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Essential dimension of some twists of μ_{p^n}

By

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Abstract

Fix an odd prime p and a field k of characteristic not p. Let C_{p^n} be a twist of the k-group of p^n-th roots of the unity μ_{p^n}. Then we give an upper bound for the essential dimension of a twisted C_{p^n}, which is expressed in terms of the dergee of its minimal splitting field over k. We also compute the essential dimension of a twisted C_8 over fields of odd characteristic.

§ 1. INTRODUCTION

Fix a field k and let G be a finite k-group scheme. Then we have an interesting numerical invariant called the *essential dimension*, which depends on the group scheme G and the base field k (cf. Reichstein [4], Rost [5] and Berhuy-Favi [2] for the definition and basic facts).

Here we consider the case where the order of a k-group scheme G is a power of a prime p, which is different from the characteristic of the base field k. There are two main results in this paper (Theorem 3.2 and Theorem 4.10).

First, we give an upper bound for the essential dimension of a twisted group G of the group scheme μ_{p^n} over a field of characteristic $\neq p$, which depends on the minimal splitting field of G. Note that we have a trivial upper bound when $p = 2$.

Secondly, we show that

$$\mathrm{ed}_k\, G = [K : k],$$

where G is a twist of μ_8 and K is the minimal splitting field of G over the base field k with $char(k) \neq 2$. For the outline of this paper, in §2, after stating conjectures which are motivations of our investigation, we give precise formulation for the main results of this paper. In the following two sections we focus on the proof of Theorems. In §3, applying the section 5,6 in [2], we will discuss a proof of a generalization for Theorem 2.2. In §4, we will show that our first conjecture holds for the case where G is a twist of the group scheme μ_8. Our computation is based on an idea of Rost and used some facts in [6]. In fact, this result is a generalization of a result of Rost in [6] for μ_4 (Theorem 2.1).

This is an announcement of the results of the author's master thesis and detailed proofs are not given.

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§ 2. STATEMENT OF RESULTS

Fix a field k of characteristic is prime to n and fix an algebraic closure $\bar{k}$ of k. Let G be a k-form of the group μ_n, the k-group scheme of the n-th roots of the unity. Denote by K the minimal splitting field of the algebraic group G. The inclusion map $\mu_n \hookrightarrow G_m$ over K induces a k-morphism $G \longrightarrow T$, where G_m is the multiplicative group scheme and $T := \mathrm{Res}_{K/k}(G_m)$. Then we have that

$$1 \longrightarrow \mathrm{G}(\bar{k}) \longrightarrow T \otimes_k \bar{k} \longrightarrow (T/\mathrm{G})(\bar{k}) \longrightarrow 1$$

where T/G is a quotient, which exists as an affine group over k. By using a property of the scalar restriction functor Res we see that the Hilbert 90 implies that the quotient map $\pi : T \longrightarrow T/G$ is a classifying G-torsor. It shows that $\mathrm{ed}_k\, G \leq [K : k]$.

Remark. The upper bound does not look like a good one. In particularly, in this paper we give a better upper bound for the case where n is a power of an odd prime p (Theorem 3.2). But we expect that it is the best possible value in the case below:
Conjecture. Let n be a power of 2 and k be a field of odd characteristic. Then

$$\mathrm{ed}_k\, G = [K : k].$$

In the case $p = 2$, our main result is to show that the above conjecture holds for $n = 8$. The main motivation of this paper is the following theorem, the case $n = 4$, proved in [6].

Theorem 2.1. **(Rost)** *Let k be a field of characteristic different from 2. Then*

$$\mathrm{ed}_k\, G = [K : k],$$

where G is a twist of μ_4 and K is the minimal splitting field of G.

Now suppose that p is an odd prime.
Denote by G_p the cyclic subgroup of order p of the group G and then it is a twisted form of the μ_p. Moreover, we have that $Gal(K_p/k)$ is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ and $Gal(K/K_p)$ is a p-group, where K_p is the corresponding field to the group G_p.
Conjecture. We expect that

$$\mathrm{ed}_k\, G = \varphi([K_p : k])[K : K_p].$$

In fact it is motivated by the following theorem which is also deduced as a corollary of a general result in [3].

Theorem 2.2. **(Buhler-Reichstein)** *The essential dimension for the cyclic group of order $\mathbb{Z}/p^n\mathbb{Z}$ over $\mathbb{Q}$ is at most $\varphi(p-1)p^{n-1}$, where φ is the Euler function.*

§ 3. AN UPPER BOUND

In this section we discuss a generalization of Theorem 2.2 to the twisted groups. We keep the notations of the previous section and our main reference is [2].

Denote by F the subfield of K/k such that $Gal(K/F) \cong Gal(K_p/k)$. Note that $Gal(K/k)$ is a cyclic group because p is an odd prime. Put $s = [K : F]$. Hence for the character group of the torus $\mathrm{Res}_{K/F}(G_m)$ one can conclude that

$$X^*(\mathrm{Res}_{K/F}(G_m)) \cong \mathbb{Z}[x]/\langle x^s - 1\rangle,$$

where the action of $Gal(\bar{k}/F)$ is identified with multiplication by x.

For any subextension L/F of K, the norm map $N_{K/L} : K^* \to L^*$ induces a map

$$N_{K/L} : \mathrm{Res}_{K/F}(G_m) \to \mathrm{Res}_{L/F}(G_m).$$

Definition 3.1. *We now define algebraic groups $T_{K/F}$ and $\Lambda_{K/F}$ to be the kernel and the image of the map $\oplus N_{K/L}$, respectively, where*

$$\oplus N_{K/L} : \mathrm{Res}_{K/F}(G_m) \longrightarrow \bigoplus_{F \subset L \subsetneq K} \mathrm{Res}_{L/F}(G_m).$$

Then we show that the algebraic group $T_{K/F}$ is a torus contaning G_p. On applying the functor $\mathrm{Res}_{F/k}$ to the torus we obtain a new G-torsor. It implies the following theorem since π is a classifying one.

Theorem 3.2.

$$\mathrm{ed}_k\, G \le \varphi([K_p : k])[K : K_p].$$

§ 4. TWIST OF μ_8

In this section, in order to give a proof of the main result we confirm the fundamental facts on the twists of μ_8. The following notational conventions will be used throughout this chapter.

$[a]$ - the image of $a \in k$ in k/k^2, $k_a := k(\sqrt{a})$, $k_{a,b} := k(\sqrt{a}, \sqrt{b})$, $H_a = \langle \sqrt{a} \otimes \frac{1}{\sqrt{a}} \rangle$, $T_a := \mathrm{Res}_{k_a/k}(G_m)$, $T := \mathrm{Res}_{k_{a,b}/k}(G_m)$, Γ - the absolute Galois group of k.

Note that for any field extension L/k one has $T(L) = (L \otimes_k k_{a,b})^*$. Therefore we may assume that any element of $T(\bar{k})$ has of the form

$$1 \otimes x + \sqrt{a} \otimes y + \sqrt{b} \otimes z + \sqrt{ab} \otimes t,$$

where $x, y, z, t \in \bar{k}$. Now assume that k is a field of characteristic $\neq 2$. Then we have

$$H^1(k, \operatorname{Aut} \mu_8) = H^1(k, (\mathbb{Z}/8\mathbb{Z})^*) \cong k^*/(k^*)^2 \times k^*/(k^*)^2.$$

We fix two generators of $\operatorname{Aut} \mu_8$, namely $\sigma : \zeta_8 \mapsto \zeta_8^3$ and $\tau : \zeta_8 \mapsto \zeta_8^7$.

Lemma 4.1. *Let G be an k-form with respect to an element $[a] \times [b] \in H^1(k, \operatorname{Aut} \mu_8)$. Then G can be identified with a cyclic subgroup of $T(\bar{k})$ generated by*

$$\langle \sqrt{a} \otimes \frac{1}{\sqrt{2}\sqrt{a}} + \sqrt{b} \otimes \frac{\sqrt{-1}}{\sqrt{2}\sqrt{b}} \rangle.$$

Note that the image of $[a] \times [b]$ is $[ab]$ under the map

$$H^1(k, \operatorname{Aut} \mu_8) \longrightarrow H^1(k, \operatorname{Aut} \mu_4),$$

which is induced by the canonical injection $\mu_4 \to \mu_8$. Equivalently, the subgroup of G of order 4 (denote by G_{ab}) is a k-form of μ_4 with respect to $[ab]$ in $H^1(k, \operatorname{Aut} \mu_4)$.

§ 4.1. The case of degenerate twists of μ_8

If $[k_{a,b} : k] = 1$, then $\operatorname{ed}_k G = 1$ because G is isomorphic to μ_8.

Proposition 4.2. *If $[k_{a,b} : k] = 2$, then* $\operatorname{ed}_k G = 2$.

If ab is a square, then $G_{ab} \cong \mu_4$ and so we need some observations on the functor $H^1(-, G)$. A classifying space of the functor $H^1(-, G)$ is induced by the following exact sequence.

Lemma 4.3. *We have*

$$1 \longrightarrow G \times H_a \longrightarrow T \otimes_k \bar{k} \xrightarrow{\pi_a} \bar{k}^* \times \frac{(T \otimes_k \bar{k})}{\bar{k}^*} \longrightarrow 1,$$

where $\pi_a(x) = (N^4_{k_a/k}(x), [x^2])$.

Now to estimate the essential dimension of versal elements for the functor $H^1(-, G)$, we need some invariants. In fact, these invariants are cohomological invariants as follows

$$\eta_k : H^1(k, G) \to H^1(k, \mu_4),$$
$$\operatorname{Res}_a : H^1(k, G) \to H^1(k_a, G) \cong k_a^*/(k_a^*)^8.$$

The invariant η_k is a map induced by the norm map $N_{k_a/k} : G \to \mu_4$ and Res_a is the canonical restriction map induced by the inclusion $Gal(\bar{k}/k_a) \hookrightarrow Gal(\bar{k}/k)$.

Lemma 4.4. *(1). One has for $t \in k^*$ and $\lambda \in k_a$:*

$$\eta_k(\delta_k(t, [\lambda])) = (t),$$
$$\mathrm{Res}_a(\delta_k(t, [\lambda])) = (t\lambda^4/N_a^2(\lambda)).$$

(2). The pair $(\delta_k(x, 1+y\ \overline{a}), k(x,y))$ is a versal element for the functor $H^1(-,G)$.

Lemma 4.5. *If (α, F) is a versal element for $H^1(-,G)$, then* $\mathrm{ed}(\alpha) \geq 2$.

§ 4.2. The case of the generic C_8

We may assume that a, b and ab are all non-square's in k, since other cases are discussed. In this case, let us start to introduce some properties of the functor $H^1(-,G)$, which are necessary to prove our main result.

Some generalities to describe the functor $H^1(-,G)$.

Let T/G denote the algebraic group $G_m \times T_a/G_m \times T_b/G_m \times T_{ab}/G_m$. Consider a morphism $\pi_{a,b}$ from T to T/G, for $x \in T$, defined as

$$\pi_{a,b}(x) = (N_{k_{a,b}/k}(x), [N_{k_{a,b}/k_b}(x)], [N_{k_{a,b}/k_a}(x)], [N^2_{k_{a,b}/k_{ab}}(x)]).$$

Thus we can state the following useful proposition.

Proposition 4.6. *There is an exact sequence of Γ-modules*

$$1 \longrightarrow G \times H_a \times H_b \longrightarrow T(\bar{k}) \xrightarrow{\pi_{a,b}} T/G(\bar{k}) \longrightarrow 1.$$

Invariants.

Applying the cohomology to the exact sequence we obtain an exact sequence

$$k^* \times k_a^*/k^* \times k_b^*/k^* \times k_{ab}^*/k^* \xrightarrow{\delta_k} H^1(k,G) \to 1,$$

i.e.

$$\delta_k(q) \in H^1(k,G)$$

where $q = (c, [\alpha], [\beta], [\gamma])$ for $c \in k^*,\ \alpha \in k_a^*,\ \beta \in k_b^*,\ \gamma \in k_{ab}^*$.

Cohomological invariants for G_{ab}, which can be found in Rost [6], are given as :

$$\eta_1 : H^1(k, G_{ab}) \to H^1(k, \mu_2),$$
$$\eta_2 : H^1(k, G_{ab}) \to H^2(k, \mu_2).$$

Recall that the invariant η_1 is the map induced from the projection $G_{ab} \to \mu_2$. The invariant μ_2 is the composition of $H^1(k,G) \to H^1(k, PGL(2))$ and the standard map $H^1(k, PGL(2)) \to H^2(k, \mu_2)$ arises from the embedding

$$G_{ab} \longrightarrow PGL(2)$$
$$\zeta \mapsto \begin{pmatrix} 1 & \overline{ab}^{\,-1} i \\ \overline{ab}\, i & 1 \end{pmatrix}$$

where $\zeta \in G_{ab}(\bar{k})$ is a generator and $i = \sqrt{-1}$. One has for $t \in k^*$ and $\lambda \in k_{ab}$:

$$\eta_1(\delta_k(t, [\lambda])) = (N_{k_{ab}/k}(\lambda)),$$
$$\eta_2(\delta_k(t, [\lambda])) = (ab) \cup (t).$$

Consider the map $H^1(k, G) \to H^1(k, G_{ab})$, which is induced from $G \xrightarrow{[2]} G_{ab}$, we get cohomological invariants for G as follows

$$\eta_1 : H^1(k, G) \longrightarrow H^1(k, \mu_2),$$
$$\eta_2 : H^1(k, G) \longrightarrow H^2(k, \mu_2).$$

Here η_1 and η_2 are new notations and these are induced by old ones.

Proposition 4.7. *One has*

$$\eta_1(\delta_k(q)) = (N_{k_{ab}/k}(\gamma)),$$
$$\eta_2(\delta_k(q)) = (ab) \cup (c).$$

Note that the group G can be identified with $\langle 1 \otimes \frac{1}{\sqrt{2}} + \sqrt{b} \otimes \frac{i}{\sqrt{2}\sqrt{b}} \rangle$ over k_a. Thus, for the canonical restriction map $\mathrm{Res}_a : H^1(k, G) \longrightarrow H^1(k_a, G)$, we get

$$\mathrm{Res}_a(\delta_k(q)) = \delta_{k_a}\Big(\frac{\alpha^2}{N_a(\alpha)}c, [\beta^2\gamma]\Big).$$

Similarly, for the restriction map $\mathrm{Res}_b : H^1(k, G) \to H^1(k_b, G)$, we also have

$$\mathrm{Res}_b(\delta_k(q)) = \delta_{k_b}\Big(\frac{\beta^2}{N_b(\beta)}c, [\alpha^2\gamma]\Big).$$

Thus we obtain another invariants which are defined as an invariant η_2 for G:

$$\eta_a : H^1(k, G) \longrightarrow H^2(k_a, \mu_2),$$
$$\eta_b : H^1(k, G) \longrightarrow H^2(k_b, \mu_2).$$

More precisely, for instance, the map η_a is the composition of the map $H^1(k, G) \to H^1(k_a, G_a)$ and the map $H^1(k_a, G_a) \to H^2(k_a, \mu_2)$. The above considerations allow us to write the following proposition.

Proposition 4.8. *We have*

$$\eta_a(\delta_k(q)) = (b) \cup \Big(\frac{\alpha^2}{N_a(\alpha)}c\Big),$$
$$\eta_b(\delta_k(q)) = (a) \cup \Big(\frac{\beta^2}{N_b(\beta)}c\Big).$$

Remark. It would not be useful to consider the map $H^1(k, G) \longrightarrow H^1(k_a, \mu_2)$, because it gives the same result as η_1 in Proposition 4.7.

Proposition 4.9. *Assume that t_1, t_2, t_3 and t_4 are independent variables over k. Set $E := k(t_1, t_2, t_3, t_4)$ and $\omega := (t_1, [1 + t_2 \sqrt{a}], [1 + t_3 \sqrt{b}], [1 + t_4 \sqrt{ab}])$. Then the pair*

$$(\delta_E(\omega), E)$$

is a versal element for $H^1(-, G)$.

We are now ready to state the Main Theorem.

Theorem 4.10. $\mathrm{ed}_k\, G = 4$ *if a, b and ab are nonsquare elements of k.*

For the proof, we show that the smallest possible value of the essentail dimension of any element for G is 4 by using invariants η_1, η_2, η_a and η_b.

Corollary 4.11. *. Let k be a field of characteristic $\neq 2$. Then*

$$\mathrm{ed}_k(\mathbb{Z}/8\mathbb{Z}) = \begin{cases} 1, & \textit{if } 2 \textit{ and } -1 \textit{ are both squares in } k \\ 4, & \textit{if } 2, -2 \textit{ and } -1 \textit{ are not squares in } k \\ 2, & \textit{otherwise} \end{cases}$$

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