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# Asymptotic analysis to Goursat problems

By

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**Abstract**

This paper deals with a new proof of solvability and uniqueness of solutions of initial-boundary value problems via an analytic continuation with respect to a certain parameter in the equation.

## § 1. Introduction

Let $x = (x_1, x_2) \in \mathbb{C}^2$ and $\partial_{x_j} = \frac{\partial}{\partial x_j}$. Let $N \geq 1$ be an integer. For real constants $a_j$ $(\pm j = 1, 2, \ldots, N)$ let the operator $L_0$ be given by

$$L_0 := \partial_{x_1}^N \partial_{x_2}^N - \sum_{j=1}^{N} \left( a_j \partial_{x_1}^{N-j} \partial_{x_2}^{N+j} + a_{-j} \partial_{x_1}^{N+j} \partial_{x_2}^{N-j} \right). \tag{1.1}$$

We consider the Goursat problem

$$L_0 v = f(x), \ \ \partial_{x_\nu}^j v\big|_{x_\nu = 0} = 0, \ \ (j = 0, 1, \ldots, N-1; \nu = 1, 2), \tag{1.2}$$

where $f(x)$ is a given holomorphic function at the origin $x = 0$. The problem (1.2) has a unique holomorphic solution in some neighborhood of the origin if there exists $t > 0$ such that

$$\sum_{j=1}^{N} (|a_j| t^j + |a_{-j}| t^{-j}) < 1. \tag{1.3}$$

The condition (1.3) is called a spectral condition. This theorem was proved by a majorant method in [3]. On the other hand, if (1.3) does not hold, then the solvability

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of (1.2) is a delicate problem because small denominator difficulties may enter in the analysis, which was shown by Leray's pioneering work (cf. [5]). We also note that, if a small denominator difficulty occurs, then a majorant method does not work well. In this note, we present a new method based on an analytic continuation with respect to a certain parameter in the equation in order to show the solvability and uniqueness of solutions of (1.2) in case (1.3) does not hold. We note that the above argument can also be applied to the Diophantine case if we generalize the notion of the so-called Leray-Pisot function of number theory appropriately. Finally I would like to appreciate the anonymous referee for suggestions towards the improvement of the redundant argument.

## § 2. Spectral radius and asymptotic solutions

In order to construct an asymptotic solution we first transform (1.2) to an equivalent integro-differential equation, and we introduce an asymptotic parameter $\varepsilon$ in the equation. For a holomorphic function $w(x)$ near the origin, we define the integration operator $\partial_{x_j}^{-1}$ by $\partial_{x_j}^{-1} w(x) = \int_0^{x_j} w ds_j$, where the integral in the right-hand side is done with respect to the $j$-th variable from the origin 0 to $x_j$. In order to make an asymptotic analysis we introduce a new parameter $\varepsilon$ in front of $\partial_{x_1}^N \partial_{x_2}^N$ in (1.2). Next we introduce an unknown function $u(x)$ by

$$v(x) = \partial_{x_1}^{-N} \partial_{x_2}^{-N} u(x), \quad \partial_{x_j}^{-N} = (\partial_{x_j}^{-1})^N.$$

Then (1.2) is written in the following form

$$(\varepsilon - \mathcal{L})\, u = f, \tag{2.1}$$

where

$$\mathcal{L} = \sum_{j=1}^{N} \left( a_j \partial_{x_1}^{-j} \partial_{x_2}^{j} + a_{-j} \partial_{x_1}^{j} \partial_{x_2}^{-j} \right). \tag{2.2}$$

We assume that $f$ has the following convergent expansion in $\varepsilon^{-1}$

$$f = f_0 + f_1 \varepsilon^{-1} + f_2 \varepsilon^{-2} + \cdots, \tag{2.3}$$

where we assume that $f_j = f_j(x)$ are holomorphic in some common neighborhood of the origin $x = 0$. We construct an asymptotic solution $u$ in the following form

$$u = u_0(x)\varepsilon^{-1} + u_1(x)\varepsilon^{-2} + u_2(x)\varepsilon^{-3} + \cdots \tag{2.4}$$

By substituting (2.4) into (2.1) we obtain the following recurrence relations

$$u_0 = f_0, \quad u_1 = f_1 + \mathcal{L} f_0, \ \cdots,$$
$$u_n = f_n + \mathcal{L} f_{n-1} + \cdots + \mathcal{L}^n f_0, \quad n = 1, 2, \ldots \tag{2.5}$$

In order to study the convergence of the series (2.4) we introduce a function space. For power series $u = \sum_\alpha u_\alpha x^\alpha/\alpha!$ and $\phi(x) = \sum_\alpha \phi_\alpha x^\alpha/\alpha!$ we say that $\phi$ is a majorant series of $u$ and denote it by $u \ll c\phi$ if there exists $c \geq 0$ such that for every $\nu$, $\nu = 0, 1, 2, \ldots$,

$$(\sum_{|\alpha|=\nu} |u_\alpha|^2)^{1/2} \leq c(\sum_{|\alpha|=\nu} |\phi_\alpha|^2)^{1/2}.$$

In the following we take

$$\phi(x) = \sum_\alpha \frac{|\alpha|!}{R^{|\alpha|}(|\alpha|+1)} \frac{x^\alpha}{\alpha!}.$$

For $R > 0$ let $X \equiv X_R$ be the set of holomorphic functions defined by

$$X_R := \left\{ u = \sum_\alpha u_\alpha \frac{x^\alpha}{\alpha!} \ll c\phi(x) \text{ for some } c \geq 0 \right\}. \tag{2.6}$$

Then $\|u\| := \inf\{c; u \ll c\phi(x)\}$ is the norm of $u$. The space $X_R$ is a Banach space with the norm $\|\cdot\|$. It is easy to see that $\mathcal{L}$ is a continuous linear operator on $X_R$.

Suppose that $f_j \in X$ $(j = 0, 1, 2, \ldots)$ and $\sum_{j=0}^\infty |\varepsilon|^{-j}\|f_j\| < \infty$. Then we want to show that $u = \sum_{j=0}^\infty \varepsilon^{-j-1}u_j$ converges in $X$ if $|\varepsilon| > \|\mathcal{L}\|$. Indeed, in terms of (2.5) we have

$$\begin{aligned} \sum |\varepsilon|^{-j-1}\|u_j\| &\leq \sum |\varepsilon|^{-j-1}(\|f_j\| + \|\mathcal{L}\|\|f_{j-1}\| + \cdots + \|\mathcal{L}\|^j\|f_0\|) \\ &\leq \left(\sum |\varepsilon|^{-j-1}\|f_j\| + |\varepsilon|^{-1}\|\mathcal{L}\| \sum |\varepsilon|^{-(j-1)-1}\|\|f_{j-1}\| + \cdots\right) \\ &\leq \sum |\varepsilon|^{-j-1}\|f_j\| \left(1 + |\varepsilon|^{-1}\|\mathcal{L}\| + \cdots + |\varepsilon|^{-j}\|\mathcal{L}\|^j + \cdots\right). \end{aligned} \tag{2.7}$$

The right-hand side of (2.7) converges if $|\varepsilon| > \|\mathcal{L}\|$. It follows that (2.1) has an analytic solution for every analytic $f$. Especially, for every homogeneous polynomial $f = f_0$ the equation (2.1) has an analytic solution. Because $\mathcal{L}$ maps every homogeneous polynomial to the one with the same degree, $\varepsilon - \mathcal{L}$ is surjective on the set of homogeneous polynomials. It follows that it is injective. Therefore every analytic solution of (2.1) is unique if $|\varepsilon| > \|\mathcal{L}\|$. Summing up the above we have

**Theorem 2.1.** *Suppose that $|\varepsilon| > \|\mathcal{L}\|$. Then (2.1) has a unique analytic solution in some neighbourhood of the origin $x = 0$ given by (2.4).*

**Example 2.2.** We consider the operator

$$\mathcal{L} = \partial_{x_1}^{-1}\partial_{x_2} + \partial_{x_1}\partial_{x_2}^{-1}. \tag{2.8}$$

We can show that $\|\mathcal{L}\| = 2$. As we will see in the following the spectral radius of $\mathcal{L}$ is equal to 2. (cf. [3]).

In the following we call $\|\mathcal{L}\|$ the spectral radius of the Goursat problem (2.1). We see from the above argument that the solution (2.4) is analytic with respect to $\varepsilon$ in the domain $|\varepsilon| > \|\mathcal{L}\|$. We shall study the analytic properties of the solution (2.4) in the set, $|\varepsilon| \le \|\mathcal{L}\|$.

## § 3. Asymptotic solution and solvability

In order to show the solvability inside the spectral radius, $|\varepsilon| \le \|\mathcal{L}\|$, we make the analytic continuation of $u$ in (2.4) with respect to $\varepsilon$. From now we assume that there exists $n_0 \ge 0$ such that

$$f = f_0(x) + f_1(x)\varepsilon^{-1} + \cdots + f_{n_0}(x)\varepsilon^{-n_0}, \tag{3.1}$$

for the sake of simplicity. By (2.5), $u$ can be written in the form

$$\begin{aligned} u &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \varepsilon^{-n-1}\mathcal{L}^{n-j} f_j = \sum_{j=0}^{n_0}\sum_{n\ge j} \varepsilon^{-n-1}\mathcal{L}^{n-j} f_j \\ &= \sum_{j=0}^{n_0}\sum_{n\ge j} \varepsilon^{-(n-j)-j-1}\mathcal{L}^{n-j} f_j = \sum_{j=0}^{n_0} \varepsilon^{-j} \sum_{\nu\ge 0} \varepsilon^{-\nu-1}\mathcal{L}^{\nu} f_j. \end{aligned} \tag{3.2}$$

By simple calculations we can easily see that, if $|\varepsilon| > \|\mathcal{L}\|$, then the right-hand side of (3.2) is equal to

$$\sum_{j=0}^{n_0} \varepsilon^{-j}(\varepsilon - \mathcal{L})^{-1} f_j. \tag{3.3}$$

We shall make the analytic continuation of $u$ with respect to $\varepsilon$ inside the spectral radius. For this purpose we study the spectrum of $\mathcal{L}$ on $X$. In the following we assume that $a_j = a_{-j}$ for $j = 1, 2, \ldots, N$ and $a_j$'s are real. The operator $\mathcal{L}$ maps the set of homogeneous polynomials of degree $\nu$, $H_\nu$ to itself. Hence we consider the restriction $\mathcal{L}_\nu$ of $\mathcal{L}$ to $H_\nu$. We remark that $\mathcal{L}_\nu$ has the same expression (2.2). We expand $u \in H_\nu$ and $v \in H_\nu$ as $u = \sum_{|\alpha|=\nu} u_\alpha x^\alpha/\alpha!$ and $v = \sum_{|\alpha|=\nu} v_\alpha x^\alpha/\alpha!$, and we define the inner product $\langle u, v\rangle := \sum_\alpha u_\alpha \overline{v_\alpha}$. Here, for the sake of simplicity we think $u_\alpha = 0$ if $\alpha \notin \mathbb{Z}_+^2$. In view of (2.2) the operator $\partial_{x_1}^{-n}\partial_{x_2}^{n}$ induces a shift operator $S_\sigma : u_\alpha \mapsto u_{\alpha+\sigma}$, where $\sigma = (-n, n)$. Hence we have

$$\langle \partial_{x_1}^{-n}\partial_{x_2}^{n} u, v\rangle = \sum_\alpha S_\sigma u_\alpha \overline{v_\alpha} = \sum_\alpha u_\alpha \overline{S_{-\sigma} v_\alpha} = \langle u, \partial_{x_1}^{n}\partial_{x_2}^{-n} v\rangle. \tag{3.4}$$

Therefore, it follows from (2.2) and the reality of $a_j$ that $\mathcal{L}_\nu$ is a self-adjoint operator on $H_\nu$. Hence the eigenvalues of $\mathcal{L}_\nu$ are real.

In view of the proof of Theorem 2.1, $\mathcal{L}_\nu$ has no eigenvalues if $|\varepsilon| > \|\mathcal{L}\|$. Hence $\mathcal{L}$ has no point spectrum outside the closed interval $-\|\mathcal{L}\| \le \varepsilon \le \|\mathcal{L}\|$. Next we will show that if $\varepsilon \in \mathbb{C} \setminus [-\|\mathcal{L}\|, \|\mathcal{L}\|]$, then it is in the resolvent set. Because $\mathcal{L}$ is a Hermitian operator on $H_\nu$, the eigenvalues are contained in the interval $-\|\mathcal{L}\| \le \varepsilon \le \|\mathcal{L}\|$. By diagonalizing $\varepsilon - \mathcal{L}$ with a unitary operator we see that there exists $C > 0$ independent of $\nu(\nu = 0, 1, 2, \ldots)$ such that for every homogeneous polynomial $f_\nu$ of degree $\nu$ we have

$$\|(\varepsilon - \mathcal{L})^{-1} f_\nu\| \le C\|f_\nu\|. \tag{3.5}$$

By multiplying both sides of (3.5) with $R^\nu/\nu!$ and by summing up with respect to $\nu$ we have, for $f = \sum_\nu f_\nu$

$$\|(\varepsilon - \mathcal{L})^{-1} f\| = \sup_\nu \frac{R^\nu}{\nu!}\|(\varepsilon - \mathcal{L})^{-1} f_\nu\| \le C \sup_\nu \frac{R^\nu}{\nu!}\|f_\nu\| = C\|f\|. \tag{3.6}$$

Therefore $u$ in (3.3) can be analytically continued with respect to $\varepsilon$ to $\mathbb{C}\setminus[-\|\mathcal{L}\|, \|\mathcal{L}\|]$. We note that (1.2) has a unique holomorphic solution in some neighborhood of the origin. Summing up the above we have proved

**Theorem 3.1.** *Suppose that (3.1) holds. Moreover, assume that $a_j = a_{-j}$ for $j = 1, 2, \ldots, N$ and $a_j$'s are real numbers. Then $u$ in (3.2) can be analytically continued with respect to $\varepsilon$ into $\mathbb{C} \setminus [-\|\mathcal{L}\|, \|\mathcal{L}\|]$ with having values in $X$, and it gives the unique holomorphic solution of (2.1) in some neighborhood of the origin $x = 0$.*

## § 4. Uniform estimates of Toeplitz matrices

In this section we shall prove a theorem which plays an important role in this paper. Let $\sigma_{\mathcal{L}}(t) = \sum_j (a_j e^{ijt\pi} + a_{-j} e^{-ijt\pi})$ be the Toeplitz symbol of $\mathcal{L}$ given by (2.2). We set $f(z) = \varepsilon - \sum_j (a_j z^j + a_{-j} z^{-j})$. Then we have $f(e^{i\theta\pi}) = \varepsilon - \sigma_{\mathcal{L}}(\theta)$. In this section we assume that $a_j$ is a real number. Let $\mathcal{L}_\nu$ be the restriction of $\mathcal{L}$ to $H_\nu$. Let $\Omega_0$ denote the connected unbounded component of the set $\mathbb{C} \setminus \{\sigma_{\mathcal{L}}(t); 0 \le t \le 2\}$. Then we have

**Theorem 4.1.** *For $\eta > 0$, let $V_\eta$ be an $\eta$-neighborhood of $\sigma_{\mathcal{L}}([0, 2])$. Let $\ell > 0$ be such that $(\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\} \ne \emptyset$. Then there exists an integer $\nu_0 \ge 0$ independent of $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ such that for every $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ and every $\nu \ge \nu_0$, $\varepsilon E - \mathcal{L}_\nu$ is invertible, where $E$ is the identity operator. Moreover, there exists a constant $M \ge 0$ independent of $\nu$ such that for every $\nu \ge \nu_0$ and every homogeneous polynomial $f_\nu$ of degree $\nu$ we have*

$$\|(\varepsilon E - \mathcal{L}_\nu)^{-1} f_\nu\| \le M\|f_\nu\|, \tag{4.1}$$

*where the norm is a $L^2$-norm on a finite dimensional Euclidian space. The constant $M$ can be chosen uniformly with respect to $\varepsilon$ in $(\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$.*

*Proof.* We make use of the argument in [1]. Let $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$. The matrix representation of $\varepsilon E - \mathcal{L}_\nu$ coincides with the one for the so-called discrete Wiener-Hopf equation

$$\frac{1}{2}\int_{-1}^{1} u(\theta) f(e^{i\theta\pi}) e^{-ik\theta\pi} d\theta = g_k \quad (0 \le k \le \nu). \tag{4.2}$$

We set $u(\theta) = \sum_{k=0}^{\nu} u_k e^{ik\theta\pi}$ and $g(\theta) = \sum_{k=0}^{\nu} g_k e^{ik\theta\pi}$. We will show that there exist $\nu_0 \ge 0$ and $M > 0$ such that for all $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ we have

$$\|u\| \le M\|g\|, \quad \forall \nu \ge \nu_0. \tag{4.3}$$

This shows that the matrix given by (4.2) is invertible. Hence we have the assertion.

By assumption we have $f(z) \ne 0$ for $|z| = 1$ and that the winding number of $f(z)$ at the origin along $|z| = 1$ is zero. Hence we have the factorization

$$f(e^{i\theta\pi}) = a_N \prod_{j=1}^{N} (e^{i\theta\pi} - \lambda_j) \prod_{j=1}^{N} (1 - e^{-i\theta\pi}\mu_j), \tag{4.4}$$

where $\lambda_j, \mu_j \in \mathbb{C}$ satisfy

$$|\mu_1| \le \cdots \le |\mu_N| \le \delta_0^{-1} < 1 < \delta_0 \le |\lambda_1| \le \cdots \le |\lambda_N|, \tag{4.5}$$

for some $\delta_0 > 1$ independent of $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$. We set

$$A := a_N \prod_{j=1}^{N} (e^{i\theta\pi} - \lambda_j), \quad B := \prod_{j=1}^{N} (1 - e^{-i\theta\pi}\mu_j), \tag{4.6}$$

and define

$$\tilde{A} := A^{-1}, \quad \tilde{B} := B^{-1}. \tag{4.7}$$

We note that $f = BA$. The equation (4.2) can be written in the form

$$uf = G_1 + g + G_2, \tag{4.8}$$

where

$$G_1 := \sum_{\nu+1}^{\infty} g_k e^{ik\theta\pi}, \quad G_2 := \sum_{-\infty}^{-1} g_k e^{ik\theta\pi}. \tag{4.9}$$

We will show that there exist $\nu_0 \ge 0$ and $K_0 > 0$ independent of $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ such that for all $\nu \ge \nu_0$ we have

$$\|G_1 \tilde{A}\| \le K_0\|g\|, \quad \|G_2 \tilde{B}\| \le K_0\|g\|. \tag{4.10}$$

Indeed, if we can show (4.10), then, by multiplying (4.8) with $\tilde{A}\tilde{B}$ we obtain

$$\begin{aligned}\|u\| &\le \|G_1\tilde{A}\tilde{B}\| + \|g\tilde{A}\tilde{B}\| + \|G_2\tilde{A}\tilde{B}\| \\ &= \|\tilde{B}\|_{L^1}\|G_1\tilde{A}\| + \|\tilde{A}\tilde{B}\|_{L^1}\|g\| + \|\tilde{A}\|_{L^1}\|G_2\tilde{B}\|,\end{aligned} \tag{4.11}$$

where $\|h\|_{L^1} := \sum_\nu |h_\nu|$ for $h = \sum_\nu h_\nu e^{i\nu\theta\pi}$. This proves (4.3).

We set

$$\tilde{A}(\nu) := \sum_{m=\nu+1}^{\infty} \tilde{A}_m e^{im\theta\pi}, \quad \tilde{B}(\nu) := \sum_{-\infty}^{-\nu-1} \tilde{B}_m e^{im\theta\pi}. \tag{4.12}$$

It follows from (4.8) that

$$uA = G_1\tilde{B} + g\tilde{B} + G_2\tilde{B}, \quad uB = G_1\tilde{A} + g\tilde{A} + G_2\tilde{A}. \tag{4.13}$$

By the first equation of (4.13) we have

$$G_2\tilde{B} = -(g\tilde{B})^- - (G_1\tilde{B})^- = -(g\tilde{B})^- - (G_1\tilde{B}(\nu))^-, \tag{4.14}$$

where $h^-$ means the negative part of a Fourier expansion. Hence we have

$$\begin{aligned}\|G_2\tilde{B}\| &\le \|g\tilde{B}\| + \|G_1\tilde{B}(\nu)\| \\ &\le \|\tilde{B}\|_{L^1}\|g\| + \|\tilde{B}(\nu)A\|_{L^1}\|G_1\tilde{A}\|.\end{aligned} \tag{4.15}$$

For any small $\alpha > 0$ there exists $\nu_0$ independent of $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ such that for every $\nu \ge \nu_0$ we have $\|\tilde{B}(\nu)A\|_{L^1} < \alpha$. In order to see this it is sufficient to estimate $\|\tilde{B}(\nu)\|_{L^1}$. We have

$$\tilde{B} = \prod_{j=1}^{N}(1 - e^{-i\theta\pi}\mu_j)^{-1}.$$

Because $\delta_0$ in (4.5) can be taken independent of $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ we see that $\|\tilde{B}(\nu)\|_{L^1}$ tends to zero uniformly in $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$ when $\nu$ tends to infinity. It follows from (4.15) that

$$\|G_2\tilde{B}\| \le \|\tilde{B}\|_{L^1}\|g\| + \alpha\|G_1\tilde{A}\|. \tag{4.16}$$

By the similar argument, using the fact that $uB$ in (4.13) has zero Fourier coefficients for $k > \nu$ we have, when $\|\tilde{A}(\nu)B\|_{L^1} \le \alpha$ $(\nu \ge \nu_0)$,

$$\|G_1\tilde{A}\| \le \|\tilde{A}\|_{L^1}\|g\| + \alpha\|G_2\tilde{B}\|. \tag{4.17}$$

The number $\nu_0$ can be chosen uniformly with respect to $\varepsilon \in (\Omega_0 \setminus V_\eta) \cap \{|\varepsilon| \le \ell\}$. By (4.16) and (4.17) we have the desired estimates if we take $\alpha < 1$. We note that the above argument also shows that $M$ has the desired property. This ends the proof.

## § 5. Spectral property

In this section we continue to use the same notation as in §3. We assume that $a_j = a_{-j}$ for $j = 1, 2, \dots, N$ and $a_j$'s are real numbers. In §3 we proved that $u$ in (2.4) can be analytically continued with respect to $\varepsilon$ to $\mathbb{C} \setminus [-\|\mathcal{L}\|, \|\mathcal{L}\|]$. We study the singularity of $u$ in the set $[-\|\mathcal{L}\|, \|\mathcal{L}\|]$. Let $\mathcal{L}_\nu$ denote the restriction of $\mathcal{L}$ to $H_\nu$. We denote the eigenvalues of $\mathcal{L}_\nu$ by $\lambda_\nu^{(j)}$ with multiplicity where $j = 1, 2, \dots, \nu + 1$. We define $\rho_0 := \sup_{\nu,j} |\lambda_\nu^{(j)}|$. Then we have

**Lemma 5.1.** *We have* $\|\mathcal{L}\| = \rho_0$.

*Proof.* Because $\mathcal{L}_\nu$ is a linear Hermitian operator on a finite dimensional space we easily see that $\|\mathcal{L}_\nu\| = \max_j |\lambda_\nu^{(j)}| \le \rho_0$. We expand $x = \sum_\nu x_\nu \in X$ as the sum of homogeneous polynomials $x_\nu$ of degree $\nu$. Because $\mathcal{L}$ preserves the set of homogeneous polynomials, we have

$$\begin{aligned} \|\mathcal{L}x\| = \|\sum_\nu \mathcal{L}x_\nu\| &\le \sup_\nu \frac{R^\nu}{\nu!} \|\mathcal{L}_\nu x_\nu\| \\ &\le \rho_0 \sup_\nu \frac{R^\nu}{\nu!} \|x_\nu\| \le \rho_0 \|x\|. \end{aligned} \tag{5.1}$$

Hence we have $\|\mathcal{L}\| \le \rho_0$. On the other hand, because $\mathcal{L}$ preserves the set of homogeneous polynomials, $\lambda_\nu^{(j)}$ is an eigenvalue of $\mathcal{L}$. It follows that $|\lambda_\nu^{(j)}| \le \|\mathcal{L}\|$. By taking the supremum of the left-hand side we obtain the converse inequality. This ends the proof.

We study the spectral set $[-\rho_0, \rho_0]$. We define the Toeplitz symbol $\sigma_{\mathcal{L}}(t)$ corresponding to $\mathcal{L}$ in (2.2) by replacing the integro-differential operator $\partial_{x_1}^{-k} \partial_{x_2}^{k}$ with $e^{i\pi kt}$ $(0 \le t \le 2)$, namely

$$\sigma_{\mathcal{L}}(t) = 2 \sum_{j=1}^{N} a_j \cos \pi jt, \quad 0 \le t \le 2, \tag{5.2}$$

where we recall that $a_j = a_{-j}$ and $a_j$ is a real number. We set $\sigma_{\mathcal{L}}([0,2]) := \{\sigma_{\mathcal{L}}(t); 0 \le t \le 2\}$. Then we have

**Lemma 5.2.** *We have* $\sigma_{\mathcal{L}}([0,2]) \subset [-\rho_0, \rho_0]$.

*Proof.* We can easily see that the matrix representation of $\mathcal{L}_\nu$ on the basis introduced in (3.4) is a Toeplitz matrix. It follows from Szegö's theorem (cf. [4]) that the set of all eigenvalues of $\mathcal{L}_\nu$ $(\nu = 0, 1, 2, \dots)$ forms a dense subset of the set $\sigma_{\mathcal{L}}([0,2])$. This ends the proof.

We study the analytic continuation of $u$ into $[-\rho_0, \rho_0]$.

**Theorem 5.3.** *The function $u = u(x, \varepsilon)$ in (3.2) is a meromorphic function of $\varepsilon$ on the open set $\mathbb{C} \setminus \sigma_{\mathcal{L}}([0, 2])$ with having values in $X$.*

*Proof.* Let $\mathcal{L}_\nu$ be the restriction of $\mathcal{L}$ to $H_\nu$. For every $\eta > 0$ sufficiently small, let $V_\eta$ be an $\eta$-neighborhood of $\sigma_{\mathcal{L}}([0, 2])$. Because the Toeplitz symbol is real-valued for $0 \le t \le 2$, the connected component of $\mathbb{C} \setminus \sigma_{\mathcal{L}}([0, 2])$ is equal to $\mathbb{C} \setminus \sigma_{\mathcal{L}}([0, 2])$. It follows from Theorem 4.1 that there exists $\nu_0$ such that for every $\nu \ge \nu_0$ the eigenvalues of $\mathcal{L}_\nu$ is contained in the set $V_\eta$. Because $\mathcal{L}$ preserves the set of homogeneous polynomials, every eigenvalue of $\mathcal{L}_\nu$ is also an eigenvalue of $\mathcal{L}$, and conversely, every eigenvalue of $\mathcal{L}$ is an eigenvalue of $\mathcal{L}_\nu$ for some $\nu$. This implies that the set of the point spectrum of $\mathcal{L}$ in $\mathbb{C} \setminus V_\eta$ is a finite one. Therefore the set of the point spectrum of $\mathcal{L}$ may accumulate only on $\sigma_{\mathcal{L}}([0, 2])$.

Next we will show that if $\varepsilon \in \mathbb{C} \setminus V_\eta$ is not an eigenvalue of $\mathcal{L}$, then it is in the resolvent set. It follows from Theorem 4.1 that there exists $C > 0$ independent of $\nu (\nu = 0, 1, 2, \ldots)$ such that for every homogeneous polynomial $f_\nu$ of degree $\nu$ we have

$$\|(\varepsilon - \mathcal{L})^{-1} f_\nu\| \le C\|f_\nu\|. \tag{5.3}$$

Then by the same argument as in (3.6) we can show that $\varepsilon$ is in the resolvent set. In order to show the analyticity in $\varepsilon$, we first note that $(\varepsilon - \mathcal{L})^{-1} f_\nu$ is analytic with respect to $\varepsilon$. We also note that the constant $C$ in (5.3) is uniform on every compact set in the complement of the point spectrum of $\mathcal{L}$ in $\mathbb{C} \setminus V_\eta$. It follows that $(\varepsilon - \mathcal{L})^{-1} f = \sum_\nu (\varepsilon - \mathcal{L})^{-1} f_\nu$ is an analytic function of $\varepsilon$. This ends the proof.

**Example 5.4.** It follows from Theorem 5.3 that every point in $[-\rho_0, \rho_0] \setminus \sigma_{\mathcal{L}}([0, 2])$ belongs either to the resolvent of $\mathcal{L}$ or to the point spectrum of $\mathcal{L}$, which may accumulate only on $\sigma_{\mathcal{L}}([0, 2])$.

If $\mathcal{L}$ has the Toeplitz symbol $\sigma_{\mathcal{L}}(t) = 2 \cos \pi t$, then we have $\sigma_{\mathcal{L}}([0, 2]) = [-2, 2] = [-\rho_0, \rho_0]$, (cf. Lemma 5.1 and Example 2.2.) If $\sigma_{\mathcal{L}}(t)$ has the Toeplitz symbol $\sigma_{\mathcal{L}}(t) := \cos \pi t + 2 \cos 2\pi t - \cos 3\pi t$, then we have $[-\rho_0, \rho_0] \setminus \sigma_{\mathcal{L}}([0, 2]) \ne \emptyset$.

Finally we study the spectral set $\sigma_{\mathcal{L}}([0, 2])$ of $\mathcal{L}$ on $X$.

**Theorem 5.5.** *The set $\sigma_{\mathcal{L}}([0, 2])$ consists of the point spectrum of $\mathcal{L}$ and the residual spectrum of $\mathcal{L}$. The point spectrum is a countable dense subset of $\sigma_{\mathcal{L}}([0, 2])$.*

*Proof.* The density of a point spectrum is proved in the proof of Lemma 5.2. Suppose that $\varepsilon \in \sigma_{\mathcal{L}}([0, 2])$ is not an eigenvalue of $\mathcal{L}$. By Szegö's theorem we can choose a sequence of eigenvalues $\{\varepsilon_k\}$ of $\mathcal{L}$ such that $\mathcal{L}\phi_k = \varepsilon_k \phi_k$ for some homogeneous polynomial $\phi_k$ of degree $\nu = \nu(k)$, $\|\phi_k\| = 1$, and

$$\lim_k \varepsilon_k = \varepsilon, \quad \nu(1) < \nu(2) < \cdots < \nu(k) < \nu(k+1) < \cdots \tag{5.4}$$

We define $u = \sum_k c_k\phi_k(x) \in X$ for some $c_k \neq 0$. Then we have

$$(\varepsilon - \mathcal{L})^{-1}u = \sum c_k(\varepsilon - \varepsilon_k)^{-1}\phi_k(x).$$

Hence, if $u$ is in the domain of $(\varepsilon - \mathcal{L})^{-1}$, then it follows that

$$\|(\varepsilon - \mathcal{L})^{-1}u\| = \sup_k \frac{R^{\nu(k)}}{\nu(k)!}|c_k||\varepsilon - \varepsilon_k|^{-1}\|\phi_k\| = \sup_k \frac{R^{\nu(k)}}{\nu(k)!}|c_k||\varepsilon - \varepsilon_k|^{-1} < \infty. \tag{5.5}$$

Because of (5.4), there exists $v = \sum_k d_k\phi_k$ such that $\sup_k |d_k|R^{\nu(k)}/\nu(k)! < \infty$ and $v$ cannot be arbitrarily approximated in $X$ by the set of $u$ satisfying (5.5). Indeed, the set of sequences $\{c_kR^{\nu(k)}/\nu(k)!\}$ which tend to zero is not dense in the set of bounded sequences. This proves that the domain of $(\varepsilon - \mathcal{L})^{-1}$ is not dense. This ends the proof.

## § 6. Diophantine phenomena

In this section we continue to use the same notation as in §5. Let $\varepsilon_0 \in \mathbb{R}$ and $0 < \theta_0 < \pi/2$. We define the sector $S_\pm(\varepsilon_0)$ in the upper (lower ) half plane with vertex at $\varepsilon_0$ by

$$S_\pm(\varepsilon_0) := \{\varepsilon; |\arg(\varepsilon - \varepsilon_0) \mp \pi/2| < \theta_0\}. \tag{6.1}$$

Let $\mathcal{L}_\nu$ be the restriction of $\mathcal{L}$ to $H_\nu$. We denote by $\varepsilon_{\nu,j}$ $(j = 1, 2, \ldots, \nu + 1)$ the eigenvalues of $\mathcal{L}_\nu$ counted with multiplicity. We define the Leray-Pisot function by

$$\rho(\varepsilon_0) := \liminf_{\nu\to\infty} \min_j |\varepsilon_0 - \varepsilon_{\nu,j}|^{1/\nu}. \tag{6.2}$$

We can easily see that $0 \le \rho(\varepsilon_0) \le 1$. Now we assume that $\varepsilon_0 \in \sigma_{\mathcal{L}}([0, 2])$ and we want to study the limit

$$\lim_{\varepsilon\to\varepsilon_0, \varepsilon\in S_\pm(\varepsilon_0)} (\varepsilon - \mathcal{L})^{-1}f, \quad f \in X.$$

**Example 6.1.** In the case $\sigma_{\mathcal{L}}(t) = 2\cos\pi t$, we have

$$\varepsilon_{\nu,j} = 2\cos\frac{\pi j}{\nu + 2} \qquad (j = 1, 2, \ldots, \nu + 1).$$

Writing $\varepsilon_0 = 2\cos\pi s$ we have that

$$\rho(\varepsilon_0) = \liminf_{\nu\to\infty} \inf_j \left|\cos\frac{\pi j}{\nu + 2} - \cos\pi s\right|^{\frac{1}{\nu}}. \tag{6.3}$$

By simple computations we have

$$\rho(\varepsilon_0) = \liminf_{\nu\to\infty} \inf_j \left|\frac{j}{\nu + 2} - s\right|^{\frac{1}{\nu}}. \tag{6.4}$$

The Diophantine function (6.4) was introduced by Leray and Pisot. (cf. [6].)

We will prove

**Theorem 6.2.** *Suppose that $\rho(\varepsilon_0) > 0$. Moreover, assume that $\varepsilon_0$ is not an eigenvalue of $\mathcal{L}$. Let $R$ and $R'$ satisfy that $R' < R\rho(\varepsilon_0)^2$. Then, for every $f \in X_R$ we have*

$$\lim_{\varepsilon\to\varepsilon_0, \varepsilon\in S_\pm(\varepsilon_0)} (\varepsilon - \mathcal{L})^{-1} f = (\varepsilon_0 - \mathcal{L})^{-1} f, \quad \text{in } X_{R'}. \tag{6.5}$$

*Proof.* We take $R''$ such that $R'/\rho(\varepsilon_0) < R'' < \rho(\varepsilon_0)R$. Next we take $\eta > 0$ sufficiently small such that $(R''/R)(\rho(\varepsilon_0)-\eta)^{-1} \le 1$ and $(R'/R'')(\rho(\varepsilon_0)-\eta)^{-1} \le 1$. By the definition of $\rho(\varepsilon_0)$ there exists $\nu_0$ such that, for every $\nu \ge \nu_0$ we have

$$|\varepsilon_0 - \varepsilon_{\nu,j}| > (\rho(\varepsilon_0) - \eta)^\nu, \quad j = 1, 2, \ldots, \nu+1. \tag{6.6}$$

Because $\varepsilon_0$ is not an eigenvalue, there exists $K > 0$ such that for all $\nu \ge 0$ we have

$$|\varepsilon_0 - \varepsilon_{\nu,j}| > K(\rho(\varepsilon_0) - \eta)^\nu, \quad j = 1, 2, \ldots, \nu+1. \tag{6.7}$$

On the other hand we can easily see that there exists $\delta_0 > 0$ independent of $j$ and $\nu$ such that

$$|\varepsilon - \varepsilon_{\nu,j}| \ge \delta_0 |\varepsilon_0 - \varepsilon_{\nu,j}| \quad \text{for all } \varepsilon \in S_\pm(\varepsilon_0), j = 1, \ldots, \nu+1, \nu = 0, 1, \ldots \tag{6.8}$$

Let $f \in X_R$. We expand $f$ as the sum of homogeneous polynomials, $f = \sum_\nu f_\nu$, where $f_\nu$ is homogeneous of degree $\nu$. We expand $f_\nu$ as $f_\nu = \sum_j c_{\nu,j}\phi_{\nu,j}(x)$, where $\phi_{\nu,j}$ ($\|\phi_{\nu,j}\| = 1$) is the eigenfunction of the eigenvalue $\varepsilon_{\nu,j}$ for the Hermitian operator $\mathcal{L}_\nu$. We have

$$(\varepsilon - \mathcal{L})^{-1} f_\nu = \sum_j c_{\nu,j} (\varepsilon - \mathcal{L}_\nu)^{-1}\phi_{\nu,j} = \sum_j c_{\nu,j}(\varepsilon - \varepsilon_{\nu,j})^{-1}\phi_{\nu,j}. \tag{6.9}$$

Because $\phi_{\nu,j}$ and $\phi_{\nu,k}$ $(k \ne j)$ are orthogonal to each other, we have

$$\|(\varepsilon - \mathcal{L})^{-1} f_\nu\|^2 = \sum_j |c_{\nu,j}|^2 |\varepsilon - \varepsilon_{\nu,j}|^{-2} \|\phi_{\nu,j}\|^2 = \sum_j |c_{\nu,j}|^2 |\varepsilon - \varepsilon_{\nu,j}|^{-2}. \tag{6.10}$$

In terms of (6.7) and (6.8) the right-hand side of (6.10) can be estimated by the following quantity

$$(K\delta_0)^{-2}(\rho(\varepsilon_0) - \eta)^{-2\nu} \sum_j |c_{\nu,j}|^2 = (K\delta_0)^{-2}(\rho(\varepsilon_0) - \eta)^{-2\nu} \|f_\nu\|^2, \tag{6.11}$$

for every $\varepsilon \in S_\pm(\varepsilon_0)$. It follows that

$$\|(\varepsilon - \mathcal{L})^{-1} f_\nu\| \le (K\delta_0)^{-1}(\rho(\varepsilon_0) - \eta)^{-\nu} \|f_\nu\|, \quad \nu \ge 0. \tag{6.12}$$

Therefore we have

$$(6.13)\qquad \|(\varepsilon-\mathcal{L})^{-1}f\|_{X_{R''}} = \sup_\nu \frac{(R'')^\nu}{\nu!}\|(\varepsilon-\mathcal{L})^{-1}f_\nu\| \le (K\delta_0)^{-1}\sup_\nu \frac{(R'')^\nu}{\nu!}(\rho(\varepsilon_0)-\eta)^{-\nu}\|f_\nu\| = (K\delta_0)^{-1}\sup_\nu \left(\frac{R^\nu}{\nu!}\|f_\nu\|\frac{(R'')^\nu}{R^\nu}(\rho(\varepsilon_0)-\eta)^{-\nu}\right).$$

Because $(R''/R)(\rho(\varepsilon_0)-\eta)^{-1}\le 1$, we get, from (6.13) that

$$(6.14)\qquad \|(\varepsilon-\mathcal{L})^{-1}f\|_{X_{R''}} \le (K\delta_0)^{-1}\sup_\nu \frac{R^\nu}{\nu!}\|f_\nu\| = (K\delta_0)^{-1}\|f\|_{X_R},\quad \varepsilon\in S_\pm(\varepsilon_0).$$

Hence we have, for $\varepsilon\in S_\pm(\varepsilon_0)$

$$(6.15)\qquad \|(\varepsilon-\mathcal{L})^{-1}f-(\varepsilon_0-\mathcal{L})^{-1}f\|_{X_{R'}} = \|(\varepsilon_0-\varepsilon)(\varepsilon-\mathcal{L})^{-1}(\varepsilon_0-\mathcal{L})^{-1}f\|_{X_{R'}}.$$

In terms of (6.14) with $\varepsilon=\varepsilon_0$ and the one with $R$ and $R''$ replaced by $R''$ and $R'$, respectively, we get, from (6.15)

$$(6.16)\qquad \|(\varepsilon-\mathcal{L})^{-1}f-(\varepsilon_0-\mathcal{L})^{-1}f\|_{X_{R'}} \le |\varepsilon_0-\varepsilon|(K\delta_0)^{-2}\|f\|_{X_R}.$$

Hence we obtain (6.5). This ends the proof.

## § 7. Nonsymmetric case

We shall extend Theorem 3.1 to the non Hermitian operator $\mathcal{L}$ with Toeplitz symbol given by

$$(7.1)\qquad \sigma_{\mathcal{L}}(t)=\sum_{j=1}^n (a_j e^{i\pi jt}+a_{-j}e^{-i\pi jt}),$$

where $a_j$'s are real numbers. We write $\sigma_{\mathcal{L}}(t)=\sigma_R(t)+i\sigma_I(t)$ with $\sigma_R(t)$ and $\sigma_I(t)$ given by

$$(7.2)\qquad \sigma_R(t)=\sum_{j=1}^n (a_j+a_{-j})\cos\pi jt,\quad \sigma_I(t)=\sum_{j=1}^n (a_j-a_{-j})\sin\pi jt.$$

Let $\mathcal{L}_{\sigma_R}$ and $\mathcal{L}_{\sigma_I}$ be the operators (2.2) with Toeplitz symbols $\sigma_R$ and $\sigma_I$, respectively. Then we have

$$(7.3)\qquad \mathcal{L}=\mathcal{L}_{\sigma_R}+i\mathcal{L}_{\sigma_I}.$$

**Theorem 7.1.** *Suppose that (3.1) holds. Moreover, assume that $a_j$ is a real number. Then the solution $u$ in (3.2) is analytically continued with respect to $\varepsilon$ to the union of the sets $|\Im\varepsilon| > \|\mathcal{L}_{\sigma_I}\|$ and $|\Re\varepsilon| > \|\mathcal{L}_{\sigma_R}\|$ with having values in $X$.*

*Remark.* If $\mathcal{L}_{\sigma_I} = 0$, then Theorem 7.1 reduces to Theorem 3.1.

*Proof.* We assume that $|\Im\varepsilon| > \|\mathcal{L}_{\sigma_I}\|$. For $0 \le s \le 2$ we introduce a geodesic parameter $\eta$ ($\eta \ge 0$) by

$$\varepsilon = \|\mathcal{L}_{\sigma_R}\| \left((\eta+1)\cos(\pi s) + i\eta\sin(\pi s)\right). \tag{7.4}$$

Then we have

$$\begin{aligned} \varepsilon E - \mathcal{L} &= \|\mathcal{L}_{\sigma_R}\|(\eta+1)\cos(\pi s)E - \mathcal{L}_{\sigma_R} + i\left(\|\mathcal{L}_{\sigma_R}\|\eta\sin(\pi s)E - \mathcal{L}_{\sigma_I}\right) \\ &=: H_1 + iH_2, \end{aligned} \tag{7.5}$$

where $E$ is the identity operator, and where $H_1$ and $H_1$ are Hermitian operators. Let us assume that $\sin\pi s \ne 0$. Then we take $\eta > 0$ sufficiently large that $\|\mathcal{L}_{\sigma_R}\|\eta|\sin(\pi s)| > \|\mathcal{L}_{\sigma_I}\|$. In view of (7.4) this is possible, because $\varepsilon$ satisfies that $|\Im\varepsilon| > \|\mathcal{L}_{\sigma_I}\|$. We can easily see that $H_2$ is invertible. It follows that $\varepsilon E - \mathcal{L} = H_1 + iH_2$ is invertible because $H_2$ is invertible. Hence $\varepsilon$ is in the resolvent set.

We will prove that $(\varepsilon E - \mathcal{L})^{-1}f$ coincides with the analytic continuation of the solution (3.2). In view of (7.4) it is sufficient to show this for the parameter $\eta$. Because the inverse can be constructed by a Neumann series, $H_2^{-1}$ is analytic with respect to $\eta$. Indeed we have

$$H_2^{-1} = (c\eta E - \mathcal{L}_{\sigma_I})^{-1} = \sum_{\nu=0}^{\infty} \frac{(\mathcal{L}_{\sigma_I})^\nu}{(c\eta)^{\nu+1}}, \tag{7.6}$$

where $c = \|\mathcal{L}_{\sigma_R}\|\sin(\pi s)$. On the other hand, noting that $H_1 = (\tilde{c}\eta + \tilde{c})E - \mathcal{L}_{\sigma_R}$ with $\tilde{c} = \|\mathcal{L}_{\sigma_R}\|\cos\pi s$ we have

$$\begin{aligned} L_0 := H_1H_2^{-1} &= (\tilde{c}\eta + \tilde{c} - \mathcal{L}_{\sigma_R})\sum_{\nu=0}^{\infty}\frac{(\mathcal{L}_{\sigma_I})^\nu}{(c\eta)^{\nu+1}} = \frac{\tilde{c}}{c} + \tilde{c}\eta\sum_{\nu=1}^{\infty}\frac{(\mathcal{L}_{\sigma_I})^\nu}{(c\eta)^{\nu+1}} \\ &+ (\tilde{c} - \mathcal{L}_{\sigma_R})\sum_{\nu=0}^{\infty}\frac{(\mathcal{L}_{\sigma_I})^\nu}{(c\eta)^{\nu+1}} = \frac{\tilde{c}}{c} + \left(\frac{\tilde{c}}{c}\mathcal{L}_{\sigma_I} + \tilde{c} - \mathcal{L}_{\sigma_R}\right)\sum_{\nu=0}^{\infty}\frac{(\mathcal{L}_{\sigma_I})^\nu}{(c\eta)^{\nu+1}}. \end{aligned} \tag{7.7}$$

We note that $L_0$ is an Hermitian operator if $\eta$ is a real number. Moreover $L_0$ is an analytic function of $\eta$ in some (complex) neighborhood of every point $\eta_0 > 0$ such that $\varepsilon_0$ which corresponds to $\eta_0$ by (7.4) satisfies $|\Im\,\varepsilon_0| > \|\mathcal{L}_{\sigma_I}\|$.

Next we will prove that $(iE + L_0)^{-1}$ exists and it is given by

$$\int_0^{\infty(-i)} e^{-i\zeta}e^{\zeta L_0}d\zeta = -i\int_0^{\infty} e^{-\xi}\exp(-i\xi L_0)d\xi, \tag{7.8}$$

where the integral in the left-hand side is taken along the path which starts from the origin and goes to infinity along the negative imaginary axis. Because $L_0$ is an Hermitian operator, we see that the integral converges. Moreover, we will show that the integral converges for $\eta$ in some (complex) neighborhood of $\eta_0 > 0$ such that $\varepsilon_0$ which corresponds to $\eta_0$ by (7.4) satisfies $|\Im\varepsilon_0| > \|\mathcal{L}_{\sigma_I}\|$. In order to see this we consider the imaginary part of the term $\eta^{-\nu-1}$ in the right-hand side sum of (7.7). By setting $\eta = \eta' + i\eta''$, we have

$$\frac{1}{\eta^{\nu+1}} = \frac{(\eta' - i\eta'')^{\nu+1}}{|\eta|^{2\nu+2}} = \frac{1}{|\eta|^{2\nu+2}} \sum_{k\geq 0} \binom{\nu+1}{k} (\eta')^{\nu+1-k}(-i\eta'')^k. \tag{7.9}$$

Hence the imaginary part of the right-hand side of (7.9) can be bounded by

$$\frac{1}{|\eta|^{\nu+1}} \sum_{k\geq 1} \binom{\nu+1}{k} \left(\frac{|\eta''|}{|\eta|}\right)^k = \frac{|\eta''|}{|\eta|^{\nu+2}} \sum_{k\geq 0} \binom{\nu+1}{k+1} \left(\frac{|\eta''|}{|\eta|}\right)^k. \tag{7.10}$$

We can easily show that for any $r_0 > 1$ there exist $\delta_0 > 0$ and $K_0 > 0$ such that if $|\eta''|/|\eta| < \delta_0$, then the right-hand side of (7.10) can be estimated by $|\eta''||\eta|^{-\nu-2}K_0 r_0^{\nu}$. It follows that there exists $K_1 > 0$ such that modulo Hermitian operators the operator norm of the right-hand side of (7.7) can be bounded by $K_1|\eta''| < K_1\delta_0$. This proves that if we take $\delta_0$ sufficiently small, then the right-hand side integral of (7.8) converges when $|\eta''| < \delta_0$. This proves the assertion.

Next we will show that (7.8) is equal to $(iE + L_0)^{-1}$. Indeed, we have

$$\begin{aligned}(iE + L_0)\int_0^{\infty(-i)} e^{-i\zeta}e^{\zeta L_0}d\zeta &= \int_0^{\infty(-i)} (iE + L_0)e^{-\zeta(i+L_0)}d\zeta \\ &= -[e^{-\zeta(i+L_0)}]_{\zeta=0}^{\zeta=-i\infty} = E.\end{aligned} \tag{7.11}$$

Indeed, we have $\lim_{\zeta\to -i\infty} e^{-\zeta(i+L_0)}f = \lim_{\xi\to\infty} e^{-\xi+i\xi L_0}f = 0$. Hence we have

$$(\varepsilon E - \mathcal{L})^{-1} = H_2^{-1}(iE + L_0)^{-1} = -iH_2^{-1}\int_0^{\infty} e^{-\xi}\exp(-i\xi L_0)d\xi. \tag{7.12}$$

We take $\eta > 0$ sufficiently large so that $\varepsilon$ given by (7.4) satisfies $|\varepsilon| > \|\mathcal{L}\|$. Because both sides of (7.12) are analytic functions of $\varepsilon$ by Theorem 2.1 and what we have proved in the above, it follows that (7.12) gives the analytic continuation of $(\varepsilon E - \mathcal{L})^{-1}$ along the path given by (7.4) to the domain $|\Im\varepsilon| > \|\mathcal{L}_{\sigma_I}\|$. This proves the assertion.

In order to show the existence and the analyticity of $(\varepsilon E - \mathcal{L})^{-1}$ with respect to $\varepsilon$ in the domain $|\Re\varepsilon| > \|\mathcal{L}_{\sigma_R}\|$, we consider $i\mathcal{L} = i\mathcal{L}_{\sigma_R} - \mathcal{L}_{\sigma_I}$ instead of $\mathcal{L}$. By what we have proved in the above, we see that $(\varepsilon' E - i\mathcal{L})^{-1} = -i(-i\varepsilon' E - \mathcal{L})^{-1}$ exists and it is analytic when $|\Im\varepsilon'| > \|\mathcal{L}_{\sigma_R}\|$. Setting $\varepsilon = -i\varepsilon'$, we see that $(\varepsilon E - \mathcal{L})^{-1}$ exists and is analytic when $|\Re\varepsilon| > \|\mathcal{L}_{\sigma_R}\|$. This ends the proof.

**Example 7.2.** For $a > 0$ we consider the operator $\mathcal{L}_\sigma$ with the following Toeplitz symbol

$$\sigma(t) = 2\cos\pi t + 2ai\sin(2t\pi). \tag{7.13}$$

Then $\mathcal{L}_{\sigma_R}$ and $\mathcal{L}_{\sigma_I}$ are the operators with symbols given by $2\cos\pi t$ and $2a\sin(2t\pi)$, respectively. We know that $\|\mathcal{L}_{\sigma_R}\| = 2$, and by the similar argument we see that $\|\mathcal{L}_{\sigma_I}\| = 2a$. (cf. Example 2.2.) We have the analytic continuation of the resolvent to the set of $\varepsilon$ such that either $|\Im\varepsilon| > 2a$ or $|\Re\varepsilon| > 2$ is satisfied.

## § 8. Analytic continuation and the Fredholm property

In this section we study the Fredholmness and the extension of Theorem 5.3 to non Hermitian operators. (cf. [7]). Let $\sigma_{\mathcal{L}}(t)$ be the Toeplitz symbol corresponding to the operator (2.2). Then we have

**Theorem 8.1.** *The solution $u$ in (3.2) can be continued as a meromorphic function of $\varepsilon$ in the connected unbounded component of $\mathbb{C} \setminus \{\sigma_{\mathcal{L}}(t); 0 \le t \le 2\}$. For every $\varepsilon \in \mathbb{C} \setminus \{\sigma_{\mathcal{L}}(t); 0 \le t \le 2\}$ the operator $\varepsilon E - \mathcal{L}$ is a Fredholm operator of index zero on $X$, where $E$ denotes the identity operator. Namely, the image of $\varepsilon E - \mathcal{L}$ is closed and the dimension of the kernel and the cokernel of $\varepsilon E - \mathcal{L}$ is finite.*

*Proof.* The former half follows from the same argument as in the proof of Theorem 5.3 by using Theorem 4.1. We will show the latter half. It follows from Theorem 4.1 that the uniform estimates (4.1) hold for all $\nu \ge \nu_0$ with $\nu_0$ given by Theorem 4.1. By the definition of the norm in $X$ we see that $\varepsilon E - \mathcal{L}$ is invertible on the subspace of $X$ with homogeneous degree greater than $\nu_0$. Hence $\varepsilon E - \mathcal{L}$ is the sum of an invertible operator and a finite-dimensional operator. Hence it is a Fredholm operator. The index property also follows if we recall that $\varepsilon E - \mathcal{L}$ preserves the set of homogeneous polynomials of degree $k \ge 0$. This ends the proof.

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