RIMS Kôkyûroku Bessatsu
B39 (2013), 051–061

Spaces of equivariant algebraic maps from real projective spaces into complex projective spaces

By

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Abstract

We study the homotopy types of certain spaces closely related to the space of algebraic (rational) maps from the m dimensional real projective space into the n dimensional complex projective space for $2 \leq m \leq 2n$ (we conjecture that this relation to be a homotopy equivalence). In [10] we proved that natural maps from these spaces to the spaces of all continuous maps are $\mathbb{Z}/2$-equivariant homotopy equivalences, where the $\mathbb{Z}/2$-equivariant action is induced from the conjugation on $\mathbb{C}$. In the same article we also proved that the homotopy types of the terms of the natural degree filtration approximate closer and closer the homotopy type of the space of continuous maps and obtained bounds that describe the closeness of the approximation in terms of the degrees of the maps. In this paper, we improve the bounds by using new methods used in [11]. In addition, in the the last section, we reprove a special case ($m = 1$) of the conjecture stated in [1] that our spaces are homotopy equivalent to the spaces of algebraic maps.

§ 1. Introduction.

Summary of the contents. In [13] Mostovoy showed (modulo certain errors that were corrected in [14]) that if $2 \leq m \leq n$ then the space $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ of holomorphic maps from $\mathbb{C}\mathrm{P}^m$ to $\mathbb{C}\mathrm{P}^n$ has the same homotopy type of the space $\mathrm{Map}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ of corresponding continuous maps up to a certain dimension, which generalizes the classical result of Segal [16] for $m = 1$. In [1] we used a variant of Mostovoy's method

Received September 5, 2011. Revised December 5, 2011.
2000 Mathematics Subject Classification(s): Primary 55P10, 55R80; Secondly 55P35
Key Words: *Simplicial resolution, algebraic map, homotopy equivalence.*
The second author is partially supported by Grant-in-Aid for Scientific Research (No. 23540079), The Ministry of Education, Culture, Sports Science and Technology, Japan
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to the analogous problem for the space $\mathrm{Alg}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ of algebraic maps from $\mathbb{R}\mathrm{P}^m$ to $\mathbb{R}\mathrm{P}^n$ for $2 \leq m < n$. These our results can also be seen as generalizations of the results of [6] and [12]. In [10] we used analogous methods to prove a homotopy (or homology) approximation theorem for the space $\mathrm{Alg}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ of real algebraic maps from $\mathbb{R}\mathrm{P}^m$ to $\mathbb{C}\mathrm{P}^n$ when $2 \leq m \leq 2n$. Combing this result with the main theorem of [1], we obtained a $\mathbb{Z}/2$-equivariant homotopy approximation result in [10] (where the $\mathbb{Z}/2$-action on $\mathbb{C}\mathrm{P}^n$ is induced by complex conjugation), which is itself a generalization of [6, Theorem 3.7].

In his recent paper [14], Mostovoy, in addition to correcting the mistakes in [13], introduced a new idea that allowed him to improve the bounds on the degree of homotopy groups in his homotopy approximation theorem. With the help of analogous methods and some additional techniques of ours, in the recent work [11], we improved the bounds of the homotopy approximation theorem given in [1].

The first purpose of this article is to obtain an improved version of the equivariant homotopy approximation theorem of [10]. Since the arguments are analogous to those in [11], we only state our new results and refer to other literature for detailed proofs (cf. Remark 2 below).

The second purpose is to consider the possibility of approximating the space of continuous maps by its subspaces of algebraic maps of a fixed degree, analogously to the results of Segal [16] and Mostovoy [14]. Because the situation is analogous, in this paper we only consider the space of algebraic maps from $\mathbb{R}\mathrm{P}^m$ to $\mathbb{R}\mathrm{P}^n$. For this purpose we only need to prove that the projection maps

$$\Psi_d : A_d(m,n) \to \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \quad \Gamma_d : \tilde{A}_d(m,n) \to \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$$

are homotopy equivalences, where they will be defined below. It is easy to see that both of these maps have contractible fibres. However that alone is not sufficient to prove that these two maps are homotopy equivalences. In fact, in [1, Conjecture 3.8], we conjectured that this is true in general. Here we provide an informal argument which shows that this is true for $m = 1$. A proof of this fact already appeared in [12, Proposition 2.1]. However we find that the argument given there is unconvincing since it does not seem to make any use of any properties of the map Ψ_d (e.g. the fact that is a quasi-fibration).

In the remainder of this section we briefly describe the notations and the definitions.

Notation. The notation of this paper is essentially analogous to the one used in [1] and [10]. Note that the presence of $\mathbb{C}$ indicates that a complex case is being considered (i.e. maps take values in $\mathbb{C}\mathrm{P}^n$ rather than $\mathbb{R}\mathrm{P}^n$ or polynomials have coefficients in $\mathbb{C}$ rather than $\mathbb{R}$, etc.).

Let m and n be positive integers such that $1 \leq m \leq 2n$. We choose $\mathbf{e}_m = [1 : 0 : \cdots : 0] \in \mathbb{R}\mathrm{P}^m$ and $\mathbf{e}'_n = [1 : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^n$ as the base points of $\mathbb{R}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$, respectively. Let $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ denote the space consisting of all based maps $f : (\mathbb{R}\mathrm{P}^m, \mathbf{e}_m) \to (\mathbb{C}\mathrm{P}^n, \mathbf{e}'_n)$. When $m \geq 2$, we denote by $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ the corresponding path component of $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ for each $\epsilon \in \mathbb{Z}/2 = \{0,1\} = \pi_0(\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n))$ ([5]). Similarly, let $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ denote the space of all free maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ and $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ the corresponding path component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$.

A map $f : \mathbb{R}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ is called a *algebraic map of degree d* if it can be represented as a rational map of the form $f = [f_0 : \cdots : f_n]$ such that $f_0, \cdots, f_n \in \mathbb{C}[z_0, \cdots, z_m]$ are homogeneous polynomials of the same degree d with no common *real* roots except $\mathbf{0}_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$. We denote by $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$) the space consisting of all (resp. based) algebraic maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d. It is easy to see that there are inclusions

$$\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n), \quad \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n),$$

where $[d]_2 \in \mathbb{Z}/2 = \{0,1\}$ denotes the integer d mod 2.

Let $A_d(m,n)(\mathbb{C})$ denote the space consisting of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in \mathbb{C}[z_0, \cdots, z_m]^{n+1}$ of homogeneous polynomials of degree d with coefficients in $\mathbb{C}$ and without non-trivial common real roots (but possibly with non-trivial common *non-real* ones). Since $\mathbb{C}^*$ acts on $A_d(m,n)(\mathbb{C})$ freely, one can define the projectivisation $\tilde{A}^{\mathbb{C}}_d(m,n)$ by the orbit space $\tilde{A}^{\mathbb{C}}_d(m,n) = A_d(m,n)(\mathbb{C})/\mathbb{C}^*$.

Let $A^{\mathbb{C}}_d(m,n) \subset A_d(m,n)(\mathbb{C})$ be the subspace consisting of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in A_d(m,n)(\mathbb{C})$ such that the coefficient of z_0^d in f_0 is 1 and 0 in the other f_k's $(k \neq 0)$. Then there are natural projection maps

$$\Psi^{\mathbb{C}}_d : A^{\mathbb{C}}_d(m,n) \to \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n), \quad \Gamma^{\mathbb{C}}_d : \tilde{A}^{\mathbb{C}}_d(m,n) \to \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n).$$

For $m \geq 2$ and $g \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{C}\mathrm{P}^n)$ a fixed algebraic map, we denote by $\mathrm{Alg}^{\mathbb{C}}_d(m,n;g)$ and $F(m,n;g)$ the spaces defined by

$$\begin{cases} \mathrm{Alg}^{\mathbb{C}}_d(m,n;g) & = \{f \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}, \\ F(m,n;g) & = \{f \in \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}. \end{cases}$$

It is well-known that there is a homotopy equivalence $F(m,n;g) \simeq \Omega^m \mathbb{C}\mathrm{P}^n$ ([15]).

Let $\mathcal{H}^m_d$ denote the space of all homogenous polynomials $h \in \mathbb{C}[z_0, \cdots, z_m]$ of degree d. We choose a fixed tuple $\mathrm{g} = (g_0, \cdots, g_n) \in A^{\mathbb{C}}_d(m-1,n)$ such that $\Psi^{\mathbb{C}}_d(\mathrm{g}) = g$. In this situation, we denote by $A^*_d(\mathbb{C}) \subset (\mathcal{H}^m_d)^{n+1}$ the subspace given by

$$A^*_d(\mathbb{C}) := \Big\{(g_0 + z_m h_0, \cdots, g_n + z_m h_n) : h_k \in \mathcal{H}^m_{d-1} \ (0 \leq k \leq n)\Big\},$$

and define the subspace $A^{\mathbb{C}}_d(m,n;g)\subset A^{\mathbb{C}}_d(m,n)$ by $A^{\mathbb{C}}_d(m,n;g)=A^{\mathbb{C}}_d(m,n)\cap A^*_d(\mathbb{C})$.

Because $\Psi^{\mathbb{C}}_d(f_0,\cdots,f_n)\in \mathrm{Alg}^{\mathbb{C}}_d(m,n;g)$ for any $(f_0,\cdots,f_n)\in A^{\mathbb{C}}_d(m,n;g)$, one can define the projection $\Psi^{\mathbb{C}\prime}_d : A^{\mathbb{C}}_d(m,n;g)\to \mathrm{Alg}^{\mathbb{C}}_d(m,n;g)$ by the restriction $\Psi^{\mathbb{C}\prime}_d=\Psi^{\mathbb{C}}_d|A^{\mathbb{C}}_d(m,n;g)$. Let

$$(1.1)\qquad \begin{cases} i_{d,\mathbb{C}}:\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)\hookrightarrow \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n),\\ j_{d,\mathbb{C}}:\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n)\hookrightarrow \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n),\\ i'_{d,\mathbb{C}}:\mathrm{Alg}^{\mathbb{C}}_d(m,n;g)\hookrightarrow F(m,n;g)\simeq \Omega^m\mathbb{C}\mathrm{P}^n \end{cases}$$

denote the inclusions and

$$(1.2)\qquad \begin{cases} i^{\mathbb{C}}_d=i_{d,\mathbb{C}}\circ\Psi^{\mathbb{C}}_d : A^{\mathbb{C}}_d(m,n)\to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n),\\ j^{\mathbb{C}}_d=j_{d,\mathbb{C}}\circ\Gamma^{\mathbb{C}}_d : \tilde{A}^{\mathbb{C}}_d(m,n)\to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n),\\ i''_d=i'_{d,\mathbb{C}}\circ\Psi^{\mathbb{C}\prime}_d : A^{\mathbb{C}}_d(m,n;g)\to F(m,n;g) \end{cases}$$

the natural maps.

The notations used in this paper can be summarized in the following two diagrams, where $g\in\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1},\mathbb{K}\mathrm{P}^n)$ denotes a fixed based algebraic map of degree d and we omit their details of the notations in (1.4) and refer the reader to [1].

$$(1.3)\qquad \begin{tikzcd} \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) & \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \arrow[l, hook'] & F^{\mathbb{C}}(m,n;g) \arrow[l, hook'] \\ \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \arrow[u, hook, "j_{d,\mathbb{C}}"] & \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m,\mathbb{C}\mathrm{P}^n) \arrow[l, hook'] \arrow[u, hook, "i_{d,\mathbb{C}}"] & \mathrm{Alg}^{\mathbb{C}}_d(m,n;g) \arrow[l, hook'] \arrow[u, hook, "i'_{d,\mathbb{C}}"] \\ \tilde{A}^{\mathbb{C}}_d(m,n) \arrow[u, two heads, "\Gamma^{\mathbb{C}}_d"] \arrow[uu, bend left, "j^{\mathbb{C}}_d"] & A^{\mathbb{C}}_d(m,n) \arrow[u, two heads, "\Psi^{\mathbb{C}}_d"] \arrow[uu, bend left, "i^{\mathbb{C}}_d"] & A^{\mathbb{C}}_d(m,n;g) \arrow[u, two heads, "\Psi^{\mathbb{C}\prime}_d"] \arrow[uu, bend left, "i''_d"] \end{tikzcd}$$

$$(1.4)\qquad \begin{tikzcd} \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n) & \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n) \arrow[l, hook'] & F(m,n;g) \arrow[l, hook'] \\ \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n) \arrow[u, hook, "j_{d,\mathbb{R}}"] & \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n) \arrow[l, hook'] \arrow[u, hook, "i_{d,\mathbb{R}}"] & \mathrm{Alg}_d(m,n;g) \arrow[l, hook'] \arrow[u, hook, "i'_{d,\mathbb{R}}"] \\ \tilde{A}_d(m,n) \arrow[u, two heads, "\Gamma_d"] \arrow[uu, bend left, "j_d"] & A_d(m,n) \arrow[u, two heads, "\Psi_d"] \arrow[uu, bend left, "i_d"] & A_d(m,n;g) \arrow[u, two heads, "\Psi'_d"] \arrow[uu, bend left, "i'_d"] \end{tikzcd}$$

§ 2. The main results.

In this section we state the main results of this paper. First define the positive integers $D^*_{\mathbb{K}}(d;m,n)$ and $D_{\mathbb{K}}(d;m,n)$ by

$$D^*_{\mathbb{K}}(d;m,n)=\begin{cases}(n-m)\big(\lfloor\frac{d+1}{2}\rfloor+1\big)-1 & \text{if } \mathbb{K}=\mathbb{R},\\ (2n-m+1)\big(\lfloor\frac{d+1}{2}\rfloor+1\big)-1 & \text{if } \mathbb{K}=\mathbb{C},\end{cases}\tag{2.1}$$

$$D_{\mathbb{K}}(d;m,n)=\begin{cases}(n-m)\big(d+1\big)-1 & \text{if } \mathbb{K}=\mathbb{R},\\ (2n-m+1)\big(d+1\big)-1 & \text{if } \mathbb{K}=\mathbb{C},\end{cases}\tag{2.2}$$

where $\lfloor x\rfloor$ is the integer part of a real number x and we remark that the equality $D_{\mathbb{C}}(d;m,n)=D_{\mathbb{R}}(d;m,2n+1)$ holds.

We first recall the following two results.

Theorem 2.1 (The case $(\mathbb{K},m)=(\mathbb{R},1)$; [9], [18]). *If $m=1<n$, the natural map $i_d:A_d(1,n)\to \mathrm{Map}^*_{[d]_2}(\mathbb{RP}^1,\mathbb{RP}^n)\simeq \Omega S^n$ is a homotopy equivalence up to dimension $D_{\mathbb{R}}(d;1,n)=(n-1)(d+1)-1$.* □

Theorem 2.2 (The case $\mathbb{K}=\mathbb{R}$ and $m\geq 2$; [11]). *Let m and n be positive integers such that $2\leq m<n$.*

(i) *Let $g\in \mathrm{Alg}^*_d(\mathbb{RP}^{m-1},\mathbb{RP}^n)$ be an algebraic map of degree d. Then the natural map $i'_d:A_d(m,n;g)\to F(m,n;g)\simeq \Omega^m S^n$ is a homotopy equivalence up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+2\leq n$ and a homology equivalence up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+1=n$.*

(ii) *The natural maps*

$$\begin{cases} i_d:A_d(m,n)\to \mathrm{Map}^*_{[d]_2}(\mathbb{RP}^m,\mathbb{RP}^n)\\ j_d:\tilde{A}_d(m,n)\to \mathrm{Map}_{[d]_2}(\mathbb{RP}^m,\mathbb{RP}^n)\end{cases}$$

are homotopy equivalences up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+2\leq n$, and homology equivalences up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+1=n$. □

Remark 1. (i) Theorem 2.2 was recently proved in [11] and it is an improvement of the main result of [1].

(ii) A map $f:X\to Y$ is called *a homotopy* (resp. *a homology*) *equivalence up to dimension* D if $f_*:\pi_k(X)\to\pi_k(Y)$ (resp. $f_*:H_k(X,\mathbb{Z})\to H_k(Y,\mathbb{Z})$) is an isomorphism for any $k<D$ and an epimorphism for $k=D$.

(iii) Let G be a finite group and let $f:X\to Y$ be a G-equivariant. For a subgroup $H\subset G$, we denote by X^H the H-fixed set of X given by $X^H=\{x\in X: h\cdot x=$

x for any $h \in H\}$. A map $f : X \to Y$ is called *a G-equivariant homotopy* (resp. *homology*) *equivalence up to dimension D* if the restriction to the set of fixed points $f^H = f|X^H : X^H \to Y^H$ is a homotopy (resp. homology) equivalence up to dimension D for any subgroup $H \subset G$ (cf. [8]).

Now we state the main results of this paper as follows.

Theorem 2.3. *Let m and n be positive integers such that $2 \leq m \leq 2n$.*

(i) *Let $g \in \mathrm{Alg}_d^{\mathbb{C}}(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{C}\mathrm{P}^n)$ be a fixed algebraic map of degree d. Then the natural map $i_d'' : A_d^{\mathbb{C}}(m,n;g) \to F(m,n;g) \simeq \Omega^m S^{2n+1}$ is a homotopy equivalence up to dimension $D_{\mathbb{C}}(d;m,n)$ if $m < 2n$ and a homology equivalence up to dimension $D_{\mathbb{C}}(d;m,n)$ if $m = 2n$.*

(ii) *The natural maps*

$$\begin{cases} i_d^{\mathbb{C}} : A_d^{\mathbb{C}}(m,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_d^{\mathbb{C}} : \tilde{A}_d^{\mathbb{C}}(m,n) \to \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences up to dimension $D_{\mathbb{C}}(d;m,n)$ if $m < 2n$ and homology equivalences up to dimension $D_{\mathbb{C}}(d;m,n)$ if $m = 2n$.

Note that the complex conjugation on $\mathbb{C}$ naturally induces $\mathbb{Z}/2$-actions on the spaces $\mathrm{Alg}_d^{\mathbb{C}}(m,n;g)$ and $A_d^{\mathbb{C}}(m,n)$. In the same way, it also induces a $\mathbb{Z}/2$-action on $\mathbb{C}\mathrm{P}^n$ and it extends to $\mathbb{Z}/2$-actions on the spaces $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^{2n+1})$ and $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$, where we identify $S^{2n+1} = \{(w_0, \cdots, w_n) \in \mathbb{C}^{n+1} : \sum_{k=0}^n |w_k|^2 = 1\}$ and regard $\mathbb{R}\mathrm{P}^m$ as a $\mathbb{Z}/2$-space with the trivial $\mathbb{Z}/2$-action.

Since $(i_d'')^{\mathbb{Z}/2} = i_d'$, $(i_d^{\mathbb{C}})^{\mathbb{Z}/2} = i_d$ and $(j_d^{\mathbb{C}})^{\mathbb{Z}/2} = j_d$, by Theorem 2.2 and Theorem 2.3 we obtain the following result.

Corollary 2.4. *Let m and n be positive integers such that $2 \leq m < n$.*

(i) *Let $g \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{C}\mathrm{P}^n)$ be an algebraic map of degree d. Then the natural map $i_d'' : A_d^{\mathbb{C}}(m,n;g) \to F(m,n;g) \simeq \Omega^m S^{2n+1}$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+2 \leq n$ and a $\mathbb{Z}/2$-equivariant homology equivalence up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+1 = n$.*

(ii) *The natural maps*

$$\begin{cases} i_d^{\mathbb{C}} : A_d^{\mathbb{C}}(m,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_d^{\mathbb{C}} : \tilde{A}_d^{\mathbb{C}}(m,n) \to \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are $\mathbb{Z}/2$-equivariant homotopy equivalences up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+2 \leq n$ and are $\mathbb{Z}/2$-equivariant homology equivalences up to dimension $D_{\mathbb{R}}(d;m,n)$ if $m+1=n$. □

Remark 2. Remark that $D^*_{\mathbb{K}}(d;m,n) < D_{\mathbb{K}}(d;m,n)$ for $d \geq 2$. Thus, we may regard Theorem 2.3 and Corollary 2.4 as the improvements of [10, Theorems 1.4, 1.5 and Corollary 1.7]. The method of the proof of Theorem 2.3 is to apply the ideas used in the proof of Theorem 2.2 to the argument of the proof of [10, Theorems 1.4 and 1.5]. We shall therefore omit the details and refer the reader to [10] and [11].

Remark 3. There is a mistake in the statement of [10, Corollary 1.7]; the condition $2 \leq m \leq 2n$ should be replaced by $2 \leq m < n$ as in Corollary 2.4.

§ 3. The projections Ψ_d and Γ_d.

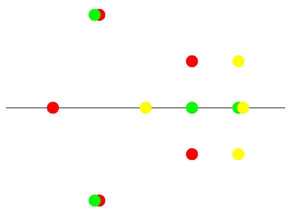
In this section, we consider the following conjecture stated in [1].

Conjecture 3.1 ([1], Conjecture 3.8)**.** *If $1 \leq m < n$, the projection maps $\Psi_d : A_d(m,n) \to \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and $\Gamma_d : \tilde{A}_d(m,n) \to \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ are homotopy equivalences.*

It is easy to see that Γ_d is a homotopy equivalence if Ψ_d is so. Thus, we only consider the projection Ψ_d. There is evidence which suggests that Ψ_d is a homotopy equivalence. Indeed, the fibre of it over $\mathrm{Alg}^*_{d-2k+2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is homeomorphic to the space of everywhere positive $\mathbb{R}$-coefficient polynomials in $(m+1)$-variables of degree $2k$ with leading coefficient 1, which is convex and then contractible. If it is a quasi-fibration, it is a homotopy equivalence. Although we cannot prove this in general, we can do it for $m=1$.

Theorem 3.2. *If $m=1$, Conjecture 3.1 is true.*

Proof. Even if we replace the point $\tilde{e}_n = [1:1:\cdots:1]$ as the base point of $\mathbb{R}\mathrm{P}^n$, all corresponding spaces are homeomorphic each other and all corresponding statements as above are equivalent. Thus, from now on, we choose $\tilde{e}_n$ as the base point of $\mathbb{R}\mathrm{P}^n$. Then by taking $z_1/z_0 = z$ we see that $A_d(1,n)$ is homeomorphic to the space consisting of all $(n+1)$-tuples $(f_0(z), \cdots, f_n(z)) \in \mathbb{R}[z]^{n+1}$ of monic $\mathbb{R}$-coefficients polynomials of one variable z with the same degree d, such that $f_0(z), \cdots, f_n(z)$ have no common *real* root (but may have a complex common root). We will exploit the convenient fact that spaces of tuples of monic polynomials in one variable can be identified with certain configuration spaces of points or particles in the complex plane. More exactly, we can also identify the space $A_d(1,n)$ with the space of d particles of each of $n+1$ different colours, in the case $n=2$, say, red, blue and yellow, located in $\mathbb{C}=\mathbb{R}^2$ symmetrically with respect to the real axis and such that no three particles of different colour lie at the same point on the real axis. Note that off the real axis the particles are completely unrestricted.



The space $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^1,\mathbb{R}\mathrm{P}^n)$ can also be thought of as a configuration space of k particles of each of $n+1$ different colours as above, where $k\leq d$, but with the additional property that when $n+1$ different particles meet (off the real line) they (and their conjugate particles) disappear (this space is topologised as the obvious quotient of the preceding one).

Finally, we need one more configuration space introduced in [12]. Let $T(d,n)$ denote the space defined as follows. Consider the d-fold symmetric product $\mathrm{SP}^d(\mathbb{R})$ of the real line $\mathbb{R}$, and we write the points of $\mathrm{SP}^d(\mathbb{R})$ as linear combinations with non-negative integer coefficients (divisors) $\sum k_ix_i$, where x_i is a point in $\mathbb{R}$ and k_i is a non-negative integer with $\sum_i k_i=d$. Now consider the subspace of the $(n+1)$-th cartesian power of $\mathrm{SP}^d(\mathbb{R})^{n+1}$ consisting of $(n+1)$-tuples of divisors whose supports have empty intersection. Then we impose on this subspace the equivalence relation "$\sim$" defined by

$$(\sum_i k_i^0x_i^0,\sum_i k_i^1x_i^1,\dots,\sum_i k_i^nx_i^n)\sim(\sum_i l_i^0x_i^0,\sum_i l_i^1x_i^1,\dots,\sum_i l_i^nx_i^n)$$

if $k_i^j\equiv l_i^j \pmod 2$ for all i,j. We denote by $T(d,n)$ its quotient space (with the quotient topology). We can think of it as the space of no more than $k_i\leq d$ particles of colour i, where $k_i\equiv d \pmod 2$, on the real axis, with the property that any even number of particles of the same colour at the same point on the real axis vanish, and of course, as before no $n+1$ particles of different colours lie at the same point. In other words, $T(d,n)$ is a configuration space modulo 2. There is a map $\Phi: A_d(1,n)\to T(d,n)$ defined by sending a collection of polynomials to the collection of their real root systems with multiplicities reduced mod 2 (when the polynomials in a collection have no real roots we send the collection to $(0,\cdots,0)$). Clearly, the map Φ factors through Ψ_d,

$$\Phi=Q_d\circ\Psi_d: A_d(1,n)\xrightarrow{\Psi_d}\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^1,\mathbb{R}\mathrm{P}^n)\xrightarrow{Q_d}T(d,n).$$

Proposition 3.3 ([12], Proposition 2.1). *The maps Ψ_d and Q_d above are homotopy equivalences.*

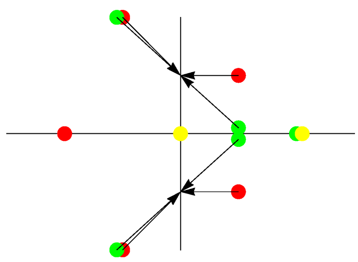
A proof of this proposition is given in [12], but as we stated earlier, it does not seem convincing to us, so we will give here a different one. More precisely, we need the following:

Lemma 3.4. *The maps Q_d and $Q_d \circ \Psi_d$ are quasi-fibrations with contractible fibres.*

From this it follows at once that Ψ_d is a homotopy equivalence, and this completes the proof of Theorem 3.2. □

Proof of Lemma 3.4. We first prove that the fibre of $Q_d \circ \Psi_d$ (and Q_d) over any point in $T(d, n)$ is contractible. Consider a configuration in $T(d, n)$.

In the fibre over this configuration, all points in the upper half plane can be sent linearly to point corresponding to the pure imaginary number i and those in the lower half plane to $-i$. For a $2k$ or $2k + 1$ fold particle lying on the real line, k of the particles are moved to $-i$ and k are moved to to $-i$ leaving 0 or 1 particles in place. This argument shows that the fibres of both $Q_d \circ \Psi_d$ and Q_d are contractible.



Next, we will show that the maps $Q_d \circ \Psi_d$ and Q_d are both quasi-fibrations. Consider a point (configuration of particles) in $T(d, n)$. By an 'obstacle for colour k' we mean a collection of n particles of distinct colours other than the k-th colour which are all located at the same point. The reason for the name is, of course, the fact that a particle of the k-th colour cannot "pass" through obstacle for the k-th colour as that would violate the requirement that there can never be $n+1$ particles of different colour in the same position.

The idea of the proof is simple enough to describe but harder to write out formally. The fibre over each configuration in $T(d, n)$ consists of configurations that are unrestricted except on the real axis. In general, fibres over nearby configurations look different because an even number of particles of the same colour can come together on the real axis and disappear. However, if the number of particles of each colour is fixed, this cannot happen. Thus the projections onto the subspace $T(d, n; p_0, \ldots, p_n)$

of $T(d,n)$ consisting of configurations with a fixed number of particles of each colour are locally trivial fibrations. Now consider the subspace $T(d,n;p_0,\dots,p_{n-1})$ of $T(d,n)$ consisting of configurations in which the numbers of particles of the first n colours are fixed at $p_0, p_2, \dots, p_{n-1}$ and filter it by the number of particles of the $n+1$-th colour (in other words, the k-th term of the filtration consists of configurations with $p_0, p_2, \dots, p_{n-1}$ particles of the first n colour and less or equal to k particles of the $n+1$-th colour). The set theoretic differences between the terms of the filtration are precisely the spaces $T(d,n;p_0,\dots,p_n)$ and we have already proved that the restriction of the maps $Q_d \circ \Psi_d$ and Q_d to the inverse images of these spaces are quasi-fibrations. Now we apply the Dold-Thom criterion [7, Lemma 4.3]. For this purpose we need to construct open neighbourhoods of spaces of configurations with no more than $k-1$ particles of the last colour in the space of configurations of no more than k-particles of the last colour. Our deformation will pull together pairs of particles of the last colour which are very close by means of a gravitational force field between particles of the last colour. For this purpose we must avoid hitting any obstacles for this colour. It is easy to see how to choose open neighbourhoods and deformations with the right properties. Applying the Dold-Thom lemma we conclude that the maps restricted to the pre-images of spaces with the number of particles of the first $n-1$ colours fixed (and an unrestricted number of particles of the last colour) are quasi-fibrations. Now fix the first $n-1$ colours and filter the resulting spaces according to the number of particles of the n-th colour. Proceeding by induction we conclude that $Q_d \circ \Psi_d$ and Q_d are quasi-fibrations. □

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