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On the composition of kernel functions of pseudo-differential operators and the compatibility with Leibniz rule

By

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Abstract

The aim of this article is the following:

(1) We show that the space of kernel functions of the pseudo-differential operators (the sections of the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of rings) is not closed in a concrete integral expression of composition; this expression is a direct analogy with the composition for kernel functions of differential operators of infinite order. We give the reason by proving a decomposition theorem.

(2) In order to revive this concrete integral expression of composition of kernel functions, we introduce the space of "formal kernel functions" as a generalization of the space of usual kernel functions. Though the space of formal kernel functions cannot be taken any inductive limit concerning their definition domains, the space of kernel functions of pseudo-differential operators are canonically embedded into our space. Then, the symbol of a formal kernel function defined in a similar way becomes a usual symbol for a pseudo-differential operator. Further the space of formal kernel functions has a concrete integral expression of composition that is compatible with the Leibniz rule for the corresponding symbols.

§ 1. Introduction and a decomposition theorem

Firstly we recall the definition of the sheaf $\mathscr{E}_X^{\mathbb{R}}$ of pseudo-differential operators by Sato-Kawai-Kashiwara [S-K-K] as follows:

$$\mathscr{E}_X^{\mathbb{R}} := \mu_X(\mathscr{O}_{X\times X})[n] \underset{p_2^{-1}\mathscr{O}_X}{\otimes} p_2^{-1}\Omega_X^n.$$

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Here X is an n-dimensional complex manifold. $\mathscr{O}_X, \mathscr{O}_{X\times X}$, and Ω_X^n denote the sheaves of holomorphic functions on X, $X \times X$, the sheaf of holomorphic n-forms on X, respectively. Further μ_X is the Kashiwara-Schapira [K-S] microlocalization functor with respect to the diagonal imbedding $X \to X \times X$, and p_2 is the second projection : $T_X^*(X \times X) \to X$. For $X = \mathbb{C}^n$ with coordinates $(z, \tilde{z}) = (x + iy, \tilde{x} + i\tilde{y}) \in X \times X$, the stalk at $p = (0; idx_1) \in T^*X \simeq T_X^*(X \times X)$ is written as

$$\mathscr{E}_X^{\mathbb{R}}|_p = \varinjlim_{\varepsilon \to +0} H_{Z_\varepsilon}^n(U_\varepsilon; \mathscr{O}_{X\times X}) \otimes d\tilde{z},$$

where $U_\varepsilon = \{(z, \tilde{z}) \in \mathbb{C}^{n+n} \,;\, |z| < \varepsilon, |\tilde{z}_j - z_j| < \varepsilon \text{ for } j = 1, ..., n\}$, $Z_\varepsilon = \{(z, \tilde{z}) \in U_\varepsilon \,;\, \tilde{y}_1 - y_1 \geq \varepsilon|\tilde{x}_1 - x_1|, |\tilde{z}_1 - z_1| \geq \varepsilon|\tilde{z}_j - z_j| \text{ for } j = 2, ..., n\}$ ([S-K-K]). Hence, under the Stein covering $\{U_\varepsilon^1, ..., U_\varepsilon^n\}$ of $U_\varepsilon \setminus Z_\varepsilon$ with

$$\begin{aligned} U_\varepsilon^1 &:= U_\varepsilon \cap \{\tilde{y}_1 - y_1 < \varepsilon|\tilde{x}_1 - x_1|\}, \\ U_\varepsilon^j &:= U_\varepsilon \cap \{|\tilde{z}_1 - z_1| < \varepsilon|\tilde{z}_j - z_j|\} \ (j = 2, ..., n), \end{aligned}$$

we have the expression

$$\mathscr{E}_X^{\mathbb{R}}|_p = \varinjlim_{\varepsilon} \Big(\mathscr{O}_{X\times X}(\bigcap_{j=1}^{n} U_\varepsilon^j) / \sum_{k=1}^{n} \mathscr{O}_{X\times X}(\bigcap_{j\neq k} U_\varepsilon^j)\Big) \otimes d\tilde{z}. \tag{1.1}$$

More explicitly, a germ P of $\mathscr{E}_X^{\mathbb{R}}|_p$ is expressed by an equivalence class $[K(z, \tilde{z} - z)d\tilde{z}]$ with

$$K(z, \tilde{z} - z) = \sum_{\alpha' \geq 0} \frac{K_{\alpha'}(z, \tilde{z}_1 - z_1)}{(\tilde{z}' - z')^{\alpha' + \mathbf{1}_{n-1}}}. \tag{1.2}$$

Here $\alpha' = (\alpha_2, ..., \alpha_n)$ is a non-negative multi-index, $\mathbf{1}_{n-1} = (\overbrace{1, \ldots, 1}^{n-1})$, and $\{K_{\alpha'}(z, w_1)\}_{\alpha'}$ are holomorphic functions on $V_{r,\varepsilon,0}$ with

$$V_{r,\varepsilon,\delta} := \{(z, w_1) \in \mathbb{C}^n \times \mathbb{C} \,;\, |z| < r, |w_1| < r, \operatorname{Im} w_1 + \delta < \varepsilon|\operatorname{Re} w_1|\} \tag{1.3}$$

for some $r, \varepsilon > 0$ satisfying the following growth condition:

$$\sup_{V_{r,\varepsilon,\delta},\, \alpha'} \left(\frac{\varepsilon}{|w_1| + \delta}\right)^{|\alpha'|} |K_{\alpha'}(z, w_1)| =: C_{K,\delta} < \infty \quad \text{for all } \delta > 0. \tag{1.4}$$

Further, $P = 0$ holds if and only if there exist small constants $r', \varepsilon' > 0$ and a constant $C'_\delta > 0$ depending only on $\delta > 0$ such that

(1.5) each $K_{\alpha'}(z, w_1)$ extends holomorphically to $V_{r',\varepsilon'}^0 := \{|z| < r', |w_1| < r'\}$ with estimates

$$|K_{\alpha'}(z, w_1)| \leq C'_\delta \left(\frac{|w_1| + \delta}{\varepsilon'}\right)^{|\alpha'|} \text{ on } V_{r',\varepsilon'}^0 \text{ for all } \delta > 0.$$

The ring structure $\mathscr{E}_X^{\mathbb{R}} \underset{\mathbb{C}}{\otimes} \mathscr{E}_X^{\mathbb{R}} \to \mathscr{E}_X^{\mathbb{R}}$ as integration operators

$$(K, K') \to K''(z, \tilde{z} - z)d\tilde{z} = \left(\int K(z, \hat{z} - z)K'(\hat{z}, \tilde{z} - \hat{z})d\hat{z}\right) d\tilde{z}.$$

is introduced by using a derived functor induced from the integration morphism for the Dolbeault complex ([S-K-K]). Here we remark that the operations of $\mathscr{E}_X^{\mathbb{R}}$ on defining holomorphic functions of microfunctions are argued in [K-K1, K-K2] based on the integration morphism. On the other hand, some explicit composition formula for kernel functions are argued in [A2, AKY2]; this formula is a direct analogy with the composition for kernel functions of differential operators with infinite order, but it has never been proven that the result of that composition becomes a defining function for $\mathscr{E}_X^{\mathbb{R}}$. For the pair (K, K') of kernel functions, this explicit composition formula argued in [A2, AKY2] is given as a complex integral for $w = \hat{z} - z$ along a chain Γ:

$$\left\{\left(w_1(t_1), \frac{|w_1(t_1)| + \varepsilon''}{\varepsilon}e^{it_2}, ..., \frac{|w_1(t_1)| + \varepsilon''}{\varepsilon}e^{it_n}\right)\middle| (t_1, ..., t_n) \in [0, 2\pi]^n\right\} \tag{1.6}$$

with

$$w_1(t_1) = \frac{(t_1 - \pi)}{\pi}r' + i\left(\varepsilon' r' \frac{|t_1 - \pi|}{\pi} - \varepsilon''\left(1 - \frac{|t_1 - \pi|}{\pi}\right)\right). \tag{1.7}$$

Here, $0 < r' < r, 0 < \varepsilon' < \varepsilon, 0 < \varepsilon'' \ll \varepsilon' r'$. Hence, we have

$$\begin{aligned} K''(z, w) = \sum_{\alpha', \beta', \gamma} \int_0^{2\pi} dw_1(t_1) \frac{K_{\alpha'}(z, w_1(t_1))\partial_z^{\gamma} K'_{\beta'}(z, w_1 - w_1(t_1))}{\gamma!} \\ \times \int_{[0,2\pi]^{n-1}} \frac{w_1(t_1)^{\gamma_1} dw'(t)}{w'(t)^{\alpha' - \gamma' + \mathbf{1}_{n-1}}(w' - w'(t))^{\beta' + \mathbf{1}_{n-1}}}. \end{aligned} \tag{1.8}$$

Therefore, the coefficients $(K''_{\alpha'}(z, w_1))_{\alpha'}$ of $K''(z, w)$ are given as follows:

$$\begin{aligned} (2\pi i)^{n-1} \sum_{\alpha' = \tilde{\alpha}' + \beta', \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}' + \beta'}{\beta'} \int_0^{2\pi} w_1(t_1)^{\gamma_1} \\ \times \frac{K_{\tilde{\alpha}' + \gamma'}(z, w_1(t_1))\partial_z^{\gamma} K'_{\beta'}(z, w_1 - w_1(t_1))}{\gamma!} dw_1(t_1). \end{aligned} \tag{1.9}$$

Since $|w_1(t_1)/w_1|$ is unbounded on $\{t_1 \in [0, 2\pi], (z, w_1) \in V_{r,\varepsilon}\}$, we cannot obtain any estimates for $K''_{\alpha'}(z, w_1)$ of type (1.4) except for the case $n = 1$. This causes an essential difficulty to get the true composition formula by using this explicit formula for kernel functions.

Example 1.1. We consider two dimensional cases. Let $K = K^1, K' = K^2$ be the kernel functions corresponding to micro-differential operators

$$P_\ell(\partial) := \sum_{j \geq 1} C_j^\ell \partial_1^{-j} \partial_2^j$$

of order 0 with constant coefficients for $\ell = 1, 2$, respectively. That is, the C_j^ℓ's are complex numbers satisfying the estimates:

$$|C_j^\ell| \le C^j \quad (\forall j = 1, 2, \ldots, \forall \ell = 1, 2)$$

for some $C > 0$, and

$$K^\ell(*, \tilde{z} - z) = \frac{1}{(2\pi i)^2} \sum_{j \ge 1} \frac{(-1)^j j C_j^\ell\, (\tilde{z}_1 - z_1)^{j-1} \log(\tilde{z}_1 - z_1)}{(\tilde{z}_2 - z_2)^{j+1}}.$$

In particular, we take

$$P_1 = \sum_{j=1}^{\infty} \frac{(-1)^j}{j(j+1)} \partial_1^{-j} \partial_2^j, \quad P_2 = \partial_1^{-1} \partial_2.$$

Then K_j'' $(j \ge 2)$ in (1.9) is written as follows:

$$\frac{-1}{(2\pi i)^3} \int_0^{2\pi} w_1(t_1)^{j-2} \log(w_1(t_1)) \log(w_1 - w_1(t_1)) dw_1(t_1) \ \ (j \ge 2).$$

Therefore our explicit composition K'' is given by

$$K''(*, w) = \frac{1}{(2\pi i)^3} \int_0^{2\pi} \frac{\log(w_1(t_1)) \log(w_1 - w_1(t_1))}{w_2^2\, (w_1(t_1) - w_2)} dw_1(t_1).$$

Here we have the following modification when $|w_2| \gg 1$:

$$\begin{aligned} \partial_{w_1} K''(*, w) = & \frac{1}{(2\pi i)^3} \int_C \frac{\log \tau}{w_2^2\, (\tau - w_2)(w_1 - \tau)} d\tau \\ & + \frac{1}{(2\pi i)^2} \frac{\log w_1}{w_2^2 (w_1 - w_2)}, \end{aligned} \tag{1.10}$$

where C is another path:

$$\tau(t) = r' \exp\left(-\frac{\pi}{2} i - \left(\frac{\pi}{2} + \arctan \varepsilon'\right) (1 - 2t)\, i\right) \ \ (0 \le t \le 1).$$

Let $E_1(w)$ be the kernel function for $P_1 P_2$ given by the coefficients

$$E_{1,j}(w_1) = \frac{-1}{(2\pi i)^2} \frac{w_1^{j-1} \log w_1}{j - 1} \quad (j \ge 2).$$

Therefore the second term of (1.10) is almost equal to $\partial_{w_1} E_1(w)$. Precisely we have

$$\partial_{w_1}(K''(*, w) - E_1(w) - E_2(w)) = \frac{1}{(2\pi i)^3} \int_C \frac{\log \tau}{w_2^2\, (\tau - w_2)(w_1 - \tau)} d\tau \tag{1.11}$$

with

$$E_2(w) = \sum_{j\geq 2} \frac{1}{(2\pi i)^2} \frac{w_1^{j-1}}{(j-1)^2 w_2^{j+1}}.$$

It is clear that E_1, E_2 are the kernel functions for some germs of $\mathscr{E}_X^{\mathbb{R}}|_p$ (indeed, E_2 is equivalent to zero). Hence if $K''(*, w)$ is the kernel function of some germ of $\mathscr{E}_X^{\mathbb{R}}|_p$, the right side of (1.11) must extend holomorphically to $\{w_1/i \in [-\delta, 0)\} \times \{r' \leq |w_2| < \infty\}$ for some small $\delta > 0$. However, since $G(w_1, w_2, \tau) := \log \tau/((2\pi i)^3 w_2^2(w_1 - \tau))$ is holomorphic on

$$\left\{|w_1| < \frac{r'}{2}, |w_2| > \frac{r'}{2}, |\tau| > \frac{r'}{2}, -3\pi/2 < \arg \tau < \pi/2\right\},$$

the right side of (1.11) is written as

$$\int_C \frac{G(w_1, w_2, \tau) - G(w_1, w_2, w_2)}{\tau - w_2} d\tau + G(w_1, w_2, w_2) \log\left(\frac{\tau(1) - w_2}{\tau(0) - w_2}\right).$$

The first term above is holomorphic on $\{|w_1| < r'/2, w_2 = \tau(0)\}$, but the second term cannot be extended to $\{|w_1| < r'/2, w_2 = \tau(0)\}$. This contradicts our assumption. Therefore, we cannot get any estimates of type (1.4) for $(K_j''(*, w_1))_j$.

One method to modify the estimation of (1.9) is to rewrite $(2\pi i)^{1-n} K''_{\alpha'}(z, w_1)$ as follows:

$$\begin{aligned}
(1.12)\qquad & \sum_{\alpha'=\tilde{\alpha}'+\beta', \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}' + \beta'}{\beta'} \partial_{w_1}^{-|\alpha'|} \cdot \partial_{w_1}^{|\alpha'|} \int_0^{2\pi} w_1(t_1)^{\gamma_1} \\
& \times \frac{K_{\tilde{\alpha}'+\gamma'}(z, w_1(t_1)) \partial_z^\gamma K'_{\beta'}(z, w_1 - w_1(t_1))}{\gamma!} dw_1(t_1) \\
& + \sum_{\alpha'=\tilde{\alpha}'+\beta', \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}' + \beta'}{\beta'} (1 - \partial_{w_1}^{-|\alpha'|} \cdot \partial_{w_1}^{|\alpha'|}) \int_0^{2\pi} w_1(t_1)^{\gamma_1} \\
& \times \frac{K_{\tilde{\alpha}'+\gamma'}(z, w_1(t_1)) \partial_z^\gamma K'_{\beta'}(z, w_1 - w_1(t_1))}{\gamma!} dw_1(t_1),
\end{aligned}$$

where $\partial_{w_1}^{-\ell} \varphi(w_1)$ (for $\ell \geq 1$) means the following integral operator

$$\int_{-i\delta_\ell}^{w_1} \frac{(w_1 - s)^{\ell-1}}{(\ell-1)!} \varphi(s) ds$$

with some small positive number $\delta_\ell \to +0$ ($\ell \to \infty$) chosen as follows:

$$(1.13)\qquad C_{K, \delta_\ell/6} C_{K', \delta_\ell/6} \leq 2^\ell \quad \text{for all } \ell \gg 1.$$

This idea, an expression by multiple indefinite integral, is due to [AKY1]. Since $\alpha' = \tilde{\alpha}' + \beta'$, after integration by parts, the first term of (1.12) is rewritten as

$$
\begin{aligned}
(1.14) \qquad & \sum_{\alpha'=\tilde{\alpha}'+\beta', \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \partial_{w_1}^{-|\alpha'|} \cdot \int_0^{2\pi} dw_1(t_1) \\
& \times \frac{\partial_\tau^{|\tilde{\alpha}'|}(\tau^{\gamma_1} K_{\tilde{\alpha}'+\gamma'}(z,\tau))|_{\tau=w_1(t_1)} \cdot \partial_{w_1}^{|\beta'|} \partial_z^{\gamma} K'_{\beta'}(z, w_1 - w_1(t_1))}{\gamma!} \\
& - \sum_{\alpha'=\tilde{\alpha}'+\beta', \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \partial_{w_1}^{-|\alpha'|} \cdot \sum_{j=0}^{|\alpha'|-1} \\
& \times \left[\frac{\partial_\tau^{j}(\tau^{\gamma_1} K_{\tilde{\alpha}'+\gamma'}(z,\tau))|_{\tau=w_1(t_1)} \cdot \partial_{w_1}^{|\tilde{\alpha}'+\beta'|-j-1} \partial_z^{\gamma} K'_{\beta'}(z, w_1 - w_1(t_1))}{\gamma!} \right]_0^{2\pi}.
\end{aligned}
$$

Further the second term of (1.12) is rewritten as

$$
\begin{aligned}
(1.15) \qquad & \sum_{\alpha'=\tilde{\alpha}'+\beta' \neq 0, \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \sum_{j=0}^{|\alpha'|-1} \frac{(w_1 + i\delta_{|\alpha'|})^j}{j!} \int_0^{2\pi} w_1(t_1)^{\gamma_1} \\
& \times \frac{K_{\tilde{\alpha}'+\gamma'}(z, w_1(t_1)) \partial_{w_1}^{j} \partial_z^{\gamma} K'_{\beta'}(z, -i\delta_{|\alpha'|} - w_1(t_1))}{\gamma!} dw_1(t_1).
\end{aligned}
$$

We divide this sum on j into $0 \leq j \leq \min\{|\beta'|, |\alpha'| - 1\}$ and $|\beta'| + 1 \leq j \leq |\alpha'| - 1$, and apply the integration by parts to the latter terms as in the case (1.14). Hence the second term of (1.12) is equal to

$$
\begin{aligned}
(1.16) \qquad & \sum_{\alpha'=\tilde{\alpha}'+\beta' \neq 0, \tilde{\alpha}', \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \sum_{j=0}^{\min\{|\beta'|, |\alpha'|-1\}} \frac{(w_1 + i\delta_{|\alpha'|})^j}{j!} \int_0^{2\pi} w_1(t_1)^{\gamma_1} \\
& \times \frac{K_{\tilde{\alpha}'+\gamma'}(z, w_1(t_1)) \partial_{w_1}^{j} \partial_z^{\gamma} K'_{\beta'}(z, -i\delta_{|\alpha'|} - w_1(t_1))}{\gamma!} dw_1(t_1) \\
& + \sum_{\alpha'=\tilde{\alpha}'+\beta', |\tilde{\alpha}'| \geq 2, \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \sum_{j=|\beta'|+1}^{|\alpha'|-1} \frac{(w_1 + i\delta_{|\alpha'|})^j}{j!} \int_0^{2\pi} dw_1(t_1) \\
& \times \frac{\partial_\tau^{j-|\beta'|}(\tau^{\gamma_1} K_{\tilde{\alpha}'+\gamma'}(z,\tau))|_{\tau=w_1(t_1)} \cdot \partial_{w_1}^{|\beta'|} \partial_z^{\gamma} K'_{\beta'}(z, -i\delta_{|\alpha'|} - w_1(t_1))}{\gamma!} \\
& - \sum_{\alpha'=\tilde{\alpha}'+\beta', |\tilde{\alpha}'| \geq 2, \beta' \geq 0, \gamma \geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \sum_{j=|\beta'|+1}^{|\alpha'|-1} \frac{(w_1 + i\delta_{|\alpha'|})^j}{j!} \sum_{\ell=0}^{j-|\beta'|-1} \\
& \times \left[\frac{\partial_\tau^{\ell}(\tau^{\gamma_1} K_{\tilde{\alpha}'+\gamma'}(z,\tau))|_{\tau=w_1(t_1)} \cdot \partial_{w_1}^{j-\ell-1} \partial_z^{\gamma} K'_{\beta'}(z, -i\delta_{|\alpha'|} - w_1(t_1))}{\gamma!} \right]_0^{2\pi}.
\end{aligned}
$$

We note here that (1.16) is holomorphic at $w_1 = 0$ because it is a polynomial in w_1. To estimate these terms above, we prepare the following lemma:

Lemma 1.2. *The inequalities*

$$\sup_{V_{\frac{r}{2},\frac{\varepsilon}{2},\delta}} (|w_1| + \delta)^{j-\gamma_1-|\beta'|} |\partial^j_{w_1}(w_1^{\gamma_1} K_{\beta'}(z, w_1))| \leq \frac{C_{K,\frac{\delta}{2}} j! 6^j 2^{|\beta'|+\gamma_1}}{\varepsilon^{j+|\beta'|}},$$

$$\sup_{V_{\frac{r}{2},\frac{\varepsilon}{2},\delta}} (|w_1| + \delta)^{j-|\beta'|} |\partial^j_{w_1} \partial^\gamma_z K'_{\beta'}(z, w_1)| \leq \frac{C_{K',\frac{\delta}{2}} j! \gamma! \left(\frac{2}{r}\right)^{|\gamma|} 6^j 2^{|\beta'|}}{\varepsilon^{j+|\beta'|}}$$

hold for all $\delta > 0, \beta' \geq 0, \gamma_1 \geq 0, \gamma \geq 0, j \geq 0$.

Proof. This is directly proven by applying the Cauchy estimates to (1.4) with radius $\varepsilon|w_1|/3$ concerning w_1 for $0 < \delta \ll \varepsilon \ll 1$. Indeed, the ratio $d(w)/|w|$ takes the minimum on $\{w \in \mathbb{C}; \operatorname{Im} w + \delta = \frac{\varepsilon}{2}|\operatorname{Re} w|\}$, where $d(w)$ is the distance between $w(\in \{w \in \mathbb{C}; \operatorname{Im} w + \delta \leq \frac{\varepsilon}{2}|\operatorname{Re} w|\})$ and $\{\tau \in \mathbb{C}; \operatorname{Im} \tau + \frac{\delta}{2} = \varepsilon|\operatorname{Re} \tau|\}$. For $|u| \geq \delta$ with $w = u + iv$ we have

$$\begin{aligned} d(w)/|w| &= \frac{\varepsilon|u| + \delta}{2\sqrt{(1+\varepsilon^2)(u^2 + (\frac{\varepsilon}{2}|u| - \delta)^2)}} \\ &\geq \frac{\varepsilon|u|}{2\sqrt{(1+\varepsilon^2)(u^2 + (\frac{\varepsilon}{2}|u| + |u|)^2)}} \\ &= \frac{\varepsilon}{\sqrt{(1+\varepsilon^2)(4 + (2+\varepsilon)^2)}} > \frac{\varepsilon}{3} \end{aligned}$$

for $\varepsilon \ll 1$. For $|u| < \delta$ we have

$$\begin{aligned} d(w)/|w| &\geq \frac{(1-\varepsilon)\delta}{2\sqrt{u^2 + (\frac{\varepsilon}{2}|u| - \delta)^2)}} \\ &\geq \frac{(1-\varepsilon)\delta}{2\sqrt{\delta^2 + \delta^2}} > \frac{\varepsilon}{3}. \end{aligned}$$

Hence we obtain $d(w)/|w| > \varepsilon/3$ for $\varepsilon \ll 1$. □

We remark that $\partial^j_{w_1}(w_1^{\gamma_1} K_{\beta'}(z, w_1))$ and $\partial^j_{w_1} \partial^\gamma_z K'_{\beta'}(z, w_1)$ have good estimates for $j \leq |\beta'|$ near $w_1 = 0$. Thus we obtain our main theorem in Section 1.

Theorem 1.3. *The composition* $K'' = K * K'$ *of two kernels* K, K' *is divided into 3 parts:* $K'' = K''_1 + K''_2 + K''_3$. *Here* K''_1 *defined as the first term of* (1.14) *satisfies the estimates* (1.4), *and so this term is a kernel function of* $\mathscr{E}^{\mathbb{R}}_X$. K''_2 *defined as the second term of* (1.14) *satisfies the null estimates* (1.5), *and so this term is a* 0*-kernel function in* $\mathscr{E}^{\mathbb{R}}_X$. *The last term* K''_3 *defined as the second term of* (1.12), *which is equal* *to* (1.16)*, satisfies the following property for the coefficients* $(K''_{3,\alpha'}(z,w_1))_{\alpha'}$*: For some* $r',\varepsilon',C'>0$*,*

$$\text{each } K''_{3,\alpha'}(z,w_1) \text{ extends holomorphically to } V^0_{r',\varepsilon'} := \{|z|<r', |w_1|<\varepsilon'\} \text{ with estimates } |K''_{3,\alpha'}(z,w_1)| \le C'^{|\alpha'|+1} \text{ on } V^0_{r',\varepsilon'}. \tag{1.17}$$

Hence K'''_3 *does not satisfy the conditions for kernel functions of* $\mathscr{E}^{\mathbb{R}}_X$*, but considered as* 0*.*

Proof. We fix a point (z,w_1) of $V_{\frac{r}{2},\frac{\varepsilon}{4},\delta}$, and take the integral path (1.7) for any small $r'>0$ with $\varepsilon'=\varepsilon/3, \varepsilon''=\delta/2$ $(0<\delta\ll r'<r/2)$:

$$\operatorname{Im} w_1(t_1) = -\frac{\delta}{2} + \left(\frac{1}{3}\varepsilon + \frac{\delta}{2r'}\right)|\operatorname{Re} w_1(t_1)| \quad (|\operatorname{Re} w_1(t_1)| \le r').$$

Note that $(z,w_1(t_1)),(z,w_1-w_1(t_1)) \in V_{\frac{r}{2},\frac{\varepsilon}{2},\frac{\delta}{3}}$ because

$$\begin{aligned}\operatorname{Im}(w_1-w_1(t_1)) &< -\delta + \frac{\varepsilon}{4}|\operatorname{Re} w_1| - \left(-\frac{\delta}{2} + \left(\frac{1}{3}\varepsilon + \frac{\delta}{2r'}\right)|\operatorname{Re} w_1(t_1)|\right)\\ &\le -\frac{\delta}{2} + \left(\frac{1}{3}\varepsilon + \frac{\delta}{2r'}\right)(|\operatorname{Re} w_1| - |\operatorname{Re} w_1(t_1)|)\\ &< -\frac{\delta}{3} + \frac{\varepsilon}{2}|\operatorname{Re}(w_1-w_1(t_1))|.\end{aligned}$$

Hence we have the following estimates for any sufficiently small $\delta>0$:

$$\begin{aligned}&\left|\frac{\partial_\tau^{|\tilde{\alpha}'|}(\tau^{\gamma_1}K_{\tilde{\alpha}'+\gamma'}(z,\tau))|_{\tau=w_1(t_1)}\cdot \partial_{w_1}^{|\beta'|}\partial_z^{\gamma}K'_{\beta'}(z,w_1-w_1(t_1))}{\gamma!}\right|\\ &\le |\tilde{\alpha}'|!|\beta'|!C_{K,\delta/6}C_{K',\delta/6}\left(\frac{8r'}{r\varepsilon}\right)^{|\gamma|}\left(\frac{12}{\varepsilon^2}\right)^{|\alpha'|}.\end{aligned} \tag{1.18}$$

By taking the radius $r'=r\varepsilon/16$ of the integral path, we obtain

$$\begin{aligned}|K''_{1,\alpha'}(z,w_1)| &\le \sum_{\alpha'=\tilde{\alpha}'+\beta',\tilde{\alpha}',\beta'\ge 0,\gamma\ge 0}\binom{\tilde{\alpha}'+\beta'}{\beta'}\frac{|\tilde{\alpha}'|!|\beta'|!}{|\alpha'|!}\\ &\quad\times C_{K,\delta_{|\alpha'|}/6}C_{K',\delta_{|\alpha'|}/6}2^{-|\gamma|}\left(\frac{12(|w_1|+\delta_{|\alpha'|})}{\varepsilon^2}\right)^{|\alpha'|}\\ &\le 2^n\left(\frac{48(|w_1|+\delta_{|\alpha'|})}{\varepsilon^2}\right)^{|\alpha'|}\end{aligned} \tag{1.19}$$

for $\forall(z, w_1) \in V_{\frac{r}{2},\frac{\varepsilon}{4},\delta_{|\alpha'|}}$ and sufficiently large $|\alpha'|$. Thus, $K''_{1,\alpha'}$ satisfies (1.4). Concerning the second term of (1.14), we have the following estimates on $\{|z| < r/2, |w_1| < \varepsilon r'/4\}$ for $\tau = r'(\pm 1 + \frac{i\varepsilon}{3})$ because $(z,\tau),(z,w_1-\tau) \in V_{\frac{r}{2},\frac{\varepsilon}{2},\frac{\delta}{3}}$:

$$
(1.20)\qquad \left|\frac{\partial_\tau^j(\tau^{\gamma_1}K_{\tilde{\alpha}'+\gamma'}(z,\tau))\cdot\partial_{w_1}^{|\tilde{\alpha}'+\beta'|-j-1}\partial_z^{\gamma}K'_{\beta'}(z,w_1-\tau)}{\gamma!}\right|
$$

$$
\le (|\alpha'|-1)!C_{K,\delta/6}C_{K',\delta/6}\frac{r'\varepsilon}{3}\left(\frac{8r'}{r\varepsilon}\right)^{|\gamma|}\left(\frac{12}{\varepsilon^2}\right)^{|\alpha'|}\left(\frac{|\tau|+\frac{\delta}{3}}{|w_1-\tau|+\frac{\delta}{3}}\right)^{|\tilde{\alpha}'|-j}
$$

$$
\le (|\alpha'|-1)!C_{K,\delta/6}C_{K',\delta/6}\frac{r'\varepsilon}{3}2^{-|\gamma|}\left(\frac{36}{\varepsilon^2}\right)^{|\alpha'|}.
$$

Here we used that $3^{-1} < (|\tau|+\frac{\delta}{3})/(|w_1-\tau|+\frac{\delta}{3}) < 3$ for $\delta \ll r'$. Thus, the argument goes in a similar way to the case $K''_{1,\alpha'}$, and we obtain the estimate (1.5) for $K''_{2,\alpha'}$. Concerning the terms of (1.16), we have $(z, w_1(t_1)), (z, -i\delta - w_1(t_1)) \in V_{\frac{r}{2},\frac{\varepsilon}{2},\frac{\delta}{3}}$ with $\delta = \delta_{|\alpha'|}$. Hence, for the first and the second terms of (1.16) we get the estimates similar to (1.18) because $\partial_{w_1}^j(w_1^{\gamma_1}K_{\beta'}(z,w_1))$ and $\partial_{w_1}^j\partial_z^{\gamma}K'_{\beta'}(z,w_1)$ have good estimates for $j \le |\beta'|$ near $w_1 = 0$ as stated at the remark of Lemma 1.2. Further for the third term of (1.16) we get the estimate similar to (1.20). Consequently we obtain the estimate similar to (1.19) for $K''_{3,\alpha'}$ except that the factors $(w_1 + i\delta_{|\alpha'|})^j/j!$ contribute to the estimate as $(2r')^j/j!$ because j varies between 0 to $|\alpha'|-1$. Since each term includes w_1 only in the factors $(w_1+i\delta_{|\alpha'|})^j/j!$, we get an estimate (1.17) of $K''_{3,\alpha'}$. □

§ 2. Formal kernel functions

As seen in Introduction, the space of kernel functions for $\mathscr{E}_X^{\mathbb{R}}$ is not closed under the explicit composition rule (1.8). We show one solution to revive this explicit composition rule by giving an extension of the usual space of kernel functions. We give only the definition at a microlocal point $(0; idx_1)$.

Definition 2.1. (**Formal Kernel functions**) Let $r_0, r, \varepsilon(< 1)$ be positive numbers, $S = (A_j)_{j=1}^{\infty}$ be a sequence of positive numbers converging to 0 as $j \to \infty$ satisfying the condition:

$$
(2.1)\qquad d \le A_{j+1}/A_j \le 1 \quad (\text{for all } j \ge 1),
$$

where $d(< 1)$ is a positive constant independent of j. A sequence $(K_j(z, \tilde{z} - z))_{j=1}^{\infty}$ of holomorphic functions is called a formal kernel function of type (r, ε, S, r_0) if

(i) each $K_j(z,w)$ has the form:

$$K_j(z,w) = \sum_{\alpha'\geq 0} K_{j,\alpha'}(z,w_1)\frac{1}{(w')^{\alpha'+\mathbf{1}_{n-1}}}.$$

Here the coefficient $K_{j,\alpha'}$ is holomorphic on $V_{r,\varepsilon,0}$ with estimates:

$$\sup_{V_{r,\varepsilon,\delta,\alpha'}} \left(\frac{\varepsilon}{\max\{A_j,|w_1|\}+\delta}\right)^{|\alpha'|} |K_{j,\alpha'}(z,w_1)| =: C_{K,\delta,j} < \infty \text{ (for all } \delta > 0). \tag{2.2}$$

(see (1.3) concerning $V_{r,\varepsilon,\delta}$).

(ii) For every j, $K_{j+1,\alpha'}(z,w_1) - K_{j,\alpha'}(z,w_1)$ extends holomorphically to $V^0_{r,r_0,j} := V^0_{r,r_0A_{j+1}} = \{|z|<r, |w_1|<r_0A_{j+1}\}$ and satisfies

$$\begin{aligned} &|K_{j+1,\alpha'}(z,w_1) - K_{j,\alpha'}(z,w_1)| \\ &\leq D_{K,\delta,j}\left(\frac{\max\{A_j,|w_1|\}+\delta}{\varepsilon}\right)^{|\alpha'|} \text{ on } V^0_{r,r_0,j} \text{ (for all } \delta>0) \end{aligned} \tag{2.3}$$

with some constants $D_{K,\delta,j} > 0$ depending only on δ, j.

We denote the space of all formal kernel functions by $\hat{K}_{r,\varepsilon,S,r_0}$. Further, a formal kernel function $(K_j)_j \in \hat{K}_{r,\varepsilon,S,r_0}$ is called a null element if there exist some $r_1 > 0$, and some j_0 such that each $K_{j_0,\alpha'}$ extends holomorphically to $V^0_{r,r_1} = \{|z|<r, |w_1|<r_1\}$ and satisfies

$$|K_{j_0,\alpha'}(z,w_1)| \leq C'_{K,\delta}\left(\frac{\max\{A_{j_0},|w_1|\}+\delta}{\varepsilon}\right)^{|\alpha'|} \text{ on } V^0_{r,r_1} \text{ (for all } \delta>0) \tag{2.4}$$

with some constants $C'_{K,\delta} > 0$ depending only on δ. We denote the space of all null formal kernel functions by $\hat{N}_{r,\varepsilon,S,r_0}$. We consider the quotient space $\hat{K}_{r,\varepsilon,S,r_0}/\hat{N}_{r,\varepsilon,S,r_0}$. By the arguments in Introduction, we can embed the usual space of kernel functions of $\mathscr{E}^{\mathbb{R}}_X|_{(0;idx_1)}$ into $\hat{K}_{r,\varepsilon,S,r_0}/\hat{N}_{r,\varepsilon,S,r_0}$. Further for $r \leq r', \varepsilon \leq \varepsilon', r_0 \leq r'_0$ we have the inclusions $\hat{K}_{r,\varepsilon,S,r_0} \supset \hat{K}_{r',\varepsilon',S,r'_0}, \hat{N}_{r,\varepsilon,S,r_0} \supset \hat{N}_{r',\varepsilon',S,r'_0}$.

Definition 2.2. (**Explicit composition for formal kernel functions**) Let $K = (K_j)_j, K' = (K'_j)_j \in \hat{K}_{r,\varepsilon,S,r_0}$ be formal kernel functions. Under the condition

$$A_1 < \varepsilon r/2 \tag{2.5}$$

for the parameters, we define explicit composition

$$(K,K') \to K''(z,\tilde{z}-z)d\tilde{z} = \left(\int K(z,\hat{z}-z)K'(\hat{z},\tilde{z}-\hat{z})\right)d\tilde{z}$$

of K, K' by

$$(2.6)\qquad K''_{j,\alpha'}(z,w_1) := (2\pi i)^{n-1} \sum_{\alpha'=\tilde{\alpha}'+\beta',\tilde{\alpha}',\beta'\geq 0,\gamma\geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \int_0^{2\pi} w_1^j(t_1)^{\gamma_1} \times \frac{K_{j,\tilde{\alpha}'+\gamma'}(z,w_1^j(t_1))\partial_z^\gamma K'_{j,\beta'}(z,w_1-w_1^j(t_1))}{\gamma!} dw_1^j(t_1).$$

Here, the integral path Γ_j is taken as

$$(2.7)\qquad \Gamma_j\,:\; w_1^j(t_1) := \frac{(t_1-\pi)}{\pi} r'_j + i\left(\varepsilon' r'_j \frac{|t_1-\pi|}{\pi} - \varepsilon''\left(1-\frac{|t_1-\pi|}{\pi}\right)\right)$$

with some positive constants $\varepsilon'(<\varepsilon), \varepsilon'' \ll \varepsilon'$ and $r'_j = r'_0 A_j/\sqrt{1+\varepsilon'^2}$ for each $j \geq 1$. Here, r'_0 is some constant with $0 < r'_0 \leq r_0/2$.

This definition of composition is considered as an extension of the explicit composition (1.8) for the kernel functions for $\mathscr{E}_X^{\mathbb{R}}$. Indeed, for $[K(z,w)d\tilde{z}], [K'(z,w)d\tilde{z}] \in \mathscr{E}_X^{\mathbb{R}}$, the composition (1.8) uses 2 terminal points concerning integration on w_1 independent of j, but the formula in this definition uses 2 terminal points depending on j.

Theorem 2.3. *Let $K = (K_j)_j, K' = (K'_j)_j \in \hat{K}_{r,\varepsilon,S,r_0}$ be formal kernel functions. Then the explicit composition K'' of K, K' belongs to $\hat{K}_{\frac{r}{2},\frac{\varepsilon}{3},S,\frac{\varepsilon r'_0}{3}}$. Further if any of K, K' belongs to $\hat{N}_{r,\varepsilon,S,r_0}$, then $K'' \in \hat{N}_{\frac{r}{2},\frac{\varepsilon}{3},S,\frac{\varepsilon r'_0}{3}}$.*

Proof. It suffices to show (2.2),(2.3) for sufficiently small δ. Hereafter, we take $\varepsilon' = 2\varepsilon/3, \varepsilon'' = \delta/2$ for the integral path (2.7). Since $(z, w_1(t_1)), (z, w_1-w_1(t_1)) \in V_{\frac{r}{2},\varepsilon,\frac{\delta}{3}}$

for $(z, w_1) \in V_{\frac{r}{2},\frac{\varepsilon}{3},\delta}$, we have

$$
\begin{aligned}
&|K''_{j,\alpha'}(z, w_1)| \\
&\leq (2\pi)^{n-1} \sum_{\alpha'=\tilde{\alpha}'+\beta',\tilde{\alpha}',\beta'\geq 0,\gamma\geq 0} \binom{\tilde{\alpha}'+\beta'}{\beta'} \left| \int_0^{2\pi} w_1^j(t_1)^{\gamma_1} \right. \\
&\times \left. \frac{K_{j,\tilde{\alpha}'+\gamma'}(z, w_1^j(t_1))\partial_z^\gamma K'_{j,\beta'}(z, w_1 - w_1^j(t_1))}{\gamma!} dw_1^j(t_1) \right| \\
&\leq (2\pi)^{n-1} \sum_{\alpha'=\tilde{\alpha}'+\beta',\tilde{\alpha}',\beta'\geq 0,\gamma\geq 0} \binom{\alpha'}{\beta'} \int_0^{2\pi} |w_1^j(t_1)|^{\gamma_1} |dw_1^j(t_1)| \\
&\times \frac{C_{K,\frac{\delta}{3},j}}{\gamma!} \left(\frac{\max\{A_j, |w_1^j(t_1)|\} + \frac{\delta}{3}}{\varepsilon} \right)^{|\tilde{\alpha}'+\gamma'|} \\
&\times C_{K',\frac{\delta}{3},j}\gamma! \left(\frac{2}{r}\right)^{|\gamma|} \left(\frac{\max\{A_j, |w_1 - w_1^j(t_1)|\} + \frac{\delta}{3}}{\varepsilon} \right)^{|\beta'|} \\
&\leq (2\pi)^n C_{K,\frac{\delta}{3},j} C_{K',\frac{\delta}{3},j} \sum_{\alpha'=\tilde{\alpha}'+\beta',\tilde{\alpha}',\beta'\geq 0,\gamma\geq 0} \binom{\alpha'}{\beta'} A_j^{\gamma_1+1} \\
&\times \left(\frac{A_j+\delta}{\varepsilon}\right)^{|\tilde{\alpha}'+\gamma'|} \left(\frac{2}{r}\right)^{|\gamma|} \left(\frac{|w_1| + A_j + \delta}{\varepsilon}\right)^{|\beta'|} \\
&\leq (2\pi)^n C_{K,\frac{\delta}{3},j} C_{K',\frac{\delta}{3},j} \left(\frac{\max\{A_j, |w_1|\} + \delta}{\varepsilon/2} \right)^{|\alpha'|} \\
&\times \sum_{\alpha'=\tilde{\alpha}'+\beta',\tilde{\alpha}',\beta'\geq 0,\gamma\geq 0} \binom{\alpha'}{\beta'} \left(\frac{2A_j}{r}\right)^{\gamma_1} \left(\frac{A_j+\delta}{\varepsilon r/2}\right)^{|\gamma'|} \left(\frac{1}{2}\right)^{|\tilde{\alpha}'|}.
\end{aligned}
$$

Since $A_j < \varepsilon r/2$, there exists some constant $0 < \tilde{C} < 1$ such that for sufficiently small $\delta > 0$

$$\frac{A_j + \delta}{\varepsilon r/2} < \tilde{C}.$$

Then, we have

$$
\begin{aligned}
&|K''_{j,\alpha'}(z, w_1)| \\
&\leq (2\pi)^n C_{K,\frac{\delta}{3},j} C_{K',\frac{\delta}{3},j} \left(\frac{\max\{A_j, |w_1|\} + \delta}{\varepsilon/2} \right)^{|\alpha'|} \left(\frac{3}{2}\right)^{|\alpha'|} \frac{1}{1-\varepsilon} \left(\frac{1}{1-\tilde{C}}\right)^{n-1} \\
&\leq C_{K'',\delta,j} \left(\frac{\max\{A_j, |w_1|\} + \delta}{\varepsilon/3} \right)^{|\alpha'|}.
\end{aligned}
$$

Here we set

$$C_{K'',\delta,j} := (2\pi)^n C_{K,\frac{\delta}{3},j} C_{K',\frac{\delta}{3},j} \frac{1}{1-\varepsilon} \left(\frac{1}{1-\tilde{C}}\right)^{n-1}.$$

Therefore, K'' satisfies (2.2). Further, for $(z,w_1)\in V_{\frac{r}{2},\frac{\varepsilon}{3},\delta}$, we obtain

$$\begin{aligned}
&K''_{j+1,\alpha'}(z,w_1)-K''_{j,\alpha'}(z,w_1)\\
&=(2\pi)^{n-1}\sum_{\alpha'=\tilde\alpha'+\beta',\tilde\alpha',\beta'\ge 0,\gamma\ge 0}\binom{\tilde\alpha'+\beta'}{\beta'}\Bigg[\int_{\Gamma_{j+1}}\tau^{\gamma_1}d\tau\\
&\times\Bigg(\frac{(K_{j+1,\tilde\alpha'+\gamma'}(z,\tau)-K_{j,\tilde\alpha'+\gamma'}(z,\tau))\,\partial_z^\gamma K'_{j+1,\beta'}(z,w_1-\tau)}{\gamma!}\\
&+\frac{K_{j,\tilde\alpha'+\gamma'}(z,\tau)\partial_z^\gamma\left(K'_{j+1,\beta'}(z,w_1-\tau)-K'_{j,\beta'}(z,w_1-\tau)\right)}{\gamma!}\Bigg)\\
&+\int_{\Gamma_{j+1}-\Gamma_j}\tau^{\gamma_1}d\tau\frac{K_{j,\tilde\alpha'+\gamma'}(z,\tau)\partial_z^\gamma K'_{j,\beta'}(z,w_1-\tau)}{\gamma!}\Bigg].
\end{aligned}$$

Hence we can modify the integral paths as follows: For the first term, we take

$$\tau_j^1(t)=r_0'A_{j+1}\exp\left(i\pi(1-t)+i(2t-1)\arctan\frac{2\varepsilon}{3}\right)\ (0\le t\le 1).$$

For the second term,

$$\tau_j^2(t)=r_0'A_{j+1}\exp\left(-\frac{\pi}{2}i-\left(\frac{\pi}{2}+\arctan\frac{2\varepsilon}{3}\right)(1-2t)\,i\right)\ (0\le t\le 1).$$

For the third term, the union of two line segments:

$$[\tau_{j-1}^1(0),\tau_j^1(0)]\cup[\tau_j^1(1),\tau_{j-1}^1(1)].$$

Therefore the three terms extend holomorphically to $V^0_{\frac{r}{2},\frac{\varepsilon r_0'}{3},j}$.

Consequently $K''_{j+1,\alpha'}(z,w_1)-K''_{j,\alpha'}(z,w_1)$ extends holomorphically to $V^0_{\frac{r}{2},\frac{\varepsilon r_0'}{3},j}$ and satisfies the following estimate:

$$\begin{aligned}
&|K''_{j+1,\alpha'}(z,w_1)-K''_{j,\alpha'}(z,w_1)|\\
&\le\left(D_{K,\delta,j}C_{K',\frac{\varepsilon r'_{j+1}}{3},j+1}+C_{K,\frac{\varepsilon r'_{j+1}}{3},j}D_{K',\delta,j}+C_{K,\frac{\varepsilon r'_{j+1}}{3},j}C_{K',\frac{\varepsilon r'_{j+1}}{3},j}\right)\\
&\times\left(\frac{8}{5}\right)^{|\alpha'|}\frac{(2\pi)^n}{1-\varepsilon}\left(\frac{1}{1-\tilde C}\right)^{n-1}\left(\frac{\max\{A_j,|w_1|\}+\delta}{2\varepsilon/5}\right)^{|\alpha'|}\\
&\le D_{K'',\delta,j}\left(\frac{\max\{A_j,|w_1|\}+\delta}{\varepsilon/4}\right)^{|\alpha'|}.
\end{aligned}$$

Here, we set

$$\begin{aligned}
D_{K'',\delta,j}:=\Bigg(&D_{K,\delta,j}C_{K',\frac{\varepsilon r'_{j+1}}{3},j+1}+C_{K,\frac{\varepsilon r'_{j+1}}{3},j}D_{K',\delta,j}\\
&+C_{K,\frac{\varepsilon r'_{j+1}}{3},j}C_{K',\frac{\varepsilon r'_{j+1}}{3},j}\Bigg)\cdot\frac{(2\pi)^n}{1-\varepsilon}\left(\frac{1}{1-\tilde C}\right)^{n-1}.
\end{aligned}$$

Thus, K'' satisfies (2.3), and $K'' \in \hat{K}_{\frac{r}{2},\frac{\varepsilon}{4},S,\frac{\varepsilon r_0'}{3}}$. Moreover, suppose that K belongs to $\hat{N}_{r,\varepsilon,S,r_0}$. Then we can choose $r_1 > 0$ and $j_0 \in \mathbb{N}$ such that each $K_{j_0,\alpha'}$ extends holomorphically to V^0_{r,r_1} and satisfies (2.4). We set $\tilde{r_1} := \min\{r_1, r'_{j_0}\}$, and for $(z, w_1) \in V^0_{\frac{r}{2},\frac{\varepsilon\tilde{r_1}}{3}}$, by the same arguments we obtain

$$\left|K''_{j_0,\alpha'}(z,w_1)\right| \le C'_{K'',\delta}\left(\frac{\max\{A_{j_0},|w_1|\}+\delta}{\varepsilon/4}\right)^{|\alpha'|}$$

$$C'_{K'',\delta} := \max\left\{C'_{K,\delta}, C_{K,\frac{\tilde{r_1}}{2},j_0}\right\} C_{K',\frac{\varepsilon\tilde{r_1}}{3},j_0}\frac{(2\pi)^n}{1-\varepsilon}\left(\frac{1}{1-\tilde{C}}\right)^{n-1}.$$

Therefore, $K'' \in \hat{N}_{\frac{r}{2},\frac{\varepsilon}{4},S,\frac{\varepsilon r_0'}{3}}$ and the other case is the same. □

Following Aoki's symbol theory of $\mathscr{E}^{\mathbb{R}}_X$ ([A1]), we can define a symbol (not a formal symbol!) for a formal kernel.

Definition 2.4. For a formal kernel $K = (K_j)_j \in \hat{K}_{r,\varepsilon,S,r_0}$, we define a symbol $\sigma(K)(z,\zeta)$ by

$$\begin{aligned}\sigma(K)(z,\zeta) &:= \int_\Gamma K_1(z,w)e^{\langle w,\zeta\rangle}dw \\ &= \sum_{\alpha'\ge 0}\int_\Gamma K_{1,\alpha'}(z,w_1)\frac{e^{\langle w,\zeta\rangle}}{(w')^{\alpha'+\mathbf{1}_{n-1}}}dw \\ &= (2\pi i)^{n-1}\sum_{\alpha'\ge 0}\int_0^{2\pi} K_{1,\alpha'}(z,w_1(t_1))\frac{\zeta'^{\alpha'}e^{w_1(t_1)\zeta_1}}{\alpha'!}dw_1(t_1).\end{aligned} \tag{2.8}$$

Here we take the integral path Γ defined at (1.6) with $w_1(t_1)$ ($t_1 \in [0, 2\pi]$) similar to $w_1^1(t_1)$, but we impose only the condition $0 < r_1' \le A_1$ for r_1'.

Theorem 2.5. *For a formal kernel $K = (K_j)_j \in \hat{K}_{r,\varepsilon,S,r_0}$, the symbol $\sigma(K)(z,\zeta)$ is a usual symbol for a germ of $\mathscr{E}^{\mathbb{R}}_X|_{(0;idx_1)}$; that is, $\sigma(K)(z,\zeta)$ is holomorphic on $\Omega := \{(z,\zeta) \in \mathbb{C}^{n+n};\ |z| < r, |\zeta'| + |\operatorname{Re}\zeta_1| < \varepsilon' \operatorname{Im}\zeta_1\}$ for some $\varepsilon' > 0$, and satisfies the growth condition:*

$$|\sigma(K)(z,\zeta)| \le B_\delta e^{\delta|\zeta|} \text{ on } \Omega \text{ for all } \delta > 0. \tag{2.9}$$

Here B_δ is a positive constant independent of z, ζ. Further, if $K \in \hat{N}_{r,\varepsilon,S,r_0}$, then $\sigma(K)(z,\zeta)$ is exponentially decreasing on Ω as $|\zeta| \to \infty$.

Proof. For $(z,\zeta) \in \Omega$, we can assume the followings:

1. $|\zeta_j| \le \varepsilon'|\zeta_1|$ for all $j \ge 2$,

2. $\operatorname{Re}(w_1\zeta_1) \le -\tilde{\varepsilon}|w_1||\zeta_1|$ on $\left\{w_1 \in \mathbb{C} \,;\, \frac{\varepsilon}{2}|\operatorname{Re} w_1| \le \operatorname{Im} w_1\right\}$ for some $\tilde{\varepsilon} > 0$.

Then, on Ω,

$$\begin{aligned}
&\left|\int_\Gamma K_1(z,w)e^{\langle w,\zeta\rangle}dw\right| \\
&= \left|\int_\Gamma \left\{\sum_{j=1}^{i-1}(K_j(z,w) - K_{j+1}(z,w)) + K_i(z,w)\right\} e^{\langle w,\zeta\rangle}dw\right| \\
&\le \sum_{j=1}^{i-1}\left|\int_\Gamma (K_j(z,w) - K_{j+1}(z,w))\, e^{\langle w,\zeta\rangle}dw\right| + \left|\int_\Gamma K_i(z,w)e^{\langle w,\zeta\rangle}dw\right|.
\end{aligned}$$

Here we change Γ into a path $\widetilde{\Gamma}_j :=$

$$\left\{\left(w_1^j(t_1), \frac{2\max\{A_j, |w_1^j(t_1)|\}}{\varepsilon}e^{it_2}, ..., \frac{2\max\{A_j, |w_1^j(t_1)|\}}{\varepsilon}e^{it_n}\right)\right\}$$

$(t_1, ..., t_n \in [0, 2\pi])$ with

$$w_1^j(t_1) := \begin{cases} r_1' e^{i(\pi-\theta_\varepsilon)}(1-t_1) & (t_1 \in [0,\delta_j]), \\ r_1'(1-\delta_j)\exp\left(i\frac{(\pi-\theta_\varepsilon)(1-\delta_j-t)+\theta_\varepsilon(t-\delta_j)}{1-2\delta_j}\right) & (t_1 \in [\delta_j, 1-\delta_j]), \\ r_1' e^{i\theta_\varepsilon} t_1 & (t_1 \in [1-\delta_j, 1], \end{cases}$$

where $\theta_\varepsilon = \arctan(\varepsilon/2), \delta_j = 1 - (\min\{r_0 A_{j+1}/2, r_1'\}/r_1')$. Then we can find some $M_j > 0$ such that

$$\begin{aligned}
(2.10)\qquad &\left|\int_{\tilde{\Gamma}_j} (K_j(z,w) - K_{j+1}(z,w))\, e^{\langle w,\zeta\rangle}dw\right| \\
&\le M_j \int_{\tilde{\Gamma}_j} \exp\left(-\tilde{\varepsilon}|w_1^j(t_1)||\zeta_1| + \frac{2\max\{A_j, |w_1^j(t_1)|\}}{\varepsilon}(n-1)\varepsilon'|\zeta_1|\right)|dw|.
\end{aligned}$$

We divide this integral into $\int_{\tilde{\Gamma}_j \cap \{|w_1| < A_j\}} + \int_{\tilde{\Gamma}_j \cap \{|w_1| \ge A_j\}}$. Hence, if $-\tilde{\varepsilon} + (2(n-1)\varepsilon'/\varepsilon) < 0$, we have

$$\begin{aligned}
(2.11)\qquad (2.10) \le &(2\pi)^n M_j \left(\exp\left(\left(-\tilde{\varepsilon}\min\left\{\frac{r_0 A_{j+1}}{2}, r_1'\right\} + \frac{2A_j(n-1)\varepsilon'}{\varepsilon}\right)|\zeta_1|\right)\right. \\
&\left. + \exp\left(\left(-\tilde{\varepsilon} + \frac{2(n-1)\varepsilon'}{\varepsilon}\right)A_j|\zeta_1|\right)\right).
\end{aligned}$$

Since $A_j \to 0$ as $j \to \infty$, we can take some j_0 such that $r_0 A_{j_0} < 2r_1'$. Hence we can choose ε' such that the right side of (2.11) is exponentially decreasing as $|\zeta_1| \to \infty$; that is, ε' is a positive number smaller than the following ε_0':

$$\frac{\varepsilon\tilde{\varepsilon}}{2(n-1)}\min\left\{\inf_j \frac{\min\{r_0A_{j+1}, 2r_1'\}}{2A_j}, 1\right\}$$
$$\geq \frac{\varepsilon\tilde{\varepsilon}}{2(n-1)}\min\left\{\inf_{1\leq j<j_0-1} \frac{\min\{r_0A_{j+1}, 2r_1'\}}{2A_j}, \frac{r_0}{2d}, 1\right\} =: \varepsilon_0' > 0.$$

On the other hand, we have for some $M_i > 0$ such that

$$\left|\int_\Gamma K_i(z,w)e^{\langle w,\zeta\rangle}dw\right| \leq M_i \exp\left(\left(1+\frac{2(n-1)\varepsilon'}{\varepsilon}\right)A_i|\zeta_1|\right).$$

Since, for any $\delta > 0$, we can choose i as

$$\left(1+\frac{2(n-1)\varepsilon'}{\varepsilon}\right)A_i < \delta,$$

we find that $\sigma(K)(z,\zeta)$ satisfies (2.9). Moreover, if $K \in \hat{N}_{r,\varepsilon,S,r_0}$, since for some i_0, K satisfies (2.4), we obtain the following estimates:

$$\left|\int_\Gamma K_{i_0}(z,w)e^{\langle w,\zeta\rangle}dw\right| \leq M_{i_0} \exp\left(\left(\left(-\frac{\tilde{\varepsilon}\tilde{r_1}}{2}+\frac{2(n-1)\varepsilon' A_{i_0}}{\varepsilon}\right)|\zeta_1|\right)\right.$$
$$\left. + \exp\left(\left(-\tilde{\varepsilon}+\frac{2(n-1)\varepsilon'}{\varepsilon}\right)\frac{A_{i_0}}{2}|\zeta_1|\right)\right).$$

Here, we set $\tilde{r_1} := \min\{r_1, r_1'\}$ and r_1 is the constant that appears in (2.4). By taking ε' sufficiently small, we can suppose $\varepsilon' > 0$ satisfies

$$-\frac{\tilde{\varepsilon}\tilde{r_1}}{2}+\frac{2(n-1)\varepsilon' A_{i_0}}{\varepsilon} < 0\ .$$

Thus, $\sigma(K)(z,\zeta)$ is exponentially decreasing on Ω as $|\zeta| \to \infty$. □

By following the argument of [AKY2], we can show the following theorem:

Theorem 2.6. *Let $K = (K_j)_j, K' = (K_j')_j \in \hat{K}_{r,\varepsilon,S,r_0}$ be formal kernel functions, and $\sigma(K)(z,\zeta), \sigma(K')(z,\zeta)$ be their symbols respectively.*

(a) Suppose that $A_1 < \varepsilon r/4$. Then the series of the Leibniz composition

$$\sigma(K)\circ\sigma(K')(z,\zeta) := \sum_{\gamma\geq 0}\frac{1}{\gamma!}\partial_\zeta^\gamma\sigma(K)(z,\zeta)\cdot\partial_z^\gamma\sigma(K')(z,\zeta)$$

defines a symbol in a small conic neighborhood of $(0; idx_1)$.

(b) Suppose that $A_1 < \varepsilon r/4$. Let K'' be the explicit composition of K, K'. Then the difference of symbol $\sigma(K'')$ and the Leibniz composition as above is an exponentially decreasing symbol in a small conic neighborhood of $(0; idx_1)$.

Proof.

(a) Let $\sigma(K)(z,\zeta)$ and $\sigma(K')(z,\zeta)$ be holomorphic on $\{(z,\zeta)\in\mathbb{C}^{n+n};\ |z|<r,|\zeta'|+|\operatorname{Re}\zeta_1|<\varepsilon'\operatorname{Im}\zeta_1\}$ and satisfy (2.9). Then, for $(z,\zeta)\in\{|z|<r/2,|\zeta'|+|\operatorname{Re}\zeta_1|<\varepsilon'\operatorname{Im}\zeta_1\}$,

$$\begin{aligned}\left|\partial_\zeta^\gamma\sigma(K)(z,\zeta)\right| &= \left|\partial_\zeta^\gamma\int_\Gamma K_1(z,w)e^{\langle w,\zeta\rangle}dw\right|\\ &= \left|\int_\Gamma w^\gamma K_1(z,w)e^{\langle w,\zeta\rangle}dw\right|\\ &\le \left(\frac{2A_1}{\varepsilon}\right)^{|\gamma|}B_\delta e^{\delta|\zeta|} \text{ for all } \delta>0\\ |\partial_z^\gamma\sigma(K')(z,\zeta)| &\le \frac{\gamma!}{(r/2)^{|\gamma|}}B_\delta e^{\delta|\zeta|} \text{ for all } \delta>0\ .\end{aligned}$$

Therefore, we have the following estimates:

$$\sum_{\gamma\ge0}\left|\frac{1}{\gamma!}\partial_\zeta^\gamma\sigma(K)(z,\zeta)\cdot\partial_z^\gamma\sigma(K')(z,\zeta)\right|\le\left(\frac{1}{1-4A_1/\varepsilon r}\right)^n B_\delta^2e^{2\delta|\zeta|}.$$

Hence, $\sigma(K)\circ\sigma(K')(z,\zeta)$ defines a symbol.

(b) We take the radius of $\tilde\Gamma$ sufficiently smaller than that of Γ. Then, we find

$$\begin{aligned}&\sum_{\gamma\ge0}\frac{1}{\gamma!}\partial_\zeta^\gamma\sigma(K)(z,\zeta)\cdot\partial_z^\gamma\sigma(K')(z,\zeta)\\ &=\sum_{\gamma\ge0}\frac{1}{\gamma!}\int_{\tilde\Gamma}\tilde w^\gamma K_1(z,\tilde w)e^{\langle\tilde w,\zeta\rangle}d\tilde w\cdot\int_\Gamma\partial_z^\gamma K_1'(z,w)e^{\langle w,\zeta\rangle}dw\\ &=\int_{\tilde\Gamma}\int_\Gamma K_1(z,\tilde w)K_1'(z+\tilde w,w)e^{\langle w+\tilde w,\zeta\rangle}dwd\tilde w\\ &=\int_{\tilde\Gamma}\int_{\Gamma+\{\tilde w\}}K_1(z,\tilde w)K_1'(z+\tilde w,w-\tilde w)e^{\langle w,\zeta\rangle}dwd\tilde w\\ &=\int_{\tilde\Gamma}\int_\Gamma K_1(z,\tilde w)K_1'(z+\tilde w,w-\tilde w)e^{\langle w,\zeta\rangle}dwd\tilde w\\ &+\int_{\tilde\Gamma}\int_{\{\Gamma+\{\tilde w\}\}-\Gamma}K_1(z,\tilde w)K_1'(z+\tilde w,w-\tilde w)e^{\langle w,\zeta\rangle}dwd\tilde w.\end{aligned}$$

Therefore, the difference $\sigma(K)\circ\sigma(K')(z,\zeta)-\sigma(K'')(z,\zeta)$ is

$$\int_{\tilde\Gamma}\int_{\{\Gamma+\{\tilde w\}\}-\Gamma}K_1(z,\tilde w)K_1'(z+\tilde w,w-\tilde w)e^{\langle w,\zeta\rangle}dwd\tilde w.$$

Since the radius of $\tilde\Gamma$ sufficiently smaller than that of Γ, we can take two paths $[w_1(0)+\tilde w_1,w_1(0)]$ and $[w_1(2\pi)+\tilde w_1,w_1(2\pi)]$ such that on the paths $\operatorname{Re}\langle w,\zeta\rangle\le-\tilde\varepsilon|\zeta|$

for some $\tilde{\varepsilon} > 0$. Therefore, by Cauchy's integration theorem, we find that the difference is exponentially decreasing in some conic neighborhood of $(0; idx_1)$.

□

Remark 1. We constructed symbols of pseudo-differential operators $\mathscr{E}_X^{\mathbb{R}}$ from formal kernel functions (Theorem 2.5) and checked the compatibility of explicit composition and Leibniz rule (Theorem 2.6). But, since ε and r must satisfy the condition (2.5), we can not take the stalk of formal kernel functions for fixed $S = (A_j)_{j=1}^{\infty}$. In the recent work in master's thesis of Kamimoto ([K]), we defined the space of formal kernel functions by other formulation and proved that the stalk of operators defined by formal kernel functions is isomorphic to that of $\mathscr{E}_X^{\mathbb{R}}$.

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