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Discrete Integrable Systems and Cluster Algebras

By

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Abstract

We construct the directed graph (quiver) for which the associated cluster algebra gives the Hirota-Miwa equation, and prove that the difference equations obtained from its reductions have the Laurent phenomenon by means of a theorem known to hold for cluster algebras. We also prove that when the autonomous difference equations called Somos-4 and 5 are deautonomized such that they preserve the Laurent phenomenon, their coefficients satisfy the q-Painlevé I and II equations, respectively.

§ 1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in 2002 [1], and are defined by so-called quivers (directed graphs) whose vertices correspond to variables (called cluster variables) on which an the operation called *quiver mutation* acts. It is well established that cluster algebras have a lot of mathematically interesting properties. One of the most distinct properties among them is the *Laurent phenomenon*. The Laurent phenomenon or the Laurentness of a difference equation implies that a solution of the initial value problem is always expressed by Laurent polynomials of the initial entries. The 2nd order difference equations called Somos-4 and 5 are the typical equations which exhibit the Laurent phenomenon. The Laurentness of these equations was proved by considering periodic quivers, which are quivers whose graphical topologies do not change by quiver mutations [2].

In this paper, we focus on the Laurent phenomenon of difference equations. Considering some specific quivers, we show that the discrete KdV equation, the Hirota-Miwa

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equation and the difference equations obtained by reduction of the Hirota-Miwa equation all enjoy the Laurent phenomenon. The periodic quivers which give Somos-4, 5 are obtained from those for the discrete KdV equation. Furthermore, by considering cluster algebras with coefficients, we prove that the necessary and sufficient conditions for the Laurentness of nonautonomous equations are given by some difference equations for the coefficients. In particular, these difference equations turn out to be the q-Painlevé I and II equations, respectively, for the deautonomized Somos-4 and 5 equations.

§ 2. Cluster algebras

In this chapter, we briefly explain on some examples the notions of cluster algebra introduced by Fomin and Zelevinsky [1][3][4] and the Laurent phenomenon.

The n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ will be called a cluster and each element x_i a cluster variable. Consider a quiver whose vertices correspond to the cluster variables. We assume that the quiver does not have a loop or a 2-cycle (Figure 1). The pair consisting of a quiver Q and a cluster $\boldsymbol{x}$, $(Q, \boldsymbol{x})$, is called a seed. We often call a vertex x_k when its corresponding cluster variable is x_k.

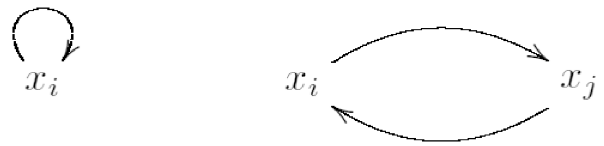

Figure 1. loop and 2-cycle

Now we define a quiver mutation μ_k $(k = 1, \ldots, n)$ at the vertex x_k of a quiver Q. Let $\mu_k(Q)$ be the new quivers obtained by the following three operations on the quiver Q. Firstly, for each pair of arrows $x_i \longrightarrow x_k \longrightarrow x_j$, we draw a new arrow $x_i \longrightarrow x_j$. Secondly, we remove all 2-cycles in the quiver thus obtained. Finally, we reverse the direction of all directed arrows at the vertex x_k. A quiver mutation μ_k satisfies ${\mu_k}^2(Q) = Q$. Figure 2 is an example of μ_1, where instead of drawing $k \in \mathbb{N}$ $(k \geq 2)$ arrows, we attached the number k to the arrow.

Next we define seed mutations μ_k $(k = 1, \ldots, n)$ of a seed $(Q, \boldsymbol{x})$. Let $(\mu_k(Q), \boldsymbol{x}') := \mu_k((Q, \boldsymbol{x}))$. The new cluster $\boldsymbol{x}' = (x'_1, \ldots, x'_n)$ is defined as

$$x'_i = x_i \quad (i \neq k),$$
$$x'_k = \frac{\prod_{j \leftarrow k} x_j + \prod_{j \rightarrow k} x_j}{x_k}.$$

This relation between $\boldsymbol{x}$ and $\boldsymbol{x}'$ is called the exchange relation. The symbol $\displaystyle\prod_{j \leftarrow k} x_j$

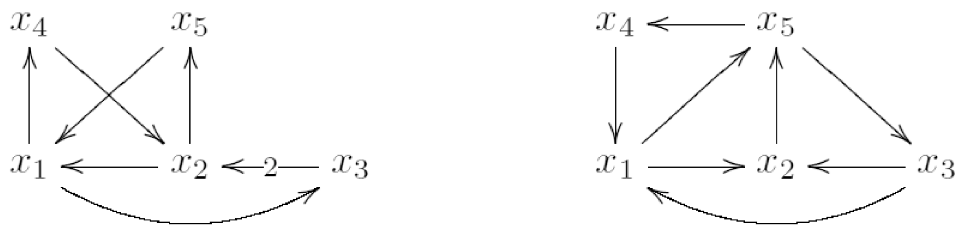

Figure 2. mutation μ_1 at the vertex x_1

$(\prod_{j\to k} x_j)$ denotes the product of all x_js whose corresponding vertices in Q have arrows from (resp. to) the vertex x_k.

Example 2.1. Let the seed $(Q, \boldsymbol{x})$ be

$$Q = x_1 \xrightarrow{2} x_2, \quad \boldsymbol{x} = (x_1, x_2).$$

The quiver mutation at x_1 gives

$$\mu_1(Q) = x_1' \xleftarrow{2} x_2', \quad \boldsymbol{x}' = (x_1', x_2'),$$

where the new cluster variables are

$$x_1' = \frac{{x_2}^2 + 1}{x_1}, \quad x_2' = x_2.$$

We apply seed mutations starting from an initial seed $(Q, \boldsymbol{x})$. Let $\mathcal{X}$ be the set of all obtained cluster variables. The cluster algebra $\mathcal{A}(Q, \boldsymbol{x})$ is then defined as

$$\mathcal{A}(Q, \boldsymbol{x}) = \mathbb{Q}\,[x | x \in \mathcal{X}] \subset \mathbb{Q}(x | x \in \mathcal{X}).$$

Example 2.2 (cluster algebra of type A_2)**.** Let $(Q, \boldsymbol{x})$ be the initial seed with

$$Q = x_1 \longrightarrow x_2, \quad \boldsymbol{x} = (x_1, x_2),$$

$$Q = Q(0), \quad \boldsymbol{x} = \boldsymbol{x(0)}, \quad x_1 = x_1(0), \quad x_2 = x_2(0).$$

We apply a mutation at x_1 to the initial seed $(Q(0), x(0))$. Denoting the new quiver and cluster variables by $Q(1), x_1(1), x_2(1)$, we find

$$Q(1) = x_1(1) \longleftarrow x_2(1), \quad \boldsymbol{x}(1) = (x_1(1), x_2(1)),$$

$$x_1(1) = \frac{x_2(0) + 1}{x_1(0)} = \frac{x_2 + 1}{x_1}, \quad x_2(1) = x_2(0) = x_2.$$

Next, we apply a mutation at x_2 to the seed $(Q(1), \boldsymbol{x}(1))$. The new quiver and cluster variables, $Q(2), x_1(2), x_2(2)$, are

$$Q(2) = x_1(2) \longrightarrow x_2(2), \quad x(2) = (x_1(2), x_2(2)),$$

$$x_1(2) = x_1(1) = \frac{x_2+1}{x_1}, \quad x_2(2) = \frac{x_1(1)+1}{x_2(1)} = \frac{x_1+x_2+1}{x_1x_2}.$$

Similarly, we apply mutations at x_1 and x_2. The new quiver and cluster variables are

$$Q(3) = x_1(3) \longleftarrow x_2(3),$$

$$x_1(3) = \frac{x_2(2)+1}{x_1(2)} = \frac{x_1+1}{x_2}, \quad x_2(3) = x_2(2) = \frac{x_1+x_2+1}{x_1x_2},$$

$$Q(4) = x_1(4) \longrightarrow x_2(4),$$

$$x_1(4) = x_1(3) = \frac{x_1+1}{x_2}, \quad x_2(4) = \frac{x_1(3)+1}{x_2(3)} = x_1,$$

$$Q(5) = x_1(5) \longleftarrow x_2(5),$$

$$x_1(5) = \frac{x_2(4)+1}{x_1(4)} = x_2, \quad x_2(4) = x_2(3) = x_1.$$

Since the last seed is equal to the initial seed with x_1 and x_2 exchanged, it is clear that the seed returns to the original seed after 5 more applications of the mutations. All possible mutations are given above, and the newly obtained cluster variables are

$$x_1(1) = \frac{x_2+1}{x_1}, \quad x_2(2) = \frac{x_1+x_2+1}{x_1x_2}, \quad x_1(3) = \frac{x_1+1}{x_2}.$$

Therefore, the cluster algebra $\mathcal{A}(Q, \boldsymbol{x})$ is found to be

$$\mathcal{A}(Q, \boldsymbol{x}) = \mathbb{Q}\,[x_1, x_2, x_1(1), x_2(2), x_1(3)] \subset \mathbb{Q}(x_1, x_2).$$

In Example 2.2 the number of cluster variables is finite and this cluster algebra is said to be of type A_2. However, the number of cluster variables will in general be infinite. The following theorem holds [4].

Theorem 2.3 (Finite type classification). *The number of different cluster variables that appears in $\mathcal{A}(Q, \boldsymbol{x})$ is finite, if and only if the undirected graph, topologically equivalent to the quiver of a seed of $\mathcal{A}(Q, \boldsymbol{x})$, coincides with a Dynkin diagram of type A, D or E.*

In the Example 2.2, all the cluster variables $x_1(1), x_2(2), x_1(3)$ are Laurent polynomials of the initial cluster variables x_1, x_2. In general, we have the following theorem [4].

Theorem 2.4 (Laurent phenomenon). *All cluster variables obtained by mutations can be expressed as Laurent polynomials of the cluster variables in the initial seed.*

§ 3. Laurent phenomenon of difference equations

In this chapter we will show that many difference equations have Laurentness by applying Theorem 2.4.

First, we define the Laurent phenomenon for difference equations.

Definition 3.1 (Laurent phenomenon). A difference equation is said to exhibit the Laurent phenomenon (or Laurentness) if the solution of its initial value problem is expressed as a Laurent polynomial of the initial entries.

Many one-dimensional difference equations can be shown to exhibit the Laurent phenomenon by using periodic quivers and Theorem 2.4 [2]. A periodic quiver is defined as follows.

Definition 3.2 (periodic quiver). A quiver Q is called a periodic quiver if there exists a quiver mutation μ_k such that $\mu_k(Q)$ coincides with Q, up to an exchange of cluster variables at the vertices.

The following is an example of a periodic quiver called the Somos-4 quiver.

Example 3.3 (Somos-4 quiver). Consider the quiver in Figure 3. We apply a mutation at ρ_0 and obtain the quiver shown in Figure 4. Then, we move the vertex ρ_0 to the right, to the vertex ρ_3, and obtain the quiver in Figure 5, which is the same as the original quiver if we shift the indices of the vertices.

We shall illustrate the above notions in the case of the difference equation called Somos-4

$$\rho_n \rho_{n+4} = {\rho_{n+2}}^2 + \rho_{n+1}\rho_{n+3}, \tag{3.1}$$

which is known to exhibit the Laurent phenomenon [2]. Let us take as initial cluster variables $\rho_0, \rho_1, \rho_2, \rho_3$ and let us consider a quiver with an initial seed as shown in Figure 3. We apply the mutations μ_0 at ρ_0 and denote the new cluster variable by $\rho_4 := \mu_0(\rho_0)$. By the exchange relation, ρ_4 is defined as

$$\rho_4 = \frac{{\rho_2}^2 + \rho_1\rho_3}{\rho_0}.$$

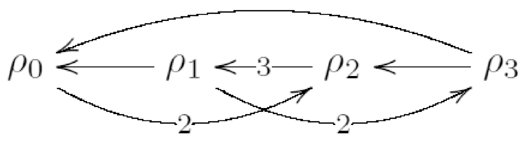

Figure 3. Somos-4 quiver

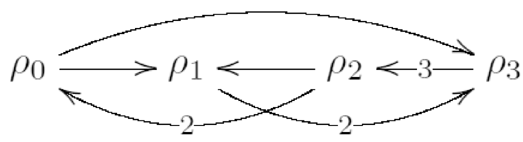

Figure 4. after mutation at ρ_0

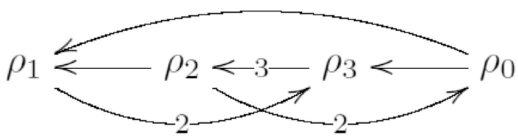

Figure 5. after replacing variables

Since the quiver is a periodic quiver, we obtain the same quiver as that shown in Figure 5. Then, we apply a mutation at ρ_1 and denote the new cluster variable by ρ_5. By the exchange relation, ρ_5 is defined as

$$\rho_5 = \frac{{\rho_3}^2 + \rho_2\rho_4}{\rho_1}.$$

As in the previous mutation, we obtain a new quiver which coincides with the initial quiver, except for a shift of the vertex indices by one. Similarly, we apply a mutation at $\rho_2, \rho_3, \cdots$ and denote the new cluster variables by $\rho_6, \rho_7, \cdots$ successively. By the exchange relation, ρ_n is defined as

$$\rho_{n+4} = \frac{{\rho_{n+2}}^2 + \rho_{n+1}\rho_{n+3}}{\rho_n}.$$

By Theorem 2.4, all cluster variables ρ_n can be expressed as Laurent polynomials of the initial variables $\rho_0, \rho_1, \rho_2, \rho_3$. Therefore, we conclude that Somos-4 (3.1) exhibits the Laurent phenomenon.

Similarly, it can be shown that the difference equation called Somos-5

$$\rho_n\rho_{n+5} = \rho_{n+2}\rho_{n+3} + \rho_{n+1}\rho_{n+4}$$

exhibits the Laurent phenomenon [2]. In this case, we take initial variables $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4$ as initial entries, and consider a quiver with initial seed as shown in Figure 6. This quiver is also a periodic quiver. We apply a mutation at $\rho_0, \rho_1, \cdots$ and denote the new cluster variables by $\rho_5, \rho_6, \cdots$, which satisfy Somos-5 and exhibit the Laurent phenomenon.

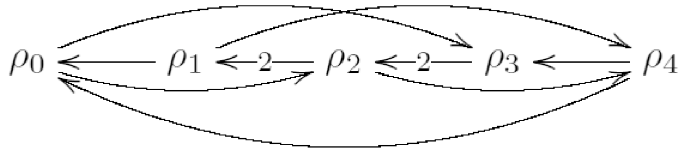

Figure 6. Somos-5 quiver

Next, we show that the discrete KdV equation, which is the two dimensional difference equation:

$$\sigma_{l-1}^{n}\sigma_{l+1}^{n+1}=\sigma_{l+1}^{n}\sigma_{l-1}^{n+1}+\sigma_{l}^{n+1}\sigma_{l}^{n} \tag{3.2}$$

exhibits the Laurent phenomenon for some specific initial value problems. In this case, unlike Somos-4 or 5 the choice of the initial entries is not unique. First, we take the initial entries as

$$\sigma_{l}^{n}\quad(l=0\text{ or }l=1\text{ or }n=0) \tag{3.3}$$

and consider a quiver with an initial seed as in Figure 7. Here l denotes the index for the horizontal direction and n denotes that for the vertical direction, and the quiver contains infinitely many vertices. We apply a mutation at σ_0^0 and denote the new cluster variable by σ_2^1. By the exchange relation, σ_2^1 is defined as

$$\sigma_2^1=\frac{\sigma_2^0\sigma_0^1+\sigma_1^1\sigma_1^0}{\sigma_0^0}$$

and we obtain the new quiver shown in Figure 8, where we have changed the position of σ_2^1. This quiver is not a periodic quiver, but it does have similar properties. The partial quiver which consists of all the links with the vertex σ_0^0 in Figure 7 and that with the vertices σ_0^1 or σ_1^0 in Figure 8 are equivalent. We apply a mutation at σ_0^1 of the quiver in Figure 8 and denote the new cluster variable by σ_2^2. By the exchange relation, σ_2^2 is defined as

$$\sigma_2^2=\frac{\sigma_2^1\sigma_0^2+\sigma_1^2\sigma_1^1}{\sigma_0^1}.$$

Furthermore, we apply a mutation at σ_1^0 of the quiver in Figure 8 and denote the new cluster variable by σ_3^1. By the exchange relation, σ_3^1 is defined as

$$\sigma_3^1=\frac{\sigma_3^0\sigma_1^1+\sigma_2^1\sigma_2^0}{\sigma_1^0}.$$

We successively apply mutations at the variables in the lower left part in the figures, to obtain new variables σ_l^n by

$$\sigma_{l+1}^{n+1}=\frac{\sigma_{l+1}^{n}\sigma_{l-1}^{n+1}+\sigma_{l}^{n+1}\sigma_{l}^{n}}{\sigma_{l-1}^{n}},$$

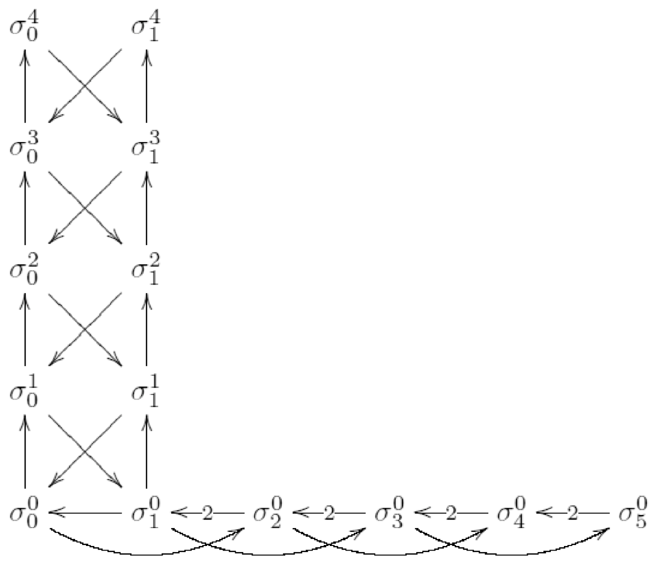

Figure 7. discrete KdV equation quiver

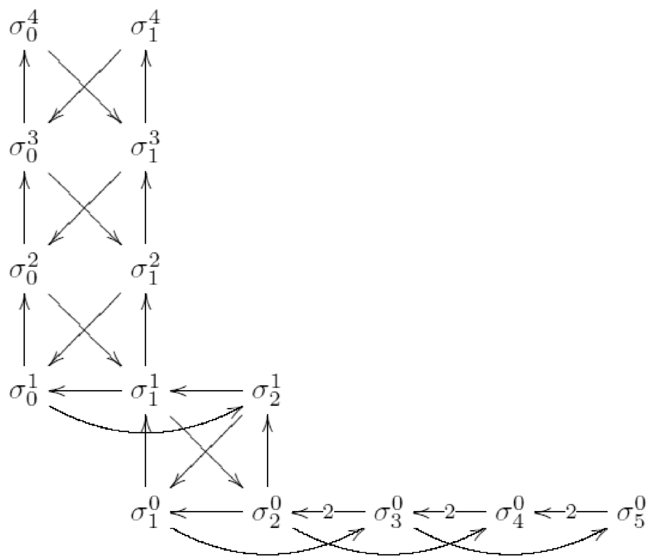

Figure 8. after mutation at σ_0^0

which is the dKdV equation (3.2). Hence, from Theorem 2.4 we find that all the cluster variables σ_l^n can be expressed as Laurent polynomials of the initial variables. Thus, we find that the Laurentness of multidimensional difference equations may be proved by using quivers (with an infinite number of vertices) that possess some good properties.

Besides (3.3), there are other sets of initial variables for which the discrete KdV equation exhibits the Laurent phenomenon such as:

$$\begin{aligned} &\sigma_l^n \quad (l+n=k, k=0,1,2), \\ &\sigma_l^n \quad (l+2n=2k, k=0,1,2,3), \\ &\sigma_l^n \quad (l+3n=3k, k=0,1,2,3,4). \end{aligned}$$

We take the quivers of initial seeds shown in Figs. 9,10,11 proving that the discrete KdV equation exhibits the Laurent phenomenon for these initial variables. These quivers, as well as that in Figure 7, transform to each other by appropriate quiver mutations.

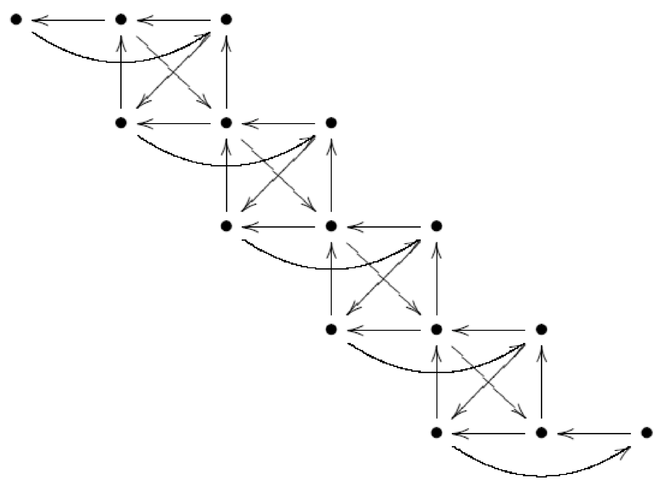

Figure 9. initial values : $\sigma_l^n \quad (l+n=k, k=0,1,2)$

The Hirota-Miwa equation

$${}^n\tau_l^{m+1} \cdot {}^{n+1}\tau_{l+1}^m = {}^n\tau_{l+1}^m \cdot {}^{n+1}\tau_l^{m+1} + {}^{n+1}\tau_l^m \cdot {}^n\tau_{l+1}^{m+1}$$

also exhibits the Laurent phenomenon [5]. The quiver of the initial seed is shown in Figure 12, where we take initial variables

$${}^n\tau_l^m \quad (l-m+n=k, k=-1,0,1).$$

As a matter of fact, the Somos-4 and Somos-5 quivers are obtained from a *reduction* of the quiver of the discrete KdV equation, and that of the discrete KdV equation is

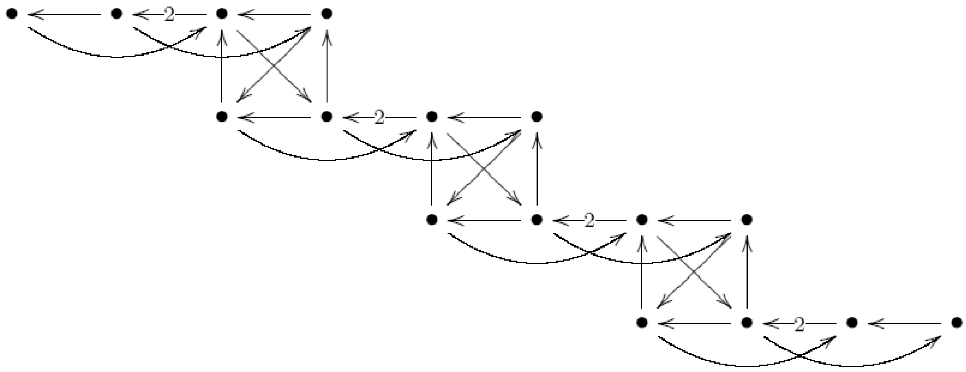

Figure 10. initial values : $\sigma_l^n \quad (l + 2n = 2k, k = 0, 1, 2, 3)$

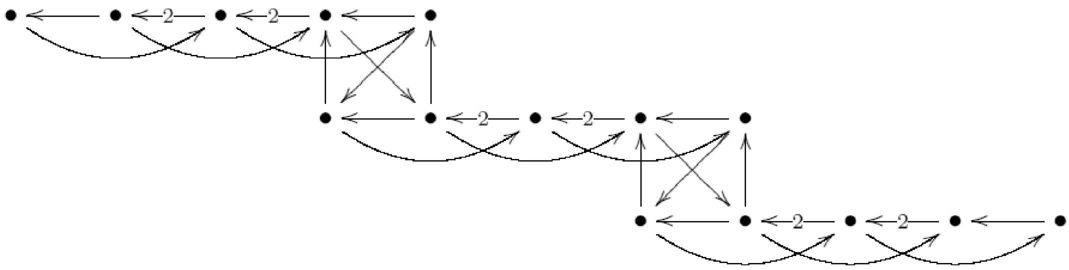

Figure 11. initial values : $\sigma_l^n \quad (l + 3n = 3k, k = 0, 1, 2, 3, 4)$

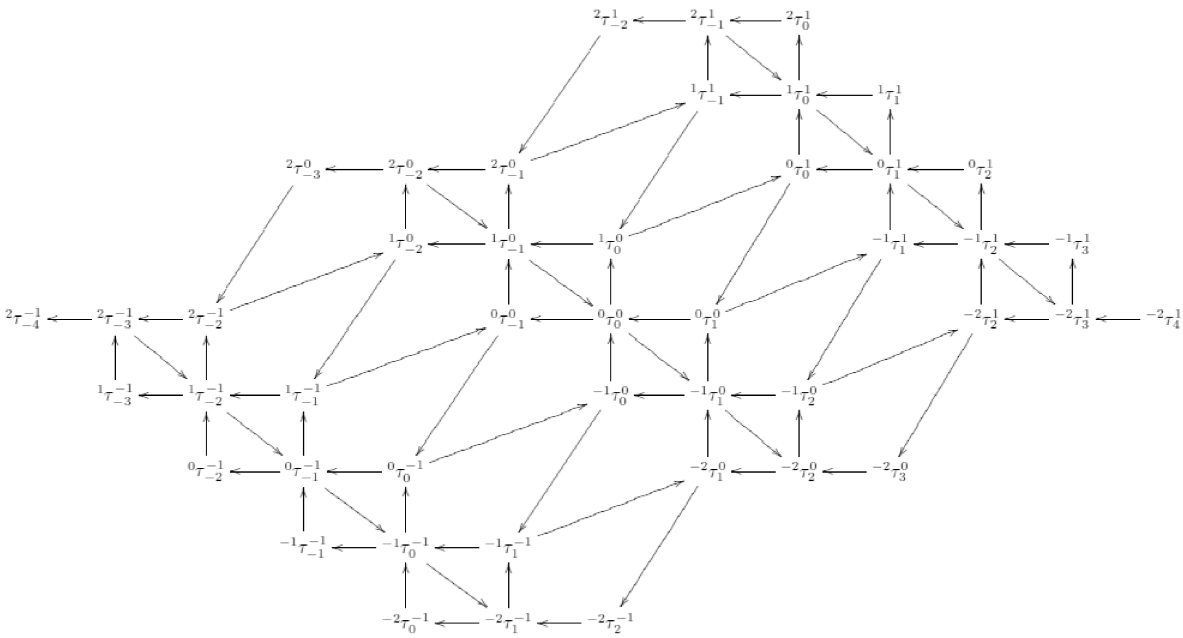

Figure 12. Hirota-Miwa equation quiver

obtained from a reduction of the quiver of the Hirota-Miwa equation. In this sense, the quivers of the discrete KdV equation and the Hirota-Miwa equation introduced in this paper can be regarded as generalizations of periodic quivers. In particular, Somos-4

$$\rho_n \rho_{n+4} = {\rho_{n+2}}^2 + \rho_{n+1}\rho_{n+3}$$

is obtained from the discrete KdV equation

$$\sigma_{l-1}^{n} \sigma_{l+1}^{n+1} = \sigma_{l+1}^{n} \sigma_{l-1}^{n+1} + \sigma_{l}^{n+1} \sigma_{l}^{n}$$

by imposing the reduction condition

$$\sigma_{l+2}^{n-1} = \sigma_l^n, \quad \rho_l := \sigma_l^0.$$

The Somos-4 quiver in Figure 3 can be obtained by *flattening* the quiver of the discrete KdV equation in Figure 10 in the $(l, n) = (2, -1)$ direction. Similarly, Somos-5

$$\rho_n \rho_{n+5} = \rho_{n+2}\rho_{n+3} + \rho_{n+1}\rho_{n+4}$$

is obtained from the discrete KdV equation by imposing the reduction condition

$$\sigma_{l+3}^{n-1} = \sigma_l^n, \quad \rho_l := \sigma_l^0.$$

Accordingly, the Somos-5 quiver in Figure 6 can be obtained by flattening the discrete KdV equation quiver in Figure 11 in the $(l, n) = (3, -1)$ direction. Moreover, the discrete KdV equation is obtained from the Hirota-Miwa equation

$${}^{n}\tau_{l}^{m+1} \cdot {}^{n+1}\tau_{l+1}^{m} = {}^{n}\tau_{l+1}^{m} \cdot {}^{n+1}\tau_{l}^{m+1} + {}^{n+1}\tau_{l}^{m} \cdot {}^{n}\tau_{l+1}^{m+1}$$

by imposing the reduction condition

$${}^{n}\tau_{l+1}^{m+1} = {}^{n}\tau_{l}^{m}, \quad \sigma_l^n := {}^{n}\tau_l^0.$$

Accordingly, the quiver of the discrete KdV equation in Figure 9 can be obtained by flattening the quiver of the Hirota-Miwa equation quiver in Figure 12 in the $(l, m, n) = (1, 1, 0)$ direction. It should be noted that, in addition to the quivers for the difference equations introduced this paper, quivers of other difference equations constructed from reductions of the Hirota-Miwa equation can be obtained by similarly flattening the quiver of the Hirota-Miwa equation, and hence they will all exhibit the Laurent phenomenon.

§ 4. Cluster algebras with coefficient

In this chapter, we will define cluster algebras with coefficients following [6].

We introduce two sets of n-tuples $\boldsymbol{x} := (x_1, \dots, x_n)$ and $\boldsymbol{y} := (y_1, \dots, y_n)$. Each x_i is called a cluster variable and each y_i a coefficient. We consider the same quivers as in the case of the cluster algebras without coefficients. Let $(Q, \boldsymbol{x}, \boldsymbol{y})$ be the triple of the quiver Q, the cluster variables $\boldsymbol{x}$, and the coefficients $\boldsymbol{y}$. $(Q, \boldsymbol{x}, \boldsymbol{y})$ will be called a seed as well.

Next, we define a seed mutation μ_k $(k = 1, \dots, n)$ of a seed $(Q, \boldsymbol{x}, \boldsymbol{y})$. Let $\mu_k((Q, \boldsymbol{x}, \boldsymbol{y})) = (\mu_k(Q), \boldsymbol{x}', \boldsymbol{y}')$, where $\mu_k(Q)$ is the same quiver as that obtained by the quiver mutation without coefficients. The new cluster variables $\boldsymbol{x}' = (x'_1, \dots, x'_n)$ are defined as

$$x'_i = x_i \quad (i \neq k),$$
$$x'_k = \frac{\prod_{j \leftarrow k} x_j + y_k \prod_{j \to k} x_j}{(y_k + 1) x_k},$$

(The products are the same as those for the case without coefficients). The new coefficients $\boldsymbol{y}' = (y'_1, \dots, y'_n)$ are defined as

$$y'_i = \begin{cases} \dfrac{1}{y_i} & (i = k), \\ y_i \, (y_k + 1)^\lambda & (i \xrightarrow{\lambda} k), \\ y_i \left(\dfrac{y_k}{y_k + 1} \right)^\lambda & (i \xleftarrow{\lambda} k), \\ y_i & (i \quad k), \end{cases}$$

where the second, third, and fourth conditions (from the top) treat the case where there exist λ arrows from x_i to x_k in Q, the case where there exist λ arrows from x_k to x_i, and the case where there do not exist any arrows between x_i and x_k.

We have the following theorem [6].

Theorem 4.1 (Laurent phenomenon). *In cluster algebras with coefficients, all cluster variables obtained by mutations can be expressed as Laurent polynomials of the cluster variables of the initial seed.*

We give the example of a cluster algebra with coefficients, of type A_2.

Example 4.2. We take an initial seed (Q, x, y)

$$Q = x_1 \longrightarrow x_2, \quad \boldsymbol{x} = (x_1, x_2), \quad \boldsymbol{y} = (y_1, y_2),$$

where $Q = Q(0), \boldsymbol{x} = \boldsymbol{x}(0), x_1 = x_1(0), x_2 = x_2(0), \boldsymbol{y} = \boldsymbol{y}(0), y_1 = y_1(0), y_2 = y_2(0)$. We then apply a mutation at x_1 to the initial seed $(Q(0), \boldsymbol{x}(0), \boldsymbol{y}(0))$. Let us denote the new quiver, cluster variables, and coefficients by $Q(1), x_1(1), x_2(1), y_1(1), y_2(1)$. These are given by

$$Q(1) = x_1(1) \longleftarrow x_2(1), \quad x(1) = (x_1(1), x_2(1)), \quad y(1) = (y_1(1), y_2(1)),$$

$$x_1(1) = \frac{x_2(0) + y_1(0)}{(y_1(0)+1)x_1(0)} = \frac{x_2 + y_1}{(y_1+1)x_1}, \quad x_2(1) = x_2(0) = x_2,$$
$$y_1(1) = \frac{1}{y_1(0)} = \frac{1}{y_1}, \quad y_2(1) = \frac{y_2(0)y_1(0)}{y_1(0)+1} = \frac{y_1y_2}{y_1+1}.$$

Then, we apply a mutation at x_2 to the seed $(Q(1), x(1), y(1))$. We denote the new quiver, cluster variables, and coefficients by $Q(2), x_1(2), x_2(2), y_1(2), y_2(2)$:

$$Q(2) = x_1(2) \longrightarrow x_2(2), \quad x(2) = (x_1(2), x_2(2)), \quad y(2) = (y_1(2), y_2(2)),$$
$$x_1(2) = x_1(1) = \frac{x_2 + y_1}{(y_1+1)x_1}, \quad x_2(2) = \frac{x_1(1) + y_2(1)}{(y_2(1)+1)x_2(1)} = \frac{y_1y_2x_1 + x_2 + y_1}{(y_1y_2 + y_1 + 1)x_1x_2},$$
$$y_1(2) = \frac{y_1(1)y_2(1)}{y_2(1)+1} = \frac{y_2}{y_1y_2 + y_1 + 1}, \quad y_2(2) = \frac{1}{y_2(1)} = \frac{y_1+1}{y_1y_2}.$$

Similarly, we apply a mutation at x_1 and x_2. The new quiver, cluster variables, and coefficients are

$$Q(3) = x_1(3) \longleftarrow x_2(3),$$
$$x_1(3) = \frac{x_2(2) + y_1(2)}{(y_1(2)+1)x_1(2)} = \frac{y_2x_1 + 1}{(y_2+1)x_2}, \quad x_2(3) = x_2(2) = \frac{y_1y_2x_1 + x_2 + y_1}{(y_1y_2 + y_1 + 1)x_1x_2},$$
$$y_1(3) = \frac{1}{y_1(2)} = \frac{y_1y_2 + y_1 + 1}{y_2}, \quad y_2(3) = \frac{y_2(2)y_1(2)}{y_1(2)+1} = \frac{1}{y_1(y_2+1)},$$

$$Q(4) = x_1(4) \longrightarrow x_2(4),$$
$$x_1(4) = x_1(3) = \frac{y_2x_1 + 1}{(y_2+1)x_2}, \quad x_2(4) = \frac{x_1(3) + y_2(3)}{(y_2(3)+1)x_2(3)} = x_1,$$
$$y_1(4) = \frac{y_1(3)y_2(3)}{y_2(3)+1} = \frac{1}{y_2}, \quad y_2(4) = \frac{1}{y_2(3)} = y_1(y_2+1),$$

$$Q(5) = x_1(5) \longleftarrow x_2(5),$$
$$x_1(5) = \frac{x_2(4) + y_1(4)}{(y_1(4)+1)x_1(4)} = x_2, \quad x_2(5) = x_2(4) = x_1,$$
$$y_1(5) = \frac{1}{y_1(4)} = y_2, \quad y_2(5) = \frac{y_2(4)y_1(4)}{y_1(4)+1} = y_1.$$

Since the last seed is equal to the initial seed when we replace x_1 and y_1 with x_2 and y_2 respectively, it is clear that a further 5 applications of these mutations make the seed return to the original seed. The above mutations exhaust all possibilities. The newly obtained cluster variables are

$$x_1(1) = \frac{x_2 + y_1}{(y_1+1)x_1}, \quad x_2(2) = \frac{y_1y_2x_1 + x_2 + y_1}{(y_1y_2 + y_1 + 1)x_1x_2}, \quad x_1(3) = \frac{y_2x_1 + 1}{(y_2+1)x_2},$$

which are all Laurent polynomials of the initial cluster variables x_1, x_2.

§ 5. Laurent phenomenon of nonautonomous difference equations

In this chapter, we give a theorem concerning the Laurent phenomenon of nonautonomous difference equations. In a special case of this theorem, q-Painlevé equations appear as the difference equations that are satisfied by the coefficients.

Let Q be a periodic quiver with the vertices $x_0, \ldots, x_{k-1}$ that coincides with the quiver $\mu_0(Q)$ by replacing x_i with x_{i-1} $(i = 1, 2, ..., k-1, k \equiv 0)$. The following theorem can be proved using Theorem 4.1. Note that the coefficients in the equations below are in fact the variables $y_n(n)$ of Example 4.2.

Theorem 5.1. *Assume that the quiver Q satisfies the above condition. Then, the nonautonomous difference equation*

$$(y_n + 1)x_n x_{n+k} = \prod_{0 \to i} x_{n+i} + y_n \prod_{0 \leftarrow i} x_{n+i},$$

where the coefficients y_n satisfy

$$y_n = \frac{1}{y_{n-k}} \prod_{0 \to i} \frac{y_{n-i}}{y_{n-i} + 1} \cdot \prod_{0 \leftarrow i} (y_{n-i} + 1),$$

exhibits the Laurent phenomenon.

The number k denotes the number of vertices of the periodic quiver. The possible types of periodic quivers are quite restricted when k is small. We explicitly give the resulting difference equations obtained from Theorem 5.1 for small k.

- case of $k = 2$ (Figure 13)

$$(y_n + 1)x_n x_{n+2} = 1 + y_n x_{n+1}^{\lambda},$$

$$y_n = \frac{(y_{n-1} + 1)^{\lambda}}{y_{n-2}}.$$

$$x_0 \xleftarrow{\lambda} x_1$$

Figure 13. periodic quiver with two vertices

- case of $k = 3$ (Figure 14)

$$(y_n + 1)x_n x_{n+3} = 1 + y_n x_{n+1}^{\lambda} x_{n+2}^{\lambda},$$

$$y_n = \frac{(y_{n-1} + 1)^{\lambda}(y_{n-2} + 1)^{\lambda}}{y_{n-3}}.$$

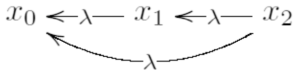

Figure 14. periodic quiver with three vertices

- case of $k = 4$ (Figure 15)

$$(y_n + 1)x_n x_{n+4} = 1 + y_n x_{n+1}^{\lambda} x_{n+2}^{\mu} x_{n+3}^{\lambda},$$

$$y_n = \frac{(y_{n-1} + 1)^{\lambda}(y_{n-2} + 1)^{\mu}(y_{n-3} + 1)^{\lambda}}{y_{n-4}},$$

or

$$(y_n + 1)x_n x_{n+4} = x_{n+2}^{\mu} + y_n x_{n+1}^{\lambda} x_{n+3}^{\lambda}, \tag{5.1}$$

$$y_n = \frac{y_{n-2}^{\mu}(y_{n-1} + 1)^{\lambda}(y_{n-3} + 1)^{\lambda}}{y_{n-4}(y_{n-2} + 1)^{\mu}}. \tag{5.2}$$

Figure 15. periodic quiver with four vertices

- case of $k = 5$ (Figure 16)

$$(y_n + 1)x_n x_{n+5} = 1 + y_n x_{n+1}^{\lambda} x_{n+2}^{\mu} x_{n+3}^{\mu} x_{n+4}^{\lambda},$$

$$y_n = \frac{(y_{n-1} + 1)^{\lambda}(y_{n-2} + 1)^{\mu}(y_{n-3} + 1)^{\mu}(y_{n-4} + 1)^{\lambda}}{y_{n-5}},$$

or

$$(y_n + 1)x_n x_{n+5} = x_{n+2}^{\mu} x_{n+3}^{\mu} + y_n x_{n+1}^{\lambda} x_{n+4}^{\lambda}, \tag{5.3}$$

$$y_n = \frac{y_{n-2}^{\mu} y_{n-3}^{\mu}(y_{n-1} + 1)^{\lambda}(y_{n-4} + 1)^{\lambda}}{y_{n-5}(y_{n-2} + 1)^{\mu}(y_{n-3} + 1)^{\mu}}, \tag{5.4}$$

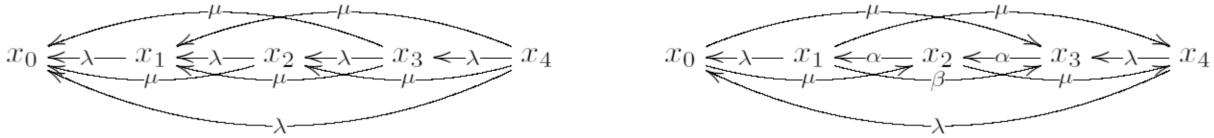

Figure 16. periodic quiver with five vertices

where, $\lambda, \mu \in \mathbb{Z}_{>0}$.

We consider the Somos-4 quiver in Figure 3. In this case, $\lambda = 1, \mu = 2$ for (5.1),(5.2). Therefore, the difference equations in Theorem 5.1 are

$$(y_n + 1)x_n x_{n+4} = x_{n+2}^2 + y_n x_{n+1} x_{n+3}, \tag{5.5}$$

$$y_n = \frac{y_{n-2}^2 (y_{n-1} + 1)(y_{n-3} + 1)}{y_{n-4}(y_{n-2} + 1)^2}.$$

The nonautonomous difference equation for x_n is nothing but Somos-4 with coefficients. When we integrate 2 times the difference equation for y_n, we obtain the q-Painlevé I equation [7]

$$y_{n-1} y_{n+1} = c\gamma^n \frac{y_n + 1}{{y_n}^2},$$

where c, γ are arbitrary constants. Hence, we find that the deautonomized Somos-4 (5.5) exhibits the Laurent phenomenon if the coefficients satisfy the q-Painlevé I equation.

Similarly, when we consider the Somos-5 quiver in Figure 6, where $\lambda = 1, \mu = 1$ for (5.3),(5.4), the difference equation in Theorem 5.1 is

$$(y_n + 1)x_n x_{n+5} = x_{n+2} x_{n+3} + y_n x_{n+1} x_{n+4},$$

$$y_n = \frac{y_{n-2} y_{n-3} (y_{n-1} + 1)(y_{n-4} + 1)}{y_{n-5}(y_{n-2} + 1)(y_{n-3} + 1)}.$$

The nonautonomous difference equation for x_n is the deautonomized Somos-5. Integrating the difference equation of y_n 3 times, we obtain the q-Painlevé II equation [8]

$$y_{2m-2} y_{2m} = c_1 \gamma^m \frac{y_{2m-1} + 1}{y_{2m-1}},$$

$$y_{2m-1} y_{2m+1} = c_2 \gamma^m \frac{y_{2m} + 1}{y_{2m}},$$

where c_1, c_2, γ are arbitrary constants. Hence, the deautonomized Somos-5 exhibits the Laurent phenomenon if the coefficients satisfy the q-Painlevé II equation.

§ 6. Conclusion

We found that the q-Painlevé I and II equations can be obtained from difference equations that exhibit the Laurent phenomenon: A necessary and sufficient condition for the deautonomized Somos-4 (5) to exhibit the Laurent phenomenon is that its coefficients solve the q-Painlevé I (II) equation. It is interesting to try to clarify whether other q-Painlevé equations can be similarly obtained from some periodic quiver. We also expect that multidimensional extensions of Theorem 5.1 may be possible, by using quivers which have similar properties to those discussed in connection to the discrete KdV equation.

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