On the other hand the latter condition in (10.4) means that condition (10.3) is satisfied so that R must belong to B_i, a contradiction. Similar arguments are feasible on the other strata of the sequence (S2). □

§ 11. The Big Open

On the big open $\mathcal{K}_\emptyset = \mathcal{K} - \mathbf{Wall}(D_4^{(1)})$ we are still distant from the complete classification, but we are already able to confine all finite orbits into a rather thin subset of the real character variety $\mathcal{S}(\theta)_{\mathbb{R}}$ (see [18]). In dealing with this stratum it is necessary to distinguish the two subsets $\mathbf{Wall}(D_4^{(1)})$ and $\mathbf{Wall}(F_4^{(1)})$ of the parameter space $\mathcal{K}$, where the former is the union of all reflecting hyperplanes for the reflection group $W(D_4^{(1)})$ and the latter is its counterpart for the group $W(F_4^{(1)})$. Note that there is the strict inclusion $\mathbf{Wall}(D_4^{(1)}) \subset \mathbf{Wall}(F_4^{(1)})$. In the parameter level almost all algebraic solutions on this stratum seem to exist on the set $\mathbf{Wall}(F_4^{(1)}) - \mathbf{Wall}(D_4^{(1)})$. In fact, Boalch's "generic" icosahedral solution [5] (see also item (2) of Remark 5.3) is the only instance outside $\mathbf{Wall}(F_4^{(1)})$ known so far (as of September 10, 2008).

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