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Isolated periodic solutions to Painlevé VI equation

By

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Abstract

In [5] we solved the problem of counting the number of isolated periodic solutions to the sixth Painlevé equation along a given loop for the generic parameters. In this article we show how to settle it for the nongeneric parameters, using the general theory of area-preserving surface dynamics developed in [6], which is applied to the birational map on a desingularized cubic surface obtained from the monodromy map of th Painlevé equation through the Riemann-Hilbert correspondence.

§ 1. Introduction

This article aims to outline a solution to the problem of counting the number of isolated periodic solutions to the sixth Painlevé equation along a given loop. In [5] we were able to solve this problem for the generic parameters by using the classical Lefschetz fixed point formula. However, it was left open for the nongeneric parameters because there may be periodic solutions parametrized by a curve and this fact prevents us from applying the classical Lefschetz formula. Even the Atiyah-Bott formula, which allows the presence of periodic curves, does not work in the current situation because the monodromy map of the Painlevé equation is area-perserving. Against this background, S. Saito's fixed point formula [7] is very appropriate for our purpose. Actually, based on his formula, we have developed a general theory of area-preserving surface dynamics in [6]. Now our problem can be settled completely by applying this theory to the birational map on a desingularized cubic surface obtained from the monodromy map of the Painlevé equation through the Riemann-Hilbert correspondence.

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singularities	$t_1 = 0$	$t_2 = 1$	$t_3 = z$	$t_4 = \infty$
first exponent	$-\lambda_1$	$-\lambda_2$	$-\lambda_3$	$-\lambda_4$
second exponent	λ_1	λ_2	λ_3	$\lambda_4 - 1$
difference	κ_1	κ_2	κ_3	κ_4

Table 1. Riemann scheme: κ_i is the difference of the second exponent from the first.

§ 2. Main Result

The sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is the following Hamiltonian system of differential equations with an independent variable $z \in Z := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and unknown functions $(q, p) = (q(z), p(z))$:

$$\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q},$$

where $\kappa \in \mathcal{K} := \{\, \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \ : \ 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \,\}$ is complex parameters and the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z-1)H(\kappa) = (q_0 q_1 q_z)p^2 - \{\kappa_1 q_1 q_z + (\kappa_2 - 1)q_0 q_1 + \kappa_3 q_0 q_z\}p + \kappa_0(\kappa_0 + \kappa_4)q_z,$$

with $q_\nu := q - \nu$ for $\nu \in \{0, 1, z\}$. Let $\mathcal{M}_z(\kappa)$ be the set of all meromorphic solution germs to $\mathrm{P_{VI}}(\kappa)$ at a base point $z \in Z$. In [2, 3, 4] the set $\mathcal{M}_z(\kappa)$ is realized as the moduli space of certain stable parabolic connections and thereby provided with the structure of a smooth quasi-projective rational surface. Here a *stable parabolic connection* is a rank 2 vector bundle with a Fuchsian connection and a parabolic structure, having the Riemann scheme as in Table 1 and satisfying a sort of stability condition in geometric invariant theory. The parameter κ_i is the difference of the second exponent from the first one at the regular singular point t_i (so λ_i is uniquely determined from κ_i). It is known that $\mathcal{M}_z(\kappa)$ is isomorphic to the space obtained from an 8-point blow-up of the Hirzebruch surface of degree 2 by removing its unique effective anti-canonical divisor.

Next, we review the concept of Riemann-Hilbert correspondence. Each stable parabolic connection restricts to a flat connection on $\mathbb{P}^1 \setminus \{0, 1, z, \infty\}$ and induces the Jordan equivalence class of its monodromy representation. In our case, the moduli space of monodromy representations is isomorphic to an affine cubic surface $\mathcal{S}(\theta)$ which depends on complex parameters $\theta \in \Theta := \mathbb{C}^4$ (see (4.1) for its explicit form) and the Riemann-Hilbert correspondence is formulated as a holomorphic map

$$\mathrm{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \mathcal{S}(\theta), \qquad \theta = \mathrm{rh}(\kappa), \tag{2.1}$$

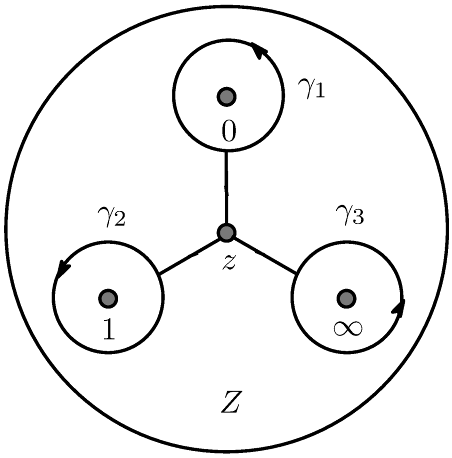

Figure 1. Three basic loops in $Z := \mathbb{P}^1 \setminus \{0, 1, \infty\}$

where rh : $\mathcal{K} \to \Theta$ is a holomorphic map between the parameter spaces, called the Riemann-Hilbert correspondence in the parameter level. The cubic surface $\mathcal{S}(\theta)$ admits at most four simple singularities and the Riemann-Hilbert correspondence (2.1) is a proper surjective map that is an analytic minimal resolution of singularities, whose exceptional set $\mathcal{E}_z(\kappa)$ parametrizes the so-called Riccati solutions to $\mathrm{P_{VI}}(\kappa)$.

Due to the Painlevé property, any solution germ $Q \in \mathcal{M}_z(\kappa)$ can be continued analytically along any loop $\gamma \in \pi_1(Z, z)$. Associating to each Q the result of its analytic continuation $\gamma_* Q$, we can define an automorphism

$$\gamma_* : \mathcal{M}_z(\kappa) \overset{\cong}{\to} \mathcal{M}_z(\kappa), \quad Q \mapsto \gamma_* Q,$$

called the (nonlinear) *monodromy map* of $\mathrm{P_{VI}}(\kappa)$ along γ. Then an element of $\mathcal{M}_z(\kappa)$ is said to be a *periodic solution* of period $n \in \mathbb{N}$ along γ if it is a periodic point of period n of the monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa)$.

Definition 2.1. A loop $\gamma \in \pi_1(Z, z)$ is said to be *elementary* if γ is conjugate to the loop γ_i^m for some index $i \in \{1, 2, 3\}$ and some integer $m \in \mathbb{Z}$, where $\gamma_1, \gamma_2, \gamma_3 \in \pi_1(Z, z)$ are loops as in Figure 1. Otherwise, γ is said to be *non-elementary*.

If the loop γ is elementary, then the map $\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa)$ preserves a fibration and exhibits a very simple dynamical behavior; this case is not so interesting whatever. Thus from now on we assume that γ is non-elementary. Then it turns out that any periodic point in $\mathcal{M}_z(\kappa) \setminus \mathcal{E}_z(\kappa)$ is isolated, but an irreducible component of $\mathcal{E}_z(\kappa)$ can be a periodic curve of the map γ_*. So it is natural to introduce the set

$$\mathrm{Per}_n^{\circ}(\gamma; \kappa) := \{\, Q \in \mathcal{M}_z(\kappa) \setminus \mathcal{E}_z(\kappa) \;:\; \gamma_*^n Q = Q \,\}$$

and call it *the set of isolated periodic solutions* of period n along γ. Then the problem is to estimate its cardinality $\#\mathrm{Per}_n^{\circ}(\gamma)$ counted with multiplicity and our main theorem is stated in the following manner.

Theorem 2.2. *Assume that the loop $\gamma \in \pi_1(Z, z)$ is non-elementary. Then the set $\mathrm{Per}_n^{\circ}(\gamma; \kappa)$ is finite for every $n \in \mathbb{N}$ and there exists a quadratic unit $\lambda(\gamma)$, called the first dynamical degree of γ, such that $\lambda(\gamma) \geq 3 + 2\sqrt{2}$ and*

$$|\#\mathrm{Per}_n^{\circ}(\gamma; \kappa) - \lambda(\gamma)^n| \leq O(1), \quad (n \to \infty),$$

where there exists an algorithm to calculate $\lambda(\gamma)$ explicitly from the reduced word of a minimal representative of the conjugacy class of γ. In particular, $\#\mathrm{Per}_n^{\circ}(\gamma; \kappa)$ grows exponentially with the exponential growth rate $\lambda(\gamma)$.

The algorithm mentioned in Theorem 2.2 is just the same as is given in [5].

§ 3. Saito's Formula and Isolated Periodic Points

As is mentioned in Introduction, Saito's fixed point formula is a powerful tool for counting the number of isolated periodic points of area-preserving surface maps with periodic curves. In this section, we recall a birational version of Saito's formula briefly and give an estimate of the number of isolated periodic points, derived from this formula. One can consult the paper [6] for a full discussion. Let X be a smooth projective surface and $f : X \to X$ be a birational map. Then f has the indeterminacy set $I(f)$, so that we need to define the notions of fixed point and fixed curve more precisely. In general, fixed point formulas guarantee that for a map $f : X \to X$ on a compact manifold X, the Lefschetz number

$$L(f) := \sum_{i=0}^{\dim X} (-1)^i \mathrm{Tr}[f^* : H^i(X, \mathbb{Z}) \to H^i(X, \mathbb{Z})]$$

is the sum of the multiplicities of intersection between the graph $\Gamma_f \subset X \times X$ of f and the diagonal $\Delta \subset X \times X$. Note that the graph $\Gamma_{f^{-1}}$ of f^{-1} is the reflection of Γ_f in the diagonal Δ. In particular, we have $\Gamma_f \cap \Delta = \Gamma_{f^{-1}} \cap \Delta$ and may express an element of $\Gamma_f \cap \Delta$ as the form $(f^{-1}(x), x) \in X \times X$ with $x \in X \setminus I(f^{-1})$. So we assume the separation condition

$$I(f) \cap I(f^{-1}) = \emptyset, \tag{3.1}$$

and adopt the following definitions of fixed point and fixed curve for a birational map $f : X \circlearrowleft$.

Definition 3.1. A point $x \in X$ is called a *fixed point* if x belongs to the set

$$X_0(f) := X_0^\circ(f) \cup X_0^\circ(f^{-1}), \tag{3.2}$$

where $X_0^\circ(f)$ is the set of all points $x \in X \setminus I(f)$ fixed by f. Moreover let $X_1(f)$ be the set of all irreducible curves C in X such that $C \setminus I(f)$ is fixed pointwise by f. An element C of $X_1(f)$ is called a *fixed curve*. It is easy to see that the definition is symmetric, namely, $X_1(f) = X_1(f^{-1})$.

In order to state Saito's fixed point formula, we need to define suitably local indices $\nu_x(f)$ and $\nu_C(f)$ at the fixed points $x \in X_0(f)$ and at the fixed curves $C \in X_1(f)$, and to divide the set $X_1(f)$ into two disjoint subset:

$$X_1(f) = X_I(f) \amalg X_{II}(f), \tag{3.3}$$

where $X_I(f)$ and $X_{II}(f)$ are the sets of fixed curves of type I and of type II respectively. These definitions are given in the paper [6] (see also [7]). Now a birational version of Saito's fixed point formula is described as follows.

Theorem 3.2 ([7]). *Let $f : X \to X$ be a nontrivial birational map on a smooth projective surface X. If the map f satisfies the separation condition (3.1), then the Lefschetz number of f is expressed as*

$$L(f) = \sum_{x \in X_0(f)} \nu_x(f) + \sum_{C \in X_I(f)} \chi_C \cdot \nu_C(f) + \sum_{C \in X_{II}(f)} \tau_C \cdot \nu_C(f), \tag{3.4}$$

where χ_C is the Euler characteristic of the normalization of $C \in X_I(f)$ and τ_C is the self-intersection number of $C \in X_{II}(f)$.

We can estimate the number of isolated periodic points of a birational surface map $f : X \circlearrowleft$ by applying the fixed point formula (3.4). Before stating this estimate, we introduce the two concepts of first dynamical degree and algebraic stability.

Definition 3.3. Let $f : X \to X$ be a birational map.

1. *The first dynamical degree* $\lambda(f)$ is defined by

   $$\lambda(f) := \lim_{n \to \infty} \|(f^n)^*|_{H^{1,1}(X)}\|^{1/n} \geq 1,$$

   where $|| \cdot ||$ is an operator norm on $\mathrm{End} H^{1,1}(X)$. It is known that the limit exists, $\lambda(f)$ is independent of the choice of the norm $|| \cdot ||$ and invariant under birational conjugation.

2. The map f is said to be *algebraically stable* (AS for short) if the condition $(f^n)^* = (f^*)^n : H^{1,1}(X) \to H^{1,1}(X)$ holds for any $n \in \mathbb{N}$. It is known that f is AS if and only if

$$f^{-m}I(f) \cap f^n I(f^{-1}) = \emptyset \qquad \text{for every} \quad m, n \geq 0. \tag{3.5}$$

Note that if f is AS, then $\lambda(f)$ can be easily calculated as the spectral radius of $f^*|_{H^{1,1}(X)}$.

Let $\mathrm{Per}_n^i(f)$ be the set of isolated periodic points of f with (not necessarily primitive) period n, and $\#\mathrm{Per}_n^i(f)$ be its cardinality counted with multiplicity, namely, it is given by

$$\#\mathrm{Per}_n^i(f) = \sum_{x \in \mathrm{Per}_n^i(f)} \nu_x(f^n). \tag{3.6}$$

Then we have the following theorem.

Theorem 3.4 ([6]). *Let X be a smooth projective surface and $f : X \to X$ be an AS birational map that satisfies the following three conditions.*

(1) The first dynamical degree of f is greater than one : $\lambda(f) > 1$.

(2) The birational map f preserves a nontrivial meromorphic 2-form ω.

(3) The pole divisor $(\omega)_\infty$ of ω does not contain a periodic curve of type I.

Then f has at most finitely many irreducible periodic curves and must have infinitely many isolated periodic points. Moreover the number of isolated periodic points of period n, counted with multiplicity, is estimated as

$$|\#\mathrm{Per}_n^i(f) - \lambda(f)^n| \leq \begin{cases} O(1) & (X \sim \textit{no Abelian surface}), \\ 4\,\lambda(f)^{n/2} + O(1) & (X \sim \textit{an Abelian surface}), \end{cases}$$

where $X \sim Y$ indicates that X is birationally equivalent to Y and $O(1)$ is a bounded function of $n \in \mathbb{N}$.

In our situation the fixed point formula (3.4) is applied to the iterates f^n of a map f, so that the separation condition (3.1) should be replaced by its iterated version:

$$I(f^n) \cap I(f^{-n}) = \emptyset \qquad \text{for every} \quad n \in \mathbb{N}.$$

It is easy to see that this condition follows from the AS condition (3.5).

Theorem 3.4 is a consequence of Theorem 3.2 and the following four ingredients.

1. For a fixed curve $C \in X_{II}(f)$ of type II, the indices $\nu_C(f^n)$ are independent of $n \in \mathbb{N}$. Similalrly, for a fixed point $x \in X_0(f)$ through which at least one fixed curve of type II passes, the indices $\nu_x(f^n)$ are independent of $n \in \mathbb{N}$.

2. If $\lambda(f) > 1$, then f has at most $\rho(X) + 1$ periodic curves of type II with mutually distinct primitive periods, where $\rho(X)$ is the Picard number of X.

3. Assume that f preserves some nontrivial meromorphic 2-form ω, and $C \in X_1(f)$ is a fixed curve such that ω has no pole of order $\nu_C(f)$ along C. Then C is of type II.

4. Assume that f is a AS birational map with $\lambda(f) > 1$. Then the following estimate holds:

$$|L(f^n) - \lambda(f)^n| \le \begin{cases} O(1) & (X \sim \text{no Abelian surface}), \\ 4\,\lambda(f)^{n/2} + O(1) & (X \sim \text{an Abelian surface}). \end{cases}$$

§ 4. Dynamics on Cubic Surface

In this section we sketch the proof of Theorem 2.2. The main idea is to turn our attention from the monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ to the birational map on the desingularized cubic surface and to apply Theorem 3.4 to the latter map.

The target of the Riemann-Hilbert correspondence (2.1) is the affine cubic surface

$$\mathcal{S}(\theta) = \{\, x = (x_1, x_2, x_3) \in \mathbb{C}^3 \,:\, f(x, \theta) = 0 \,\}, \tag{4.1}$$

where the cubic polynomial $f(x, \theta)$ of x with a parameter $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta := \mathbb{C}^4$ is given by

$$f(x, \theta) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4.$$

Since $\mathcal{S}(\theta)$ is a $(2, 2, 2)$-surface, namely, the defining equation $f(x, \theta)$ of $\mathcal{S}(\theta)$ is quadratic in each variable x_i, for any point $x = (x_i, x_j, x_k) \in \mathcal{S}(\theta)$, there is a unique second point $x' = (x_i', x_j, x_k) \in \mathcal{S}(\theta)$, which induces an involution

$$\sigma_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta), \quad x \mapsto x'. \tag{4.2}$$

The biregular map σ_i preserves a natural area form on $\mathcal{S}(\theta)$ up to sign. More precisely, we have

$$\sigma_i^* \omega_\theta = -\omega_\theta, \tag{4.3}$$

where $\omega_\theta := dx_1 \wedge dx_2 \wedge dx_3 \,/\, df(x, \theta)$ is the Poincaré residue for $\mathcal{S}(\theta)$, which is well-defined outside the singularities of $\mathcal{S}(\theta)$.

Let G be the group generated by three involutions σ_1, σ_2 and σ_3, and $G(2)$ be the index-two subgroup of G generated by three elements $\sigma_1\sigma_2$, $\sigma_2\sigma_3$ and $\sigma_3\sigma_1$. It is known that G is the universal Coxeter group of rank 3 (see [5], Theorem 4), and is of

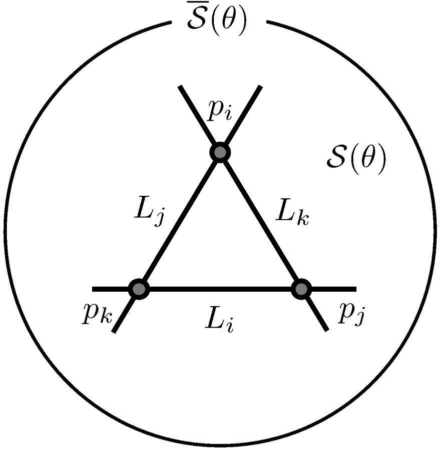

Figure 2. Tritangent lines at infinity on $\overline{\mathcal{S}}(\theta)$

finite index in the group of polynomial automorphisms of $\mathcal{S}(\theta)$ (see [1]). Each element $\sigma \in G(2)$ plays an important role because any nonlinear monodromy $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ is conjugate to some biregular map $\sigma : \mathcal{S}(\theta) \circlearrowleft$ with $\sigma \in G(2)$. More precisely, the monodromy $(\gamma_i)_* : \mathcal{M}_z(\kappa) \circlearrowleft$ along a generator γ_i of $\pi(Z, z)$ is semi-conjugate to the map $\sigma_i\sigma_{i+1} : \mathcal{S}(\theta) \circlearrowleft$ via the Riemann-Hilbert correspondence (2.1). This fact leads to the isomorphism of groups

$$\pi_1(Z, z) \to G(2). \tag{4.4}$$

We would like to apply Saito's fixed point formula to a more tractable map σ than a monodromy γ_*. However, the fact that $\mathcal{S}(\theta)$ is neither compact nor smooth prevents us from doing so directly. Therefore we carry out two procedures, that is, a compactification $\overline{\mathcal{S}}(\theta)$ of $\mathcal{S}(\theta)$ and a desingularization $\widetilde{\mathcal{S}}(\theta)$ of $\overline{\mathcal{S}}(\theta)$. First, we compactify the affine cubic surface $\mathcal{S}(\theta)$ by a standard embedding

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}}(\theta) \subset \mathbb{P}^3, \qquad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3].$$

The intersection of $\overline{\mathcal{S}}(\theta)$ with the plane at infinity is the union L of three lines

$$L_i = \{\, [X_0 : X_1 : X_2 : X_3] \in \mathbb{P}^3 \,:\, X_0 = X_i = 0 \,\} \qquad (i = 1, 2, 3).$$

The set $L = L_1 \cup L_2 \cup L_3$ is independent of the parameters, called the tritangent lines at infinity, and the intersection point of L_j and L_k is denoted by p_i for $\{i, j, k\} = \{1, 2, 3\}$ as in Figure 2. Note that the surface $\overline{\mathcal{S}}(\theta)$ is smooth in a neighborhood of L for any $\theta \in \Theta$ (see [5], Lemma 2), and thus the type of singularities of $\overline{\mathcal{S}}(\theta)$ coincides with that of $\mathcal{S}(\theta)$. In particular, the singularities of $\overline{\mathcal{S}}(\theta)$ are simple.

Next we consider an algebraic minimal resolution of singularities

$$\pi : \widetilde{\mathcal{S}}(\theta) \to \overline{\mathcal{S}}(\theta) \tag{4.5}$$

with the exceptional set $\mathcal{E}(\theta)$. In the case where $\overline{\mathcal{S}}(\theta)$ is smooth, put $\pi := \mathrm{id} : \widetilde{\mathcal{S}}(\theta) := \overline{\mathcal{S}}(\theta) \circlearrowleft$ for convenience. Each element $\sigma \in G$ uniquely extends to a birational map on $\overline{\mathcal{S}}(\theta)$ and this birational map lifts to one on $\widetilde{\mathcal{S}}(\theta)$. We shall use the same notation σ for the biregular map on $\mathcal{S}(\theta)$ and the induced birational maps on $\overline{\mathcal{S}}(\theta)$ and $\widetilde{\mathcal{S}}(\theta)$. Note that the birational map $\sigma : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ restricts to an automorphism of $\widetilde{\mathcal{S}}(\theta) \setminus L$.

Recall that the Riemann-Hilbert correspondence (2.1) is an analytic minimal resolution of simple singularities. Since a minimal desingularization is unique up to isomorphism, $\mathrm{RH}_{z,\kappa}$ lifts to an isomorphism $\widetilde{\mathrm{RH}}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \widetilde{\mathcal{S}}(\theta) \setminus L$ sending $\mathcal{E}_z(\kappa)$ to $\mathcal{E}(\theta)$ and the map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ is strictly conjugate to $\sigma : \widetilde{\mathcal{S}}(\theta) \setminus L \circlearrowleft$ via $\widetilde{\mathrm{RH}}_{z,\kappa}$, where σ is determined by the isomorphism (4.4). Moreover it is seen that there exists an element $\tau \in G$ such that $\sigma' := \tau^{-1}\sigma\tau \in G(2)$ and $\sigma' : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ is AS. Then $\gamma \in \pi(Z, z)$ is non-elementary if and only if the corresponding element σ' can not be expressed as $\sigma' = (\sigma_i\sigma_j)^l$ for some $\{i, j, k\} = \{1, 2, 3\}$ and $l \in \mathbb{N}$. Such an element $\sigma' \in G(2)$ is said to be *non-elementary*, too. Let $\mathrm{Per}_n^{\circ}(\sigma)$ be the set of periodic points of $\sigma : \widetilde{\mathcal{S}}(\theta)\setminus L \circlearrowleft$ outside the exceptional set $\mathcal{E}(\theta)$ with (not necessarily primitive) period n. Since $\sigma' : \widetilde{\mathcal{S}}(\theta) \setminus L \circlearrowleft$ is strictly conjugate to $\sigma : \widetilde{\mathcal{S}}(\theta) \setminus L \circlearrowleft$, we have $\#\mathrm{Per}_n^{\circ}(\gamma) = \#\mathrm{Per}_n^{\circ}(\sigma) = \#\mathrm{Per}_n^{\circ}(\sigma')$. Therefore our main theorem is reduced to the following theorem.

Theorem 4.1. *Assume that $\sigma \in G(2)$ is a non-elementary AS element. Then $\mathrm{Per}_n^{\circ}(\sigma)$ is a finite set for each n and the cardinality of $\mathrm{Per}_n^{\circ}(\sigma)$ counted with multiplicity is estimated as*

$$|\#\mathrm{Per}_n^{\circ}(\sigma) - \lambda(\sigma)^n| \le O(1), \quad (n \to \infty), \tag{4.6}$$

where the first dynamical degree $\lambda(\sigma)$ is a quadratic unit greater than or equal to $3+2\sqrt{2}$ and there exists an algorithm to calculate $\lambda(\sigma)$.

We will prove this theorem by applying Theorem 3.4. To this end, we need to check the three conditions in Theorem 3.4. The following are the main ideas on how to check each condition.

(1) Since σ is AS, $\lambda(\sigma)$ is calculated as the spectral radius of $\sigma^* : H^{1,1}(\widetilde{\mathcal{S}}(\theta)) \circlearrowleft$. The cohomology group admits the direct sum decomposition $H^{1,1}(\widetilde{\mathcal{S}}(\theta)) = V \oplus V^{\perp}$, where V is the subspace spanned by the lines L_1, L_2, L_3 at infinity and $V^{\perp}$ is the orthogonal complement to it with respect to the intersection form. Then it is seen that σ^* preserves the subspaces V and $V^{\perp}$, and the operator $\sigma^*|_{V^{\perp}}$ is unitary. In particular, the spectral radius of $\sigma^*|_{V^{\perp}}$ is equal to one, and thus the spectral radius of $\sigma^* : H^{1,1}(\widetilde{\mathcal{S}}(\theta)) \circlearrowleft$ coincides with that of $\sigma^*|_V$. The action σ^* on the subspace V is same whether $\widetilde{\mathcal{S}}(\theta)$

is smooth or not, and is described in the paper [5]. The description elaborated in [5] yields the result that $\lambda(\sigma)$ is a quadratic unit greater than or equal to $3+2\sqrt{2}$ and the algorithm to calculate $\lambda(\sigma)$.

(2) The area form ω_θ on $\mathcal{S}(\theta)$ induces a meromorphic 2-form $\widetilde{\omega}_\theta$ on $\widetilde{\mathcal{S}}(\theta)$, whose pole divisor $(\widetilde{\omega}_\theta)_\infty$ is $L_1+L_2+L_3$. It follows from the relation (4.3) that the birational map $\sigma : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ preserves the meromorphic 2-form $\widetilde{\omega}_\theta$.

(3) The third condition is a consequence of the fact that the periodic curves of $\sigma : \widetilde{\mathcal{S}}(\theta) \circlearrowleft$ are contained in the exceptional set $\mathcal{E}(\theta)$ of $\pi : \widetilde{\mathcal{S}}(\theta) \to \overline{\mathcal{S}}(\theta)$. In particular, no irreducible component of the pole divisor $(\widetilde{\omega}_\theta)_\infty = L_1 + L_2 + L_3$ is a periodic curve of σ.

Therefore all the conditions of Theorem 3.4 are satisfied. Since $\widetilde{\mathcal{S}}(\theta)$ is birationally equivalent to no Abelian surface, we have an estimate

$$|\#\mathrm{Per}_n^i(\sigma) - \lambda(\sigma)^n| \le O(1).$$

On the other hand, the cardinality $\#\mathrm{Per}_n^\circ(\sigma)$ is obtained from $\#\mathrm{Per}_n^i(\sigma)$ by subtracting the indices at the isolated fixed points of σ^n on the tritangent lines L and the exceptional set $\mathcal{E}(\theta)$. For any $m \in \mathbb{N}$, σ^m restricted to L has two fixed points p_i and p_j for some $\{i,j,k\} = \{1,2,3\}$, independent of m, in the sense of Definition 3.1 and the local indices $\nu_{p_i}(\sigma^m)$ and $\nu_{p_j}(\sigma^m)$ turn out to be equal to one. Moreover, since the singularities of $\overline{\mathcal{S}}(\theta)$ are simple, the exceptional set $\mathcal{E}(\theta)$ is the union of finitely many irreducible components isomorphic to the projective line $\mathbb{P}^1$. The map σ restricted to $\mathcal{E}(\theta)$ is an automorphism and has finitely many isolated periodic points. It is known that the local indices at an isolated periodic point with period m are bounded under iterations of σ^m (see [6], Theorem 7.5 and [8]). Therefore the estimate (4.6) is established by combining all these observations.

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