The Laurent Phenomenon and Discrete Integrable Systems

By

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Abstract

The Laurent phenomenon is the property that the solution to an initial value problem of a discrete equation is expressed as a Laurent polynomial of the initial values. This concept has arisen from the study of cluster algebras, for which it is known that any cluster variable is a Laurent polynomial of the initial cluster variables. In this paper, we leave the connection with cluster algebras aside and study the Laurent phenomenon for its own sake. We will explain that most of the discrete bilinear equations that appear in the field of integrable systems exhibit this phenomenon and we shall discuss its relation to integrability. Finally, we shall introduce a technique for calculating the algebraic entropies relying on this phenomenon. For reasons of brevity we shall omit most proofs of the theorems we present.

§ 1.   Introduction

In this section, we will introduce the Laurent phenomenon by means of some simple examples. Although the phenomenon itself arose from cluster algebras, specific knowledge concerning these algebras is not necessary in the present context where we shall only concern ourselves with the Laurent phenomenon itself.

§ 1.1.   The Laurent phenomenon

Definition 1.1 (Laurent phenomenon).   An initial value problem of a discrete equation exhibits the Laurent phenomenon if its solution can be expressed as a Laurent polynomial of the initial values.

Example 1.2. Consider the equation

(1.1)
$$\begin{cases} f_m = \frac{f_{m-1}^2 + \alpha}{f_{m-2}}, \\ f_0 = X, \, f_1 = Y, \end{cases}$$

where $\alpha \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ is a parameter. The first several iterates are

$$f_2 = \frac{Y^2 + \alpha}{X},$$

$$f_3 = \frac{\left(\frac{Y^2 + \alpha}{X}\right)^2 + \alpha}{Y} = \frac{(Y^2 + \alpha)^2 + \alpha X^2}{X^2 Y},$$

$$f_4 = \frac{\left(\frac{(Y^2 + \alpha)^2 + \alpha X^2}{X^2 Y}\right)^2 + \alpha}{\frac{Y^2 + \alpha}{X}} = \frac{(Y^2 + \alpha)^3 + 2\alpha X^2(Y^2 + \alpha) + \alpha X^4}{X^3 Y^2},$$

$$f_5 = \cdots = \frac{(Y^2 + \alpha)^4 + 3\alpha X^2(Y^2 + \alpha)^2 + 2\alpha X^4(Y^2 + \alpha) + \alpha X^6 + \alpha^2 X^4}{X^4 Y^3},$$

and we see that f_2, f_3, f_4, f_5 are Laurent polynomials of X and Y. We will in fact prove that all of the f_m are Laurent polynomials of X and Y, and thus, that this equation exhibits the Laurent phenomenon.

Example 1.3. Consider the equation

$$\begin{cases} f_m = \frac{f_{m-1} + f_{m-2} + 1}{f_{m-1}}, \\ f_0 = X, \, f_1 = Y. \end{cases}$$

The first two iterates are

$$f_2 = \frac{X + Y + 1}{Y},$$

$$f_3 = \frac{Y^2 + X + 2Y + 1}{X + Y + 1},$$

and f_3 is not a Laurent polynomial of X, Y. Therefore this equation does not exhibit the Laurent phenomenon, and we see that an equation defined by a Laurent polynomial does not always have the Laurent property. Since Laurent polynomials are not closed under division, the Laurent phenomenon in fact requires sufficient cancellations at each step in the iteration.

The above examples are all equations on a one dimensional lattice. It is however also possible to consider the Laurent phenomenon for multidimensional discrete systems.

Example 1.4. Consider the equation

$$f_{\ell m} = \begin{cases} \frac{f_{\ell, m-1} f_{\ell-1, m} + \alpha}{f_{\ell-1, m-1}} & (\ell, m > 0), \\ X_{\ell m} & (\ell = 0 \text{ or } m = 0), \end{cases}$$

where $\alpha \in \mathbb{C}^{\times}$ is a parameter. The first several iterates are

$$f_{11} = \frac{f_{10}f_{01} + \alpha}{f_{00}} = \frac{X_{10}X_{01} + \alpha}{X_{00}},$$

$$f_{21} = \frac{f_{20}f_{11} + \alpha}{f_{10}} = \frac{X_{10}X_{01}X_{20} + \alpha X_{20} + \alpha X_{00}}{X_{00}X_{10}},$$

$$f_{12} = \frac{f_{11}f_{02} + \alpha}{f_{01}} = \frac{X_{10}X_{01}X_{02} + \alpha X_{02} + \alpha X_{00}}{X_{00}X_{01}},$$

$$f_{22} = \frac{f_{21}f_{12} + \alpha}{f_{11}} = \frac{X_{10}X_{01}X_{20}X_{02} + \alpha X_{20}X_{02} + \alpha X_{00}X_{20} + \alpha X_{00}X_{02} + \alpha X_{00}^2}{X_{00}X_{10}X_{01}}.$$

In fact, all of the $f_{\ell m}$ are Laurent polynomials of X_{ij} and this discrete system exhibits the Laurent phenomenon.

§ 1.2. Characterizations of the Laurent phenomenon

When trying to show the "Laurentness" of some equation, we can only seldom show directly that the solutions are Laurent polynomials of the initial values. To facilitate such proofs we therefore introduce some different characterizations of the Laurent phenomenon by using Example 1.2.

Let $A = \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$ be the Laurent polynomial ring of X, Y over $\mathbb{C}$. Recall that A is a unique factorization domain (UFD).

Proposition 1.5. *For (1.1), the following four conditions are equivalent:*
(a) $f_m \in A$.
(b) $(f_{m-1}^2 + \alpha)$ *can be divided by* f_{m-2} *in the ring* A.
(c) $f_{m-1}^2 + \alpha = 0$ *in the ring* $A/(f_{m-2})$.
(d) *If we consider* f_m *as a rational function of* $(X, Y) \in \mathbb{C}^2$, *then it is holomorphic on* $(\mathbb{C}^{\times})^2$.

Proof. (a) is the definition of the Laurent phenomenon itself. (b) is a direct rewording of (a), and (c) of (b). (d) is an algebro-geometric characterization of (a), corresponding to the fact that $\operatorname{Spec} A \cong (\mathbb{C}^{\times})^2$. $\square$

Remark. It is usual to consider the Laurent phenomenon over the base ring $\mathbb{Z}[\alpha]$. However, for simplicity, we restrict the base ring (field) to $\mathbb{C}$ in this paper.

We will now prove the Laurentness of equation (1.1) by using these characterizations.

Proposition 1.6. *(1.1) exhibits the Laurent phenomenon.*

Proof. By induction on m, we shall assume $f_0, \ldots, f_{m-1} \in A$ and show that $f_m \in A$.

First, f_{m-2} and f_{m-3} must be relatively prime in A. This is because, if g is a common factor of f_{m-2} and f_{m-3}, considering both sides of $f_{m-1} f_{m-3} = f_{m-2}^2 + \alpha$ modulo g will yield $\alpha \equiv 0 \pmod{g}$, which is not allowed.

We denote by $A_{f_{m-3}}$ the localization of A at the element f_{m-3}. Since f_{m-2} and f_{m-3} are relatively prime in A, $(f_{m-1}^2 + \alpha)$ is divisible by f_{m-2} in the ring A if and only if it is so in the ring $A_{f_{m-3}}$. Furthermore, this condition is also equivalent to the condition $f_{m-1}^2 + \alpha = 0$ in the ring $A_{f_{m-3}}/(f_{m-2})$.

In the ring $A_{f_{m-3}}/(f_{m-2})$, we have

$$
\begin{aligned}
f_{m-1}^2 + \alpha &= (f_{m-2}^2 + \alpha)^2 / f_{m-3}^2 + \alpha \\
&= \alpha^2 / f_{m-3}^2 + \alpha \\
&= \alpha(\alpha + f_{m-3}^2) / f_{m-3}^2 \\
&= \alpha f_{m-4} f_{m-2} / f_{m-3}^2 \\
&= 0.
\end{aligned}
$$

Therefore $(f_{m-1}^2 + \alpha)$ is divisible by f_{m-2} and we have $f_m \in A$. Thus (1.1) exhibits the Laurent phenomenon. $\qquad\square$

By a similar argument, we can show the Laurentess of various equations by elementary methods. Indeed, the Laurentness of Example 1.4 can be shown in the same way.

## § 1.3.   Cluster Algebras and the Caterpillar Lemma

As mentioned above, the Laurent phenomenon is a concept which has arisen from cluster algebras. A cluster algebra is a commutative ring, together with some characteristic generators, called cluster variables. We shall not try to rigorously define cluster algebras here, since too much preparations would be needed for this purpose and we refer the reader to [2] for details concerning cluster algebras.

However, one important fact concerning cluster algebras that is worth mentioning is the next theorem.

Theorem 1.7 (Laurent Phenomenon [2]).   *In cluster algebras, every cluster variable is a $\mathbb{Z}$-coefficient Laurent polynomial of the initial clusters.*

The key to proving this theorem is the so-called Caterpillar Lemma. This lemma says that if we can construct a special pattern, called a caterpillar, such system has the Laurent property. Using this Caterpillar Lemma, S. Fomin and A. Zelevinsky have

shown the Laurentness of several discrete equations [1], among which several famous discrete integrable systems, for example the discrete KdV equation, the Hirota-Miwa equation and the discrete BKP equation.
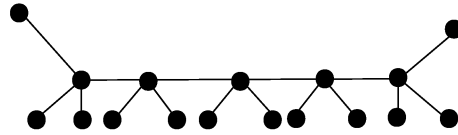
Caterpillar

§2. Initial value problems for discrete bilinear equations

There are many equations that exhibit the Laurent phenomenon. In particular, one could claim that it is quite common for discrete bilinear equations to have the property. In this section, we shall consider the Laurentness of bilinear equations. First we shall define the initial value problems for which one can consider the Laurent phenomenon, and then we shall introduce several equations that possess this property.

In this section, we consider the following equation:

$$\alpha_0 f_{h+v_0} f_{h+u_0} + \cdots + \alpha_n f_{h+v_n} f_{h+u_n} = 0,$$

where $n \geq 2$, L is a lattice, h is an independent variable moving on L, f_h is the dependent variable, $\alpha_i \in \mathbb{C}^\times$ are parameters, and $u_i, v_i \in L$ satisfy the relations

$$u_0 + v_0 = \cdots = u_n + v_n.$$

We shall regard as identical all equations that can be transformed into each other by coordinate changes of h or replacements of α_i.

We can transform the above equation as follows:

$$(2.1) \qquad f_h = \frac{\alpha_1 f_{h+v_1} f_{h+u_1} + \cdots + \alpha_n f_{h+v_n} f_{h+u_n}}{f_{h+w}},$$

where h moves on the lattice L and $u_i, v_i, w \in L$ satisfy the relations

$$v_1 + u_1 = \cdots = v_n + u_n = w.$$

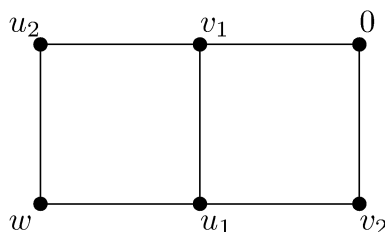Almost all the bilinear equations we shall consider can be expressed in this form.

Example 2.1 (discrete KdV equation).   Let

$$L = \mathbb{Z}^2, n = 2, v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, u_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

and we have the discrete KdV equation:

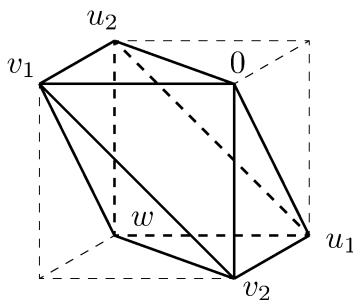$$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2}}{f_{h+w}}.$$

Example 2.2 (Hirota-Miwa equation).   Let

$$L = \mathbb{Z}^3, n = 2, v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, u_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

and we have the Hirota-Miwa equation:

$$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2}}{f_{h+w}}.$$
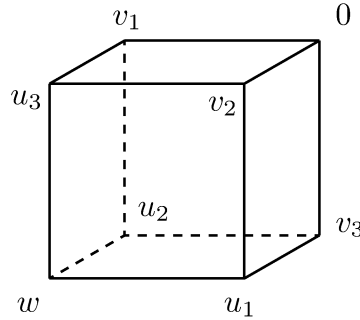
Example 2.3 (discrete BKP equation). Let

$$L = \mathbb{Z}^3, n = 3, v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, u_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

$$u_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

and we have the discrete BKP equation:

$$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2} + \gamma f_{h+v_3} f_{h+u_3}}{f_{h+w}}.$$
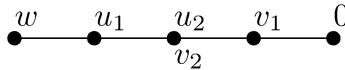

Example 2.4 (Somos-4). Let

$$L = \mathbb{Z}, n = 2, v_1 = -1, u_1 = -3, v_2 = u_2 = -2, w = -4$$

and we get the so-called Somos-4 sequence [3]:

$$f_h = \frac{\alpha f_{h-1} f_{h-3} + \beta f_{h-2}^2}{f_{h-4}}.$$

Remark. We assume that the lattice L does not have torsions. Thus, L is a free $\mathbb{Z}$-module and we can take a basis $x_1, \ldots, x_r$ in order to represent L as $\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_r$. To consider an L with torsions corresponds to considering an equation with multiple τ-functions.

§ 2.1.   Initial value problems

Studying the Laurent phenomenon first requires the introduction of an initial value problem. If the lattice is $\mathbb{Z}$, we have only to place the first several f_h as the initial values. For example, an initial value problem for Somos-4 is of the following form:

$$\begin{cases} f_h = \frac{\alpha f_{h-1} f_{h-3} + \beta f_{h-2}^2}{f_{h-4}}, \\ f_0 = X_0, \ f_1 = X_1, \ f_2 = X_2, \ f_3 = X_3. \end{cases}$$

If the lattice is multidimensional, we must however define where we shall consider the equation, where the initial values are, and which direction we shall evolve the equation in. In the following we shall define initial value problems for bilinear equations, by generalizing the methods introduced in [1].

Initial value problems often involve periodic boundary conditions or boundary conditions at infinity. However, it is usually impossible to deal with such equations algebraically and they are not congenial to the Laurent phenomenon. Therefore, in this paper we shall impose conditions on initial value problems such that f_h can be calculated using (2.1) in finitely many steps.

Two conditions are necessary to define such an initial value problem; the first condition concerns the evolution direction and the second the definition domain.

2.1.1.   A condition concerning the evolution direction

We interpret (2.1) as instructing us to calculate f_h from $f_{h+v_1}, \ldots, f_{h+w}$. The condition that allows us to evolve the equation in this direction is then the following:

Condition: Let Δ be the polytope generated by $0, v_1, u_1, \cdots, v_n, u_n, w$.
Then, we require that 0 be a vertex of Δ.

We introduce a (semi)order $\leq$ on L as follows. Let $S = \mathbb{Z}_{\geq 0} v_1 + \cdots + \mathbb{Z}_{\geq 0} u_n + \mathbb{Z}_{\geq 0} w$ be a semigroup. We introduce a binary relation $\leq$ on L by

$$h \leq h' \Leftrightarrow h \in h' + S,$$

where $h, h' \in L$. $\leq$ is an order relation on L since 0 is a vertex of Δ. We evolve the equation according to $\leq$.

Remark. $\leq$ is an order on L if and only if 0 is a vertex of Δ. Thus, strictly speaking, Δ is unnecessary if we require $\leq$ to be an order on L. However, in practice, Δ is much easier to imagine than $\leq$.

Example 2.5.   The polytopes Δ of the Hirota-Miwa equation, the discrete BKP equation, the discrete KdV equation and Somos-4 are as follows:

Hirota-Miwa   discrete BKP   discrete KdV   Somos-4
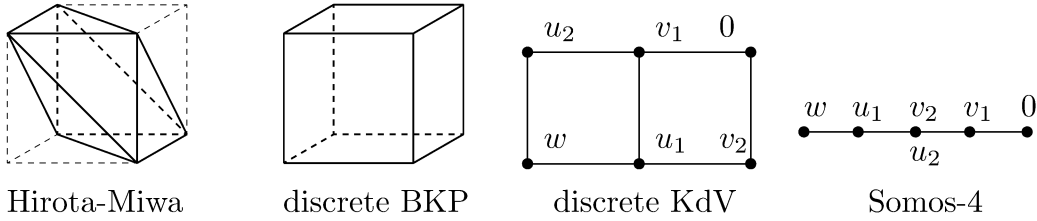
Example 2.6. Consider the discrete KdV equation (Example 2.1). In the case of configuration (I) we can evolve the equation in the direction of the upper right corner. On the other hand, 0 is not a vertex of Δ in configuration (II) and in this case it is unclear how an evolution direction can be defined.

(I)   (II)
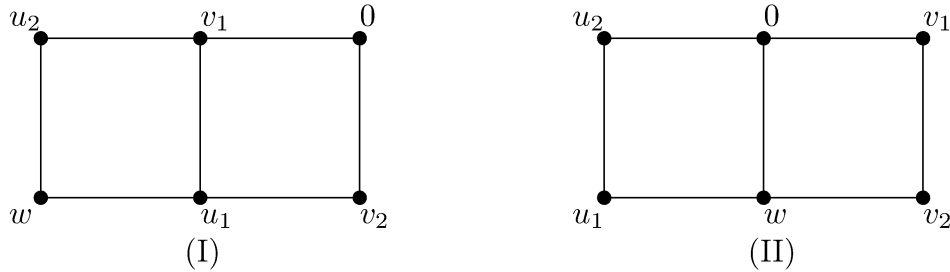
Example 2.7. For the discrete KdV equation, S is the following domain marked by dots $\bullet$. The relation $h \leq h'$ implies that, roughly speaking, h is situated at the lower left of h'.
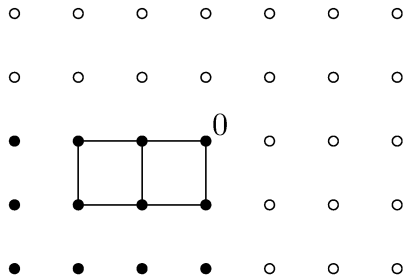
Remark. To be precise, Δ is defined as $\Delta = \mathrm{ConvexHull}\{0, v_1, u_1, \ldots, v_n, u_n, w\} \subset L_{\mathbb{R}}$, where $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ and we can think of L as $L \subset L_{\mathbb{R}}$ by $L \hookrightarrow L \otimes \mathbb{R}$. If the lattice L has torsions, $L \to L \otimes \mathbb{R}$ is not an injection and Δ cannot be properly defined. In that case, we will instead adopt the condition concerning the order $\leq$.

2.1.2. A condition concerning the definition domain

We set up a condition concerning the definition domain, in order to be able to correctly define initial value problems.

Definition 2.8 ("good domain"). $H \subset L, H \neq \emptyset$ is said to be a good domain if it satisfies the following two conditions:

(a) For any $h \in H$, the set

$$(h + S) \cap H \qquad (= \{h' \in H \mid h' \leq h\})$$

is finite.

(b) If $h \in H$, then $h - v_1, \ldots, h - u_n, h - w \in H$ (i.e. $h - S \subset H$).

Let

$$H_0 = \{h \in H \mid \text{ some of } h + v_1, \ldots, h + u_n, h + w \text{ do not belong to } H\},$$

where $H \subset L$ is a good domain. We will give initial values on $h_0 \in H_0$.

Remark. Usually, one would first choose where the initial values will be given, which then determines the domain in which the evolution will take place. However, here we do not think in this way. We first choose an evolution domain and the points in this domain that cannot be calculated from the other points are designated to be the initial values.
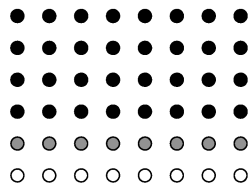
Example 2.9. Consider the discrete KdV equation (Example 2.1) on the following four domains. H_0 consists of the points marked by ◓.
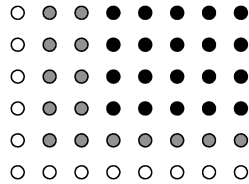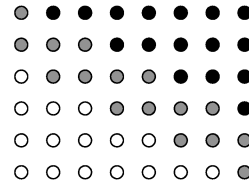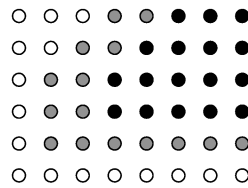
While (I) and (II) are good domains, (III) and (IV) are not. (III) does not satisfy the condition (a). In fact we cannot consider an initial value problem at all on (III). On the other hand, (IV) does not satisfy the condition (b) and, although in principle one could consider an initial value problem on (IV), this domain is incompatible with the Laurent property as will be explained shortly.

Now we can define the initial value problem.

Definition 2.10 (initial value problem for a bilinear equation). Let $H \subset L$ be a good domain. Consider

$$f_h = \begin{cases} \dfrac{\alpha_1 f_{h+v_1} f_{h+u_1} + \cdots + \alpha_n f_{h+v_n} f_{h+u_n}}{f_{h+w}} & (h \in H \setminus H_0), \\ X_h & (h \in H_0) \end{cases}$$

in the field $\mathbb{C}(X_{h_0} : h_0 \in H_0)$. We evolve f_h according to the order $\leq$. I.e. if we want to calculate f_h, we must know $f_{h'}$ for all $h' \leq h$. Thus the way to walk on H is essentially unique and we need not care about compatibility.

§ 2.2. Laurent phenomenon for discrete bilinear equations

Definition 2.11 (Laurent phenomenon for a discrete bilinear equation). An initial value problem

$$f_h = \begin{cases} \dfrac{\alpha_1 f_{h+v_1} f_{h+u_1} + \cdots + \alpha_n f_{h+v_n} f_{h+u_n}}{f_{h+w}} & (h \in H \setminus H_0), \\ X_h & (h \in H_0) \end{cases}$$

exhibits the Laurent phenomenon if

$$f_h \in \mathbb{C}[X_{h_0}, X_{h_0}^{-1} : h_0 \in H_0]$$

for all $h \in H$.

A bilinear equation exhibits the Laurent phenomenon if for every good domain $H \subset L$, the corresponding initial value problem has the Laurent property.

Theorem 2.12 (Fomin-Zelevinsky [1]). *The discrete KdV equation, the Hirota-Miwa equation, the discrete BKP equation and Somos-4 exhibit the Laurent phenomenon.*

The Laurentness of the discrete KdV equation, the Hirota-Miwa equation and the discrete BKP equation is proved in [1] by direct application of the Caterpillar Lemma. Note that in [1], different names are used for the discrete KdV equation, the Hirota-Miwa equation and the discrete BKP equation.

However, it is possible to prove the Laurentness of the discrete KdV equation, the Hirota-Miwa equation and the discrete BKP equation by elementary methods, in particular, without the Caterpillar Lemma. In doing so, we also prove the fact that the solution f_h is in fact an irreducible Laurent polynomial of the initial values.

Finally, let us give an example of the problems with respect to the Laurent phenomenon that arise for initial value problems on a domain that is not a good domain.

Example 2.13. Consider the discrete KdV equation (Example 2.1) on domain (IV) of the previous paragraph:
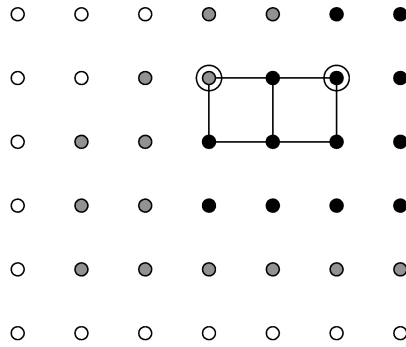
In this case, it can be easily verified that f_h at ◉ is not a Laurent polynomial since the point ◎ corresponds to an indeterminate and thus obstructs cancellations.

## §3.   The Laurent phenomenon and reductions of bilinear equations

In this section, we shall explain the relation between the Laurent phenomenon and reductions of bilinear equations. First we will show that if an equation exhibits the Laurent phenomenon, then a reduction of the equation will also possess this property. Hence, many equations with the Laurent property can be constructed in this way. Finally, we will introduce Somos sequences that have a close relation to the Laurent phenomenon and to reductions of bilinear equations.

§3.1.   Reductions

By a "reduction" we mean that we require the solutions to an equation to be invariant under a translation in some direction. In general, reductions decrease the dimension of the lattice. In the following, we shall only consider reductions of discrete bilinear equations that do not decrease the number of terms in the equation.

Example 3.1 (discrete KdV equation → Somos-4). The discrete KdV equation (Example 2.1) turns into Somos-4 (Example 2.4) by requiring that

$$f_{h+v_2-u_2} = f_h.$$
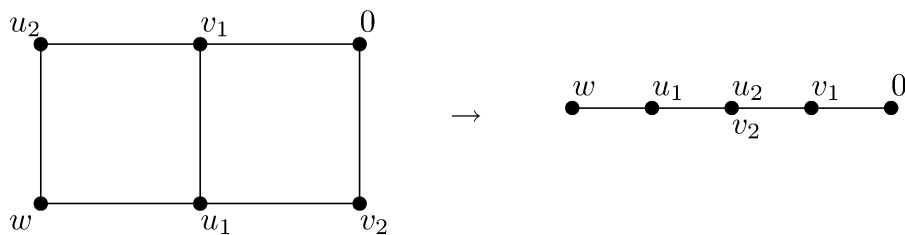
We have stated in Theorem 2.12 that the discrete KdV equation has the Laurent property. However, it is well known that the discrete KdV equation is a reduction of the Hirota-Miwa equation. We can therefore show the Laurentness of the discrete KdV equation using this fact.

Proposition 3.2. *Assume that (I) → (II) is a reduction of discrete bilinear equations. Then an initial value problem for (II) can be lifted to one for (I). Furthermore, a lift of a good domain is also a good domain. In particular, if (I) exhibits the Laurent phenomenon, (II) will do so as well.*

Example 3.3. The correspondence between domains in the example of discrete KdV → Somos-4 is as follows:
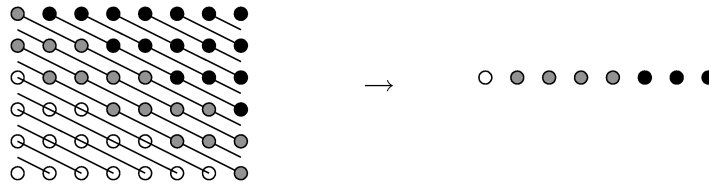
§ 3.2.   Bilinear equations with the Laurent property

By Theorem 2.12 and Proposition 3.2, we can find many equations which exhibit the Laurent phenomenon. We list some of them here.

(a) The Hirota-Miwa equation and its reductions.

They can be represented, on some lattice, as:

$$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2}}{f_{h+w}},$$

$$v_1 + u_1 = v_2 + u_2 = w.$$

For example, the discrete KdV equation, the discrete Toda equation or Somos-4, almost all the equations we usually consider when studying integrable systems, are of this type.

(b) The discrete BKP equation and its reductions.

These can be represented, on some lattice, as:

$$f_h = \frac{\alpha f_{h+v_1} f_{h+u_1} + \beta f_{h+v_2} f_{h+u_2} + \gamma f_{h+v_3} f_{h+u_3}}{f_{h+w}},$$

$$v_1 + u_1 = v_2 + u_2 = v_3 + u_3 = w,$$

$$v_1 + v_2 + v_3 = w.$$

Note that $v_1 + v_2 + v_3 = w$ is a necessary and sufficient condition for the equation to be a reduction of the discrete BKP equation. Removing this condition, we end up with

an equation on a four dimensional lattice that, however, does not possess the Laurent property.

It is not known whether there exist discrete bilinear equations with Laurentness except these.

§ 3.3.   Somos Sequences

In this section we introduce Somos sequences, which are closely related to the Laurent phenomenon.

Definition 3.4 (Somos Sequence).   Let ℓ be an integer greater than 3.   The Somos-ℓ sequence is defined by the following recurrence formula:

$$f_m = \frac{\sum_{1 \le j \le \ell/2} f_{m-\ell+j} f_{m-j}}{f_{m-\ell}},$$

$$f_0 = f_1 = \cdots = f_{\ell-1} = 1.$$

Example 3.5.   The Somos-4, 5, 6, 7, 8 sequences are as follows:

- Somos-4: $f_m = \frac{f_{m-1}f_{m-3}+f_{m-2}^2}{f_{m-4}}$,

- Somos-5: $f_m = \frac{f_{m-1}f_{m-4}+f_{m-2}f_{m-3}}{f_{m-5}}$,

- Somos-6: $f_m = \frac{f_{m-1}f_{m-5}+f_{m-2}f_{m-4}+f_{m-3}^2}{f_{m-6}}$,

- Somos-7: $f_m = \frac{f_{m-1}f_{m-6}+f_{m-2}f_{m-5}+f_{m-3}f_{m-4}}{f_{m-7}}$,

- Somos-8: $f_m = \frac{f_{m-1}f_{m-7}+f_{m-2}f_{m-6}+f_{m-3}f_{m-5}+f_{m-4}^2}{f_{m-8}}$.

A most interesting property concerning Somos sequences is:

Proposition 3.6.   *Every term of Somos-4, 5, 6, 7 is a positive integer.*

This is clearly nontrivial in view of the division in the equations. The first several terms are

- Somos-4: $1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \cdots$,

- Somos-5: $1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, \cdots$,

- Somos-6: $1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, \cdots$,

- Somos-7: $1, 1, 1, 1, 1, 1, 1, 3, 5, 9, 17, 41, 137, 769, \cdots$,

which are indeed all integers.

Of course, it is easily noticed that Somos-4, 5 are reductions of the Hirota-Miwa equation and that Somos-6, 7 are reductions of the discrete BKP equation. Therefore Somos-4, 5, 6, 7 possess the Laurent property with respect to their initial values, which are all equal to 1. Considered from this point of view, the integer nature of the Somos sequences is quite trivial.

On the other hand, f_m is not an integer, in general, if $\ell \geq 8$. Indeed, the first several terms of such sequences are

- Somos-8: $1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61,\ 187, 775, 5827, 14815, \frac{420514}{7}, \cdots,$

- Somos-9: $1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 49, 115,\ 355, 1483, 11137, 27937, \frac{755098}{7}, \cdots,$

- Somos-10: $1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 149\,413, 1473, 7073, 64785, \frac{800961}{5}, \cdots,$

and fractional numbers appear. Hence, these equations do not possess the Laurent property. As a consequence, we see that an equation that has a reduction to Somos-ℓ ($\ell \geq 8$) never possesses the Laurent property.

§ 4.   The Laurent phenomenon and nonautonomous bilinear equations

While, up to now, we have only considered autonomous equations, it is also possible to discuss the Laurent phenomenon in the case of nonautonomous equations. In this section, we shall first examine when exactly nonautonomous equations exhibit the Laurent phenomenon. After that we shall describe the relation between the Laurent phenomenon and gauge transformations.

Assume $\alpha_h^{(1)}, \ldots, \alpha_h^{(n)} \in \mathbb{C}^\times$ depend on $h \in L$. We will formulate a necessary and sufficient condition on the $\alpha_h^{(1)}, \ldots, \alpha_h^{(n)}$ for the Laurentness of the equation

$$f_h = \frac{\alpha_h^{(1)} f_{h+v_1} f_{h+u_1} + \cdots + \alpha_h^{(n)} f_{h+v_n} f_{h+u_n}}{f_{h+w}}.$$

Theorem 4.1.   *The nonautonomous Hirota-Miwa equation*

$$f_h = \frac{\alpha_h f_{h+v_1} f_{h+u_1} + \beta_h f_{h+v_2} f_{h+u_2}}{f_{h+w}} \qquad (\alpha_h, \beta_h \in \mathbb{C}^\times)$$

has the Laurent property if and only if α_h and β_h satisfy the relation:

$$(4.1) \qquad \alpha_h \alpha_{h+w} \beta_{h+v_1} \beta_{h+u_1} = \beta_h \beta_{h+w} \alpha_{h+v_2} \alpha_{h+u_2}.$$

It should be noted that (4.1) is known as the condition for integrability of the nonautonomous Hirota-Miwa equation [5, 7]. Moreover, if satisfied, we can transform the equation into an autonomous system by a gauge transformation.

Theorem 4.2. *The nonautonomous discrete BKP equation*

$$f_h = \frac{\alpha_h f_{h+v_1} f_{h+u_1} + \beta_h f_{h+v_2} f_{h+u_2} + \gamma_h f_{h+v_3} f_{h+u_3}}{f_{h+w}} \quad (\alpha_h, \beta_h, \gamma_h \in \mathbb{C}^\times)$$

possesses the Laurent property if and only if $\alpha_h, \beta_h, \gamma_h$ satisfy the following relations:

$$\alpha_{h+v_2}\beta_h\gamma_{h+u_1} = \alpha_{h+v_3}\beta_{h+u_1}\gamma_h,$$
(4.2) $$\alpha_{h+u_2}\beta_{h+v_3}\gamma_h = \alpha_h\beta_{h+v_1}\gamma_{h+u_2},$$
$$\alpha_h\beta_{h+u_3}\gamma_{h+v_1} = \alpha_{h+u_3}\beta_h\gamma_{h+v_2}.$$

As was the case for the Hirota-Miwa equation, if (4.2) is satisfied, the equation can be transformed into an autonomous system by some gauge transformation.

Theorem 4.3. *A nonautonomous version of a reduction of the Hirota-Miwa equation*
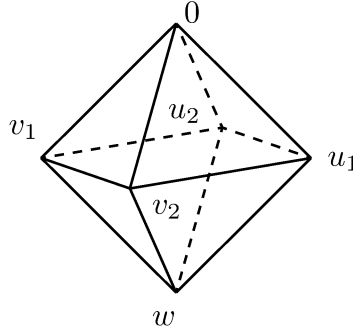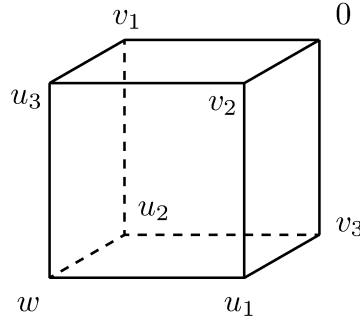
$$f_h = \frac{\alpha_h f_{h+v_1} f_{h+u_1} + \beta_h f_{h+v_2} f_{h+u_2}}{f_{h+w}} \quad (\alpha_h, \beta_h \in \mathbb{C}^\times)$$

has the Laurent property if and only if α_h and β_h satisfy the relation:

$$\alpha_h\alpha_{h+w}\beta_{h+v_1}\beta_{h+u_1} = \beta_h\beta_{h+w}\alpha_{h+v_2}\alpha_{h+u_2}.$$

In the case of the nonautonomous Hirota-Miwa equation, we have seen that the Laurentness is equivalent to the possibility of transforming the equation into an autonomous system. However, reductions decrease the dimension of the lattice and therefore also the

number of degrees of freedom for the gauge transformations. In general, it will therefore be impossible to transform such an equation into an autonomous system. Regarding gauge transformations in general, one can show the following property:

Proposition 4.4. *The Laurent phenomenon is unaffected by gauge transformations.*

§ 5. The Laurent phenomenon and integrability

In §3.2, we have seen that almost all the bilinear equations we usually consider exhibit the Laurent phenomenon. In Theorem 4.1, we have seen that the condition for the Laurentness of the nonautonomous Hirota-Miwa equation is equivalent to the condition for its integrability. Furthermore, we have seen that the Laurent phenomenon is a concept that is invariant under gauge transformations.

In an initial value problem with the Laurent property, if we substitute any nonzero value for each indeterminate corresponding to an initial value, the denominator of f_h never becomes zero. It may happen that $f_h = 0$. However, after that, f_h never diverges, nor can the iteration yield an indeterminate value. Hence, the equation evolves uniquely. This situation is very similar to the concept of singularity confinement [4]. Conversely, if the denominator of f_h never vanishes for any nonzero initial value, then f_h must be a Laurent polynomial of the initial values.

This strongly suggests a close relation between the Laurent phenomenon and integrability. Now what would be the advantage of the Laurent phenomenon as an integrability test compared to other tests?

First, the definition of the Laurent phenomenon is strict and plain. Although some preparation is needed, the Laurent phenomenon is defined as soon as we decide on an initial value problem. Since the definition is very simple, it is also quite easy to investigate properties that follow from this phenomenon.

Proposition 3.2 plays an important role in this. Since a reduction of an equation with the Laurent property also possesses the Laurent property, we have only to investigate those equations that are not expressible as reductions of other equations.

Another advantage is that we can use algebraic theories such as the theory of rings and fields, since the Laurent phenomenon is a purely algebraic property. In fact, elementary ring theory is used repeatedly when checking the Laurentness of equations. Furthermore, it is also possible to consider this property even when the base field is not $\mathbb{C}$. Here, we have restricted the base field to $\mathbb{C}$ for all arguments to be simple. However, this restriction is not essential at all. It is possible to consider the Laurent phenomenon over any base field, for example fields of positive characteristic, say finite fields.

Remark. We have seen that Laurentness is equivalent to the property that the denominator of f_h never vanishes for any nonzero initial value. It is true over any base field that if f_h is a Laurent polynomial of the initial values, the denominator of f_h never vanishes for any nonzero initial value. However, the converse is not true unless the base field is algebraically closed. $\frac{1}{1+X^2}$ over $\mathbb{R}$ constitutes a counterexample. Indeed, while $1 + t^2$ never vanishes for any real number t, $\frac{1}{1+X^2}$ is not a Laurent polynomial of X. Base field extension to its algebraic closure solves this problem since the Laurent property is invariant under field extensions.

Moreover, one can show the following:

Proposition 5.1. *In the case of the Hirota-Miwa equation, the discrete BKP equation or a reduction of the Hirota-Miwa equation onto a two dimensional lattice, the solution f_h is an irreducible or an invertible Laurent polynomial, where an irreducible Laurent polynomial means an irreducible element in the Laurent polynomial ring $\mathbb{C}[X_{h_0}, X_{h_0}^{-1} : h_0 \in H_0]$, whereas f_h is invertible if and only if f_h is an initial value (i.e. $f_h = X_h$). The same statement remains valid in the nonautonomous cases of these equations.*

From here on, we assume that f_h is an irreducible or invertible Laurent polynomial.

If we consider a situation where we substitute nonzero values for each indeterminate X_{h_0}, it is natural to regard the whole space of the initial values as $\prod_{h_0 \in H_0} \mathbb{C}^\times$. By Proposition 5.1, f_h and $f_{h'}$ are relatively prime in the Laurent polynomial ring if $h \neq h'$. Therefore, it is very rare for $f_h = f_{h'} = 0$ to occur. Here "very rare" means that the initial values for which $f_h = f_{h'} = 0$ have codimension at least two in the whole space of initial values. Note that the initial values which make one of the f_h vanish have codimension one.

Solutions to discrete integrable systems are often represented as ratios of τ-functions, where the τ-functions satisfy bilinear equations. Consider the situation where a function a_h is represented as a ratio of f_h's where f_h is a Laurent polynomial of the initial values. We say that a_h is singular if a_h becomes zero, has a pole or has an indeterminate value. Clearly, $a_h = 0$ if some f_h in the numerator vanishes; a_h has a pole if some f_h in the denominator vanishes; and a_h has an indeterminate value if f_h's in both the numerator and denominator vanish. Thus, it is very rare for multiple f_h's to cause a singularity in a_h. It is also very rare for a_h to take an indeterminate value, since such an indeterminacy requires several f_h's to vanish.

§6.   The Laurent phenomenon and algebraic entropy

In this section, we introduce a method to calculate the algebraic entropy, which is another well-established integrability test [6], using the Laurent phenomenon.

The denominator of the solution to an equation with the Laurent property is monomial and thus easy to investigate. Moreover, although the numerator can be quite complicated, it is not difficult to calculate the algebraic entropy if we can estimate the degree of the numerator in function of the degree of the denominator.

Example 6.1. Consider the equation of Example 1.2:

$$\begin{cases} f_m = \frac{f_{m-1}^2 + \alpha}{f_{m-2}}, \\ f_0 = X, \ f_1 = Y. \end{cases}$$

If $m \geq 2$, we have that

$$\text{denominator of } f_m = X^{m-1} Y^{m-2},$$

$$\deg(\text{numerator of } f_m) = \deg(\text{denominator of } f_m) + 1$$

and $\deg f_m = 2m - 2$. Thus $\deg f_m = \mathcal{O}(m^1)$ and the algebraic entropy is zero.

Example 6.2. We generalize the above example and consider the equation

$$\begin{cases} f_m = \frac{f_{m-1}^a + 1}{f_{m-2}}, \\ f_0 = X, \ f_1 = Y, \end{cases}$$

where $a \in \mathbb{Z}_{\geq 3}$. It is easy to check the Laurentness of the equation. However, decomposing f_m as $f_m = p_m / q_m$, we have

$$\begin{cases} q_m = q_{m-1}^a / q_{m-2}, \\ q_2 = X, \ q_3 = X^a Y. \end{cases}$$

Let $\lambda = \frac{a + \sqrt{a^2 - 4}}{2}$, then we have $\deg q_m = \mathcal{O}(\lambda^m)$. In particular, the algebraic entropy of the equation is greater than λ and not zero.

From here on we analyze the denominator of the solution to a bilinear equation with the Laurent property. Let

$$f_h = \begin{cases} \frac{\alpha_1 f_{h+v_1} f_{h+u_1} + \cdots + \alpha_n f_{h+v_n} f_{h+u_n}}{f_{h+w}} & (h \in H \setminus H_0), \\ X_h & (h \in H_0) \end{cases}$$

be an initial value problem of a bilinear equation with the Laurent property and let $f_h = p_h / q_h$. q_h is a monomial in the initial values with coefficient 1. Then one can show the following:

Theorem 6.3. q_h *satisfies*

$$(6.1) \qquad q_h = \begin{cases} 1 & (h \in H_0), \\ X_{h+w} \operatorname{LCM}_{1 \leq j \leq n}(q_{h+v_j} q_{h+u_j}) & (h + w \in H_0), \\ \operatorname{LCM}_{1 \leq j \leq n}(q_{h+v_j} q_{h+u_j}) / q_{h+w} & (otherwise), \end{cases}$$

where LCM *denotes the least common multiple as polynomials. I.e. for* $g_i = \prod_\ell X_\ell^{e_{i\ell}}$,
$\mathrm{LCM}_i(g_i) = \prod_\ell X_\ell^{\max_i(e_{i\ell})}$.

This theorem implies that the denominators can be evolved by themselves, and hence it is not hard to calculate these denominators using a computer.

Example 6.4. Consider the discrete KdV equation (Example 2.1) on the following domain:
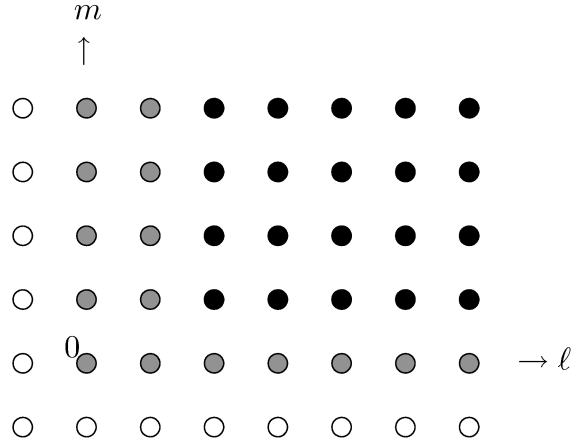
Then the successive q_h are as follows:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	$X_{00}X_{01}X_{02}$	$X_{00}^2X_{10}X_{01}^2X_{11}X_{02}X_{12}$	$X_{00}^3X_{10}^2X_{01}^2X_{20}X_{11}^2X_{02}X_{12}$	$\cdots$
1	1	$X_{00}X_{01}$	$X_{00}^2X_{10}X_{01}X_{11}$	$X_{00}^2X_{10}^2X_{01}X_{20}X_{11}$	$\cdots$
1	1	X_{00}	$X_{00}X_{10}$	$X_{00}X_{10}X_{20}$	$\cdots$
1	1	1	1	1	$\cdots$

Formula (6.1) might look complicated, but its dependence on each indeterminate is quite simple. If we use the relation between LCM and powers for monomials, (6.1) leads to the following $(\max, +)$-equation.

Corollary 6.5. *Let $h_0 \in H_0$ be fixed and let $d_h^{(h_0)}$ be the degree with respect to X_{h_0} of the denominator q_h. Then $d_h^{(h_0)}$ satisfies the following relations:*

$$(6.2) \qquad d_h^{(h_0)} = \begin{cases} 0 & (h \in H_0), \\ 1 & (h + w = h_0), \\ \max_{1 \le j \le n}(d_{h+v_j}^{(h_0)} + d_{h+u_j}^{(h_0)}) & (h + w \in H_0,\ h + w \neq h_0), \\ \max_{1 \le j \le n}(d_{h+v_j}^{(h_0)} + d_{h+u_j}^{(h_0)}) - d_{h+w}^{(h_0)} & (otherwise), \end{cases}$$

and $\deg q_h$ *is obtained by moving* $h_0 \in H_0$ *and summing up* $d_h^{(h_0)}$. *Moreover, we need not consider* $\deg p_h$ *since* $\deg p_h = \deg q_h + 1$ *is always satisfied.*

In the case of the Hirota-Miwa equation and the discrete BKP equation, we can solve (6.2) and q_h can be represented concretely.

Proposition 6.6. *In the case of the Hirota-Miwa equation and the discrete BKP equation, q_h is represented concretely as follows:*

$$q_h = \prod_{h'} X_{h'},$$

where h' moves over all elements in H_0 satisfying $h' \leq h + w$.

Now, we consider a reduction to a one dimensional lattice. The following corollary is easily obtained from Proposition 3.2.

Corollary 6.7. *Let a, b, c, ℓ be distinct positive integers and $\ell > a, b$. Consider the following two types of equations:*

$$f_m = \frac{\alpha f_{m-a} f_{m-\ell+a} + \beta f_{m-b} f_{m-\ell+b}}{f_{m-\ell}},$$

$$f_m = \frac{\alpha f_{m-a} f_{m-b-c} + \beta f_{m-b} f_{m-a-c} + \gamma f_{m-c} f_{m-a-b}}{f_{m-a-b-c}}.$$

Then the degree of the solution of these equations is at most of order $\mathcal{O}(m^2)$. In particular, the algebraic entropy of these equations is zero.

Equations of the first type are reductions of the Hirota-Miwa equation, and those of the second type are reductions of the discrete BKP equation. We can thus obtain infinitely many equations with zero algebraic entropy by choosing a, b, c appropriately.

§7. Conclusions

In this paper, we have explained the relation between the Laurent phenomenon and discrete integrable systems. We have seen that almost all the bilinear equations we usually consider exhibit this phenomenon and we described the conditions for the Laurentness for certain nonautonomous systems. All these results strongly suggest a close relation between the Laurent property and integrability. Thus, it is to be expected that the concept of Laurentness might offer a powerful tool for testing the integrability of a given discrete system.

Finally, we have introduced a method to calculate algebraic entropies using the Laurent phenomenon. In particular we have seen that for an equation with the Laurent property, explicit expressions for the denominator of the solutions to such an equation can be obtained and that the algebraic entropy for the equation can be easily calculated

from these relations. Hence the hope that the use of this phenomenon might lead to further interesting developments in the field of discrete integrable systems.

References

[1] S. Fomin and A. Zelevinsky, The Laurent Phenomenon, *Advances in Applied Mathematics,* **28** (2002), no. 2, 119–144.

[2] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, *Journal of the American Mathematical Society,* **15** (2002), no. 2, 497–529.

[3] D. Gale, Mathematical Entertainments, *Mathematical Intelligencer,* **13** (1991), no. 1, 40–43.

[4] B. Grammaticos and A. Ramani, Do integrable mapping have the Painlevé property?, *Physical Review Letters,* **67** (1991), no. 14, 1825–1828.

[5] B. Grammaticos and A. Ramani, Integrability in a discrete world, *Chaos, Solitons & Fractals,* **11** (2000), no.1–3, 7–18.

[6] J. Hietarinta and C. Viallet, Singularity Confinement and Chaos in Discrete Systems, *Physical Review Letters,* **81** (1998), no. 2, 325–328.

[7] R. Willox, T. Tokihiro and J. Satsuma, Darboux and binary Darboux transformations for the nonautonomous discrete KP equation, *Journal of Mathematical Physics,* **38** (1997), no. 12, 6455–6469.