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Construction of the exceptional orthogonal polynomials and its application to the superintegrable system

By

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Abstract

To construct systems of polynomial eigenfunctions with jump in degree by means of the theory of Darboux transformation, a class of eigenfunctions of the Sturm-Liouville operator is introduced. Then we show a systematic way to construct the system of polynomial eigenfunctions with jump in degree from the Sturm-Liouville operator of the classical orthogonal polynomial. We classify these systems of the polynomial eigenfunctions with jump in degree according to the contour of integration which determines the gauge factor of the seed solution of the Darboux transformation. Finally, we give a brief review on the superintegrable Hamiltonian derived from the exceptional orthogonal polynomials.

§ 1. Introduction

Let us consider the polynomial eigenfunctions of the general second-order ordinary differential operator,

$$A(x)\partial^2 + B(x)\partial + C(x). \tag{1.1}$$

It is well-known that Bochner[2] classified the all orthogonal polynomials which satisfy

$$A(x)p_n'' + B(x)p_n' + C(x)p_n = \lambda_n p_n, \quad \deg p_n = n, \tag{1.2}$$

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for $n = 0, 1, 2, \ldots$, and some value of the spectral parameter λ_n. In this case the polynomial eigenfunctions are given by the classical orthogonal polynomials of Jacobi, Laguerre, Hermite and the Bessel polynomials. If there exists polynomials of degree $0, 1, 2$ satisfying (1.2), then the coefficients of this operator, $A(x), B(x)$ and $C(x)$, are required to be polynomials and their degrees are as follows: $\deg A(x) \leq 2, \deg B(x) = 1$ and $\deg C(x) = 0$.

Recently, as one of the generalization of the classical orthogonal polynomials, Gomez-Ullate, Kamran and Milson (GKM) [6] have introduced the "exceptional orthogonal polynomials" which has following properties:

- eigenfunctions of a second-order differential operator,
- existence of a positive weight function $w(x)$ on an interval I of real line,
- a complete basis in the corresponding $L^2(w(x)dx, I)$-space,

but they do not contain any polynomials from degree 0 to degree $j-1$ and start with degree $j > 0$. They explicitly present the case when $j = 1$ starting from the Laguerre polynomials and the Jacobi polynomials, which are known as the X_1-Laguerre polynomials and the X_1-Jacobi polynomials, respectively. This new class of orthogonal polynomials sequences is now attracting many interests of researchers of various fields. Among them, Odake and Sasaki (OS) [11] have developed this new class of orthogonal polynomial sequences and presented many new explicit examples, such as the exceptional Jacobi polynomials for $j > 1$, and also the q-analogue of exceptional orthogonal polynomials. Recently, OS introduced the "multi-index" Jacobi polynomials and the "multi-index" Laguerre polynomials [12]. Before GKM, Dubov, Eleonskii and Kulagin (DEK) [5] has proposed similar polynomial sequences, which contain constant function as the lowest degree polynomial and the second lowest degree polynomial is given by $j+1$-degree polynomial, in 1994. All these examples can be derived from the Darboux transformation of the classical orthogonal polynomials [7, 15].

The purpose of this article is to show a systematic way to construct these classes of polynomial sequences, which can be derived from the one-time Darboux transformation, started from the Sturm-Liouville operator of the classical orthogonal polynomials. In §2, we review the theory of Darboux transformations [1, 3, 4]. In §3, we discuss a class of eigenfunctions of the Sturm-Liouville operator. In §4, we construct the system of the orthogonal polynomials with jump in degree, and then present the list of the Darboux transformed polynomial eigenfunctions, which has a positive finite weight function $w(x)$ on the real line. Finally we give brief review on the superintegrable Hamiltonian derived from the X_1-Jacobi polynomials [14].

§ 2. Darboux transformations of Sturm-Liouville operator

The Darboux transformation is one of the basic tools in our method to construct orthogonal polynomial systems with jump in degree. In this section we present the Darboux transformation of the Sturm-Liouville operator and discuss several properties of eigenfunctions of the Darboux transformed operators.

Consider the eigenvalue problem of the Sturm-Liouville equation,

$$\frac{1}{w(x)}\frac{d}{dx}\left(A(x)w(x)\frac{d\phi}{dx}\right)+C(x)\phi=\lambda\,\phi, \tag{2.1}$$

where λ is the eigenvalue, the function $A(x), w(x)$ are continuously differentiable functions on the interval $I=(x_0,x_1)$ and $C(x)$ is continuous on I. Note that the second order differential eigenvalue problem,

$$A(x)\phi''+B(x)\phi'+C(x)\phi=\lambda\phi,$$

can be turned into the form of the Sturm-Liouville eigenvalue problem (2.1) by introducing the function $w(x)$ which satisfies

$$(A(x)w(x))'=B(x)w(x). \tag{2.2}$$

Eq.(2.2) is formally solved to give $w(x)=\kappa_0\exp\left(\int^x\frac{B(y)-A'(y)}{A(y)}dy\right)$, where κ_0 is an integration constant. We may write (2.1) as

$$\mathcal{L}[\phi]=\lambda\phi, \tag{2.3}$$

where the operator $\mathcal{L}$, what we call Sturm-Liouville operator, is defined by

$$\mathcal{L}[\phi]=\frac{1}{w(x)}(A(x)w(x)\phi')'+C(x)\phi. \tag{2.4}$$

Here we would like to introduce several notations to present the Sturm-Liouville operator in short. We will present the Sturm-Liouville operator $\mathcal{L}$ defined in (2.4) as follows:

$$\mathcal{L}=\mathcal{D}_w\,A\,\mathcal{D}+C, \tag{2.5}$$

where the operators $\mathcal{D},\mathcal{D}_f$ are defined by

$$\mathcal{D}[\phi]=\phi',\quad \mathcal{D}_f[\phi]=f^{-1}\partial f\phi=\phi'+\frac{f'(x)}{f(x)}\phi, \tag{2.6}$$

respectively.

Several useful formulas in calculation are presented here:

$$\mathcal{D}_{\kappa f}[\phi]=\mathcal{D}_f[\phi]$$

for nonzero constant κ and

$$(\mathcal{D}_f \cdot g)[\phi] = (g \cdot \mathcal{D}_{fg})[\phi], \tag{2.7}$$

$$(\mathcal{D}_{fh} \cdot g \cdot \mathcal{D}_{1/h})[\phi] = (\mathcal{D}_f \cdot g \cdot \mathcal{D} + \widetilde{C})[\phi], \tag{2.8}$$

where $\widetilde{C}(x) = (\mathcal{D}_{fh} \cdot g \cdot \mathcal{D}_{1/h})[1](x) = -(\mathcal{D}_f \cdot g \cdot \mathcal{D})[h](x)/h(x)$, which can be proved by straightforward calculation. Note that by applying the operator $\mathcal{D}_{1/f}$ to the function f, then

$$\mathcal{D}_{1/f}[f] = (f\partial f^{-1})[f] = 0.$$

Darboux transformation Let us introduce a set of eigenfunctions $\{\phi_\alpha\}_{\alpha\in\mathbb{I}}$ of $\mathcal{L}$ such that

$$\mathcal{L}[\phi_\alpha](x) = \lambda_\alpha \, \phi_\alpha(x), \tag{2.9}$$

where $\mathbb{I}$ be an infinite set, which will be discussed in the next section, and λ_α be the spectral parameter corresponding to the eigenfunction ϕ_α. For simplicity here and hereafter we assume that these spectral parameters λ_α are all mutually distinct.

By fixing $d \in \mathbb{I}$, which we may call Darboux parameter, we introduce one eigenfunction ϕ_d as a seed eigenfunction of the Darboux transformation and denote that $\lambda^{(1)} = \lambda_d$ and $\chi^{(1)} = \phi_d$. From the factorization of the operator $\mathcal{L} - \lambda^{(1)}$, we obtain a pair of first order differential operators $\mathcal{B}^{(1)}$ and $\mathcal{F}^{(1)}$ such that

$$\mathcal{L} = \mathcal{B}^{(1)} \, \mathcal{F}^{(1)} + \lambda^{(1)}. \tag{2.10}$$

It is easy to see that $\mathcal{B}^{(1)}$ and $\mathcal{F}^{(1)}$ are explicitly given by

$$\mathcal{B}^{(1)} = \mathcal{D}_{w\chi^{(1)}} \, A/r^{(1)}, \quad \mathcal{F}^{(1)} = r^{(1)} \, \mathcal{D}_{1/\chi^{(1)}},$$

respectively, where $r^{(1)}(x)$ is introduced as an arbitrary decoupling function for normalization purpose. Note that $\mathcal{F}^{(1)}[\phi_\alpha]$ can be expressed in terms of Wronskian:

$$\mathcal{F}^{(1)}[\phi_\alpha] = r^{(1)} \left(\phi_\alpha' - \frac{(\chi^{(1)})'}{\chi^{(1)}} \phi_\alpha \right) = \frac{r^{(1)}}{\chi^{(1)}} \begin{vmatrix} \chi^{(1)} & \phi_\alpha \\ (\chi^{(1)})' & \phi_\alpha' \end{vmatrix}.$$

Here we introduce the new operator $\mathcal{L}^{(1)}$ by exchanging $\mathcal{B}^{(1)}$ and $\mathcal{F}^{(1)}$ of $\mathcal{L}$:

$$\mathcal{L}^{(1)} = \mathcal{F}^{(1)} \, \mathcal{B}^{(1)} + \lambda^{(1)}. \tag{2.11}$$

The Sturm-Liouville form of $\mathcal{L}^{(1)}$ is given by

$$\mathcal{L}^{(1)} = \mathcal{D}_{w^{(1)}} \, A \, \mathcal{D} + C^{(1)}, \tag{2.12}$$

where $w^{(1)} = Aw/(r^{(1)})^2$ and $C^{(1)} = (\mathcal{F}^{(1)}\,\mathcal{B}^{(1)})[1] + \lambda^{(1)}$, which can be proved by rewriting the operator $\mathcal{L}^{(1)}$ as

$$\mathcal{L}^{(1)} - \lambda^{(1)} = \mathcal{F}^{(1)}\,\mathcal{B}^{(1)} = (r^{(1)}\,\mathcal{D}_{1/\chi^{(1)}})\,(\mathcal{D}_{w\chi^{(1)}}\,A/r^{(1)}) = \mathcal{D}_{(r^{(1)}\chi^{(1)})^{-1}}\,A\,\mathcal{D}_{w\chi^{(1)}A/r^{(1)}}$$

and using the formula (2.8). Concerning the backward operator $\mathcal{B}^{(1)}$, we can view it as an "undressing" operator of the "dressed" function:

$$\mathcal{B}^{(1)}[\phi_\alpha^{(1)}] = (\lambda_\alpha - \lambda^{(1)})\phi_\alpha, \tag{2.13}$$

which can be proved by (2.10), (2.14) and (2.15).

Taking into account that the operator $\mathcal{F}^{(1)}$ annihilates the eigenfunction $\chi^{(1)}(= \phi_d)$, i.e. $\mathcal{F}^{(1)}[\chi^{(1)}] = 0$, we define

$$\phi_\alpha^{(1)} := \mathcal{T}_d^{(1)}[\phi_\alpha] := \begin{cases} r^{(1)}/(Aw\chi^{(1)}) \text{ if } \mathcal{F}^{(1)}[\phi_\alpha] = 0 \\ \mathcal{F}^{(1)}[\phi_\alpha] \qquad \text{otherwise} \end{cases} \tag{2.14}$$

for all $\alpha \in \mathbb{I}$, which is the eigenfunction of the transformed operator $\mathcal{L}^{(1)}$: if $\mathcal{F}^{(1)}[\phi_\alpha] \neq 0$, then

$$\mathcal{L}^{(1)}[\mathcal{F}^{(1)}[\phi_\alpha]] = (\mathcal{F}^{(1)}(\mathcal{L} - \lambda^{(1)}) + \lambda^{(1)}\mathcal{F}^{(1)})[\phi_\alpha] = \lambda_\alpha \mathcal{F}^{(1)}[\phi_\alpha]$$

and, if $\mathcal{F}^{(1)}[\phi_\alpha] = 0$, then

$$\mathcal{L}^{(1)}[\phi_\alpha^{(1)}] = (\mathcal{F}^{(1)}\mathcal{B}^{(1)} + \lambda^{(1)})[\phi_\alpha^{(1)}] = \lambda^{(1)}\phi_\alpha^{(1)}$$

where we have used $\mathcal{B}^{(1)}[r^{(1)}/(Aw\chi^{(1)})] = \mathcal{D}_{w\chi^{(1)}}[1/(w\chi^{(1)})] = \mathcal{D}[1] = 0$. Hence the Darboux transformation

$$(\mathcal{L}, \{\phi_\alpha\}) \overset{\lambda^{(1)}}{\longmapsto} (\mathcal{L}^{(1)}, \{\phi_\alpha^{(1)}\})$$

is an isospectral transformation:

$$\mathcal{L}^{(1)}[\phi_\alpha^{(1)}] = \lambda_\alpha\,\phi_\alpha^{(1)} \quad \text{for all } \alpha \in \mathbb{I}. \tag{2.15}$$

Orthogonality We give a formal discussion on the orthogonality relation of the system of transformed functions $\{\phi_\alpha^{(1)}\}$. Suppose that there exists the linear functional $\langle\cdot\rangle_w = \langle w\,\cdot\rangle$ such that

$$\langle\phi_\alpha\phi_{\alpha'}\rangle_w = h_\alpha\delta_{\alpha,\alpha'}, \quad \text{for} \quad \alpha, \alpha' \in S \subset \mathbb{I},$$

where $h_\alpha \neq 0$ and $\delta_{\alpha,\alpha'}$ is Kronecker's delta function. The system of transformed eigenfunctions becomes the orthogonal system with respect to the modified weight $w^{(1)} = Aw/(r^{(1)})^2$:

$$\langle\phi_\alpha^{(1)}\phi_{\alpha'}^{(1)}\rangle_{w^{(1)}} = h_\alpha^{(1)}\delta_{\alpha,\alpha'}, \tag{2.16}$$

where

$$h_\alpha^{(1)} = \begin{cases} h_\alpha(\lambda^{(1)} - \lambda_\alpha) & \text{if } \lambda^{(1)} \neq \lambda_\alpha \\ \langle A^{-1} w^{-1} (\chi^{(1)})^{-2} \rangle & \text{otherwise} \end{cases}.$$

This orthogonality relation can be proved by induction:

$$\begin{aligned} &\langle \mathcal{F}^{(1)}[\phi_\alpha^{(0)}]\, \mathcal{F}^{(1)}[\phi_{\alpha'}^{(0)}] \rangle_{w^{(1)}} = \langle \mathcal{F}^{(1)}[\phi_\alpha^{(0)}]\, (w^{(1)}\, r^{(1)}\, \mathcal{D}_{1/\chi^{(1)}})[\phi_{\alpha'}^{(0)}] \rangle \\ &= \langle \mathcal{F}^{(1)}[\phi_\alpha^{(0)}]\, (A/r^{(1)}\, \mathcal{D}_{1/w\chi^{(1)}})[w\phi_{\alpha'}^{(0)}] \rangle = -\langle \mathcal{B}^{(1)} \mathcal{F}^{(1)}[\phi_\alpha^{(0)}]\, w\phi_{\alpha'}^{(0)} \rangle \\ &= \langle (\lambda^{(1)} - \lambda_\alpha) \phi_\alpha^{(0)}\, w\phi_{\alpha'}^{(0)} \rangle = (\lambda^{(1)} - \lambda_\alpha) \langle \phi_\alpha^{(0)}\, \phi_{\alpha'}^{(0)} \rangle_w, \end{aligned}$$

where we have used that the operator $(\mathcal{B}^{(1)})^* = -A/r^{(1)}\, \mathcal{D}_{1/w\chi^{(1)}}$ is formal adjoint of $\mathcal{B}^{(1)}$, i.e., $\langle \mathcal{B}^{(1)}[u], v \rangle = \langle u, (\mathcal{B}^{(1)})^*[v] \rangle$. Similarly, we have

$$\begin{aligned} &\langle \mathcal{F}^{(1)}[\phi_\alpha^{(0)}]\, r^{(1)} (Aw\chi^{(1)})^{-1} \rangle_{w^{(1)}} = \langle (r^{(1)}\, \mathcal{D}_{1/\chi^{(1)}})[\phi_\alpha^{(0)}]\, (r^{(1)} \chi^{(1)})^{-1} \rangle \\ &= -\langle \phi_\alpha^{(0)}\, (-\mathcal{D}_{\chi^{(1)}}\, r^{(1)})[(r^{(1)} \chi^{(1)})^{-1}] \rangle = \langle \phi_\alpha^{(0)}\, (\chi^{(1)})^{-1} \mathcal{D}[1] \rangle = 0, \end{aligned}$$

and $\langle (r^{(1)})^2 (Aw\chi^{(1)})^{-2} \rangle_{w^{(1)}} = \langle (Aw)^{-1} (\chi^{(1)})^{-2} \rangle$.

§ 3. System of quasi-polynomial eigenfunctions

In this section, we will discuss a special class of eigenfunctions of the operator $\mathcal{L}$ such that

$$\mathcal{L}[\xi \tilde{p}_n](x) = \mu_n \xi(x) \tilde{p}_n(x), \quad n = 0, 1, 2, \ldots \tag{3.1}$$

where ξ is a function in x and $\tilde{p}_n$ is a polynomial in x. Here each eigenfunction is given by what we call *quasi-polynomial eigenfunction*, that is, the product of gauge factor ξ and polynomial $\tilde{p}_n$, The operator $\mathcal{L}$ may have several sequences of quasi-polynomial eigenfuncitons. We call the set of all sequences of quasi-polynomial eigenfuncitons of $\mathcal{L}$ as *the system of quasi-polynomial eigenfunctions of* $\mathcal{L}$. From this system of quasi-polynomial eigenfunctions, in the next section, we will take the seed function of the Darboux transformation. This is the key ingredient in the construction of orthogonal polynomial system with jump in degree.

First we introduce the Bochner type differential operator which provides a sequence of polynomial eigenfunctions. Let $\mathbb{L}$ be a set of second order ordinary differential operators defined by

$$\mathbb{L} = \left\{ \alpha_2 \partial^2 + \alpha_1 \partial \mid \alpha_1, \alpha_2 \text{ are polynomial in } x, 0 \leq \deg(\alpha_2) \leq 2, \deg(\alpha_1) = 1 \right\}. \tag{3.2}$$

Suppose that $\mathcal{L} = A(x)\partial^2 + B(x)\partial \in \mathbb{L}$ and the condition

$$kA''/2 + B' \neq 0 \text{ for } k \in \mathbb{Z}_{>0}, \tag{3.3}$$

holds, then, for each $n > 0$, the eigenvalue problem

$$\mathcal{L}[p_n](x) = \lambda_n \, p_n(x), \tag{3.4}$$

has a unique polynomial eigenfunction of degree n up to the scaling constant. We may denote the nth degree monic polynomial eigenfunction of $\mathcal{L} = A\partial^2 + B\partial \in \mathbb{L}$ by $P_n(x; A, B)$, that is,

$$\mathcal{L}[P_n(x; A, B)] = \lambda_n^{(A,B)} P_n(x; A, B),$$

where the value of spectral parameter, $\lambda_n^{(A,B)}$, is given by $\lambda_n^{(A,B)} = n[(n-1)A''/2 + B']$. The condition (3.3) assures that $\lambda_n \neq \lambda_m$ for any non-negative integer $n \neq m$[8].

In this section, we give all possible gauge factors $\xi(x)$ which enable us to derive a sequence of quasi-polynomial eigenfunctions of $\mathcal{L} \in \mathbb{L}$. Here we introduce the conjugated operator by

$$\tilde{\mathcal{L}} = \xi^{-1} \, \mathcal{L} \, \xi - \mu_0, \tag{3.5}$$

and its corresponding spectral parameter $\tilde{\lambda}_n = \mu_n - \mu_0$, then (3.1) is rewritten as

$$\tilde{\mathcal{L}}[\tilde{p}_n] = \tilde{\lambda}_n \tilde{p}_n, \quad n = 0, 1, 2, \ldots, \tag{3.6}$$

which means that $\tilde{\mathcal{L}}$ has a sequence of polynomial eigenfunctions. Then it follows that $\tilde{\mathcal{L}} \in \mathbb{L}$ from Bochner's theorem. Here the zero eigenfunction is given by $\tilde{p}_0 = 1$: $\tilde{\mathcal{L}}[1] = \tilde{\lambda}_0 = 0$. Now we can show that all possible gauge factors can be derived from the following proposition.

Proposition 3.1. *Let $\mathcal{L} = A(x)\partial^2 + B(x)\partial \in \mathbb{L}$ and let η_γ be the function defined by*

$$\eta_\gamma(x) = \eta(x; A, B, \gamma) = \frac{1}{2\pi i} \int_\gamma \frac{B(z) - A'(z)}{A(z)} \frac{dz}{z - x}, \tag{3.7}$$

where γ is a positively oriented closed curve in $\mathbb{C} \backslash \{$zeros of $A(z), x\}$ which does not enclose the point $x \in \mathbb{C}$. In the case when $(A, B) \neq (a_0(x-a)^2, b_0(x-a))$ with some constants $a_0, b_0, a \in \mathbb{C}$ satisfying $a_0 b_0 \neq 0$, there exists some function $\xi(x)$ such that

$$\xi^{-1} \, \mathcal{L} \, \xi - \mu \in \mathbb{L}, \tag{3.8}$$

where μ denotes the constant part of $\xi^{-1}\mathcal{L}\xi$, if and only if,

$$\xi'(x)/\xi(x) = \eta_\gamma(x), \tag{3.9}$$

with some γ. In the case when $(A, B) = (a_0(x-a)^2, b_0(x-a))$, it holds that $\xi^{-1} \, \mathcal{L} \, \xi - \mu \in \mathbb{L}$, if and only if, $\xi'/\xi = \kappa/(x-a)$ with an arbitrary constant κ.

First we prove that this transformation $\xi^{-1}\mathcal{L}\xi$ preserves the polynomiality of the coefficients of $\mathcal{L}$.

Lemma 3.2. *Let $A(x), B(x), C(x)$ be polynomials in x and define the operator $\mathcal{L}$ by $\mathcal{L} = A(x)\partial^2 + B(x)\partial + C(x)$. Let γ be a positively oriented closed curve in $\mathbb{C}\backslash\{$zeros of $A(x)$, $x\}$. We introduce functions $\tilde{A}(x), \tilde{B}(x), \tilde{C}(x)$ as the coefficients of the operator $\xi_\gamma^{-1}\,\mathcal{L}\,\xi_\gamma = \tilde{A}(x)\partial^2 + \tilde{B}(x)\partial + \tilde{C}(x)$, where $\xi_\gamma(x)$ is determined up to the multiplier constant from $\xi'_\gamma/\xi_\gamma = \eta_\gamma = \eta(x; A, B, \gamma)$. Then the coefficients $\tilde{A}(x), \tilde{B}(x), \tilde{C}(x)$ are the polynomials in x such that $\tilde{A} = A$ and*

$$\deg(\tilde{B}) \le \max(\deg(A) - 1, \deg(B)), \tag{3.10}$$

$$\deg(\tilde{C}) \le \max(\deg(A) - 2, \deg(B) - 1, \deg(C)). \tag{3.11}$$

Proof. If there are no zeros of $A(x)$ inside the curve γ, then this lemma trivially holds. Here we suppose that the curve γ encircles several number of zeros of $A(x)$, denoted by $a_1, a_2, \ldots, a_n$. Let k_j be the order of the zero a_j for $j = 1, 2, \ldots, n$. Then we have $A(x) = p(x)\prod_{j=1}^{n}(x - a_j)^{k_j}$ where $p(x)$ is a polynomial. It follows that

$$\eta_\gamma(x) = \sum_{j=1}^{n} \operatorname*{Res}_{z=a_j}\left[\frac{B(z) - A'(z)}{A(z)\,(z - x)}\right],$$

from the residue theorem. Let us introduce $g_{\gamma_j}(x)$ $(j = 1, 2, \ldots, n)$ by

$$\frac{g'_{\gamma_j}(x)}{g_{\gamma_j}(x)} = \operatorname*{Res}_{z=a_j}\left[\frac{B(z) - A'(z)}{A(z)\,(z - x)}\right],$$

which can be rewritten as

$$\frac{g'_{\gamma_j}(x)}{g_{\gamma_j}(x)} = \frac{k_j}{x - a_j} + \frac{1}{(k_j^{(0)} - 1)!}\frac{d^{k_j^{(0)}-1}}{dz^{k_j^{(0)}-1}}\left((z - a_j)^{k_j^{(0)}}\frac{B(z)}{A(z)(z - x)}\right)\Bigg|_{z=a_j},$$

where $k_j^{(0)}$ denotes the order of pole at the point $x = a_j$ of $B(x)/A(x)$. Then one can find that $\xi_\gamma(x)$ is proportional to $\prod_{j=1}^{n} g_{\gamma_j}(x)$. Hence it is enough to prove that each conjugate transformation, $\mathcal{L} \to g_{\gamma_j}^{-1}\mathcal{L}g_{\gamma_j}$, preserves the polynomiality of the coefficients and the inequalities (3.10) and (3.11).

Let us consider the following conjugated operator:

$$g_{\gamma_j}^{-1}\mathcal{L}g_{\gamma_j} = \tilde{A}\partial^2 + \tilde{B}\partial + \tilde{C}.$$

By equating coefficients on both sides of equation above, we find that $\tilde{A} = A$, $\tilde{B} = B + 2A\,g'_{\gamma_j}/g_{\gamma_j}$, $\tilde{C} = C + B\,g'_{\gamma_j}/g_{\gamma_j} + Ag''_{\gamma_j}/g_{\gamma_j}$. The function $g'_{\gamma_j}(x)/g_{\gamma_j}(x)$ can be expanded into $g'_{\gamma_j}(x)/g_{\gamma_j}(x) = \sum_{\mu=1}^{k_j^{(0)}} \alpha_\mu (x - a_j)^\mu$ with some constant α_μ. Since the order $k_j^{(0)}$ is less than or equal to k_j, $A(x)\eta_\gamma(x)$ is regular at $x = a_j$. Therefore $\tilde{B}(x)$ is a polynomial of degree less than or equal to $\max(\deg(A)) - 1, \deg(B))$. Substituting $(\log g_{\gamma_j})' = \sum_{\mu=1}^{k_j^{(0)}} \alpha_\mu (x-a_j)^\mu$ into $\tilde{C} = C+(\log g_{\gamma_j})'B+((\log g_{\gamma_j})''+((\log g_{\gamma_j})')^2)A$, one can find that $\tilde{C}(x)$ can be expanded into $\tilde{C} = \sum_{\mu=-k_j^{(0)}}^{M} \beta_\mu (x-a_j)^\mu$ where $M = \max(\deg(C), \deg(B)-1, \deg(A)-2)$. After some calculations, we find that $\operatorname{Res}_{x=a}(\tilde{C}(x)) = \operatorname{Res}_{x=a}((x-a)\tilde{C}(x)) = \cdots = \operatorname{Res}_{x=a}((x-a)^{k-1}\tilde{C}(x)) = 0$, which mean that $\tilde{C}(x)$ is a polynomial of degree less than or equal to M. This completes the proof. $\square$

Let $\mathbf{Q}_{A,B}$ be the set of all poles of the integrand $(B(z) - A'(z))A(z)^{-1}(z-x)^{-1}$ in $\mathbb{C}\cup\{\infty\}$. The number of elements of $\mathbf{Q}_{A,B}$, denoted by $\sharp\mathbf{Q}_{A,B}$, is taken from 0 to 3, since the degree of A is less than or equal to 2. One can find that the integrand is identically zero if $\sharp\mathbf{Q} = 0$ and also $\mathbf{Q}_{A,B} = \{x\}$ if $\sharp\mathbf{Q}_{A,B} = 1$. The function $\eta_\gamma = \eta(x; A, B, \gamma)$ can be evaluated in terms of the residue at the pole. Since $\deg(A) \leq 2$ and $\deg(B) = 1$, then we have at most two or four types of γ. For any pair of nonzero polynomial A of degree at most 2, and linear polynomial B, there are two types of closed curve γ, denoted by I and II, as follows:

- I does not enclose any points in $\mathbf{Q}_{A,B}$
- II encloses all points but x in $\mathbf{Q}_{A,B}$

Note that, if $\sharp\mathbf{Q}_{A,B}$ is equal to 0 or 1, then the γ_{I} and γ_{II} can be treated as being the same curve. Additionally, if the set $\mathbf{Q}_{A,B}\backslash\{x\}$ contains two elements, say a_1 and a_2, then two more additional closed curves III and IV can be introduced as

- III encloses a_1, but not a_2 and x
- IV encloses a_2, but not a_1 and x

Let us define the finite set $\mathbb{G}_{A,B}$ by

$$\mathbb{G}_{A,B} = \begin{cases} \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\} & \text{if } \sharp\mathbf{Q}_{A,B} = 3 \\ \{\mathrm{I}, \mathrm{II}\} & \text{otherwise} \end{cases}.$$

Note that the operator $\mathcal{L} = A\partial^2 + B\partial \in \mathbb{L}$ and the path of integration $\gamma \in \mathbb{G}_{A,B}$ determine the conjugated operator:

$$\tilde{\mathcal{L}}_\gamma = \xi_\gamma^{-1}\, \mathcal{L}\, \xi_\gamma - \mu_\gamma \in \mathbb{L}, \tag{3.12}$$

where the gauge function ξ_γ is formally given by

$$\xi_\gamma(x) = \xi(x; A, B, \gamma) = \exp\left(\int^x \eta(z; A, B, \gamma) dz \right),$$

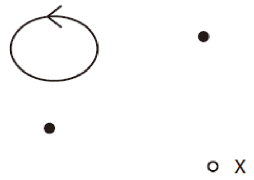

Figure 1. Case $\sharp\mathbf{Q}_{A,B} = 3$, curve I

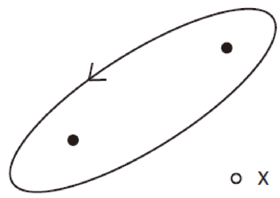

Figure 2. Case $\sharp\mathbf{Q}_{A,B} = 3$, curve II

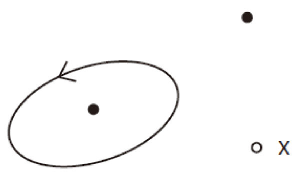

Figure 3. Case $\sharp\mathbf{Q}_{A,B} = 3$, curve III

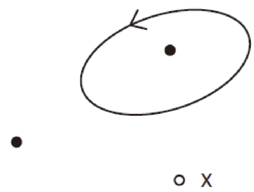

Figure 4. Case $\sharp\mathbf{Q}_{A,B} = 3$, curve IV

and the constant μ_γ can be calculated from $\mu_\gamma = (\eta_\gamma' + \eta_\gamma^2)A + \eta_\gamma B$. Direct calculation shows that

$$\tilde{\mathcal{L}}_\gamma = A(x)\partial^2 + \tilde{B}_\gamma(x)\partial,$$

where $\tilde{B}_\gamma(x) = B(x) + 2A(x)\,\eta(x; A, B; \gamma)$. It is easy to see that $\tilde{B}_\gamma(x)$ is indeed a linear polynomial in x. By using (2.2) we can rewrite the equation of ξ_γ as follows:

$$\frac{\xi_\gamma'(x)}{\xi_\gamma(x)} = \eta_\gamma(x) = \frac{1}{2\pi i}\int_\gamma \frac{B(z) - A'(z)}{A(z)}\frac{dz}{z-x} = \frac{1}{2\pi i}\int_\gamma \frac{w'(z)}{w(z)}\frac{dz}{z-x}. \tag{3.13}$$

Suppose that $\tilde{\mathcal{L}}_\gamma$ satisfies the condition (3.3), then, for each $n \geq 0$, the nth degree polynomial eigenfunction $p_{(\gamma,n)}$ is uniquely determined up to the scaling constant from

$$\tilde{\mathcal{L}}_\gamma[p_{(\gamma,n)}] = (\mathcal{L} + 2A\eta_\gamma\partial)[p_{(\gamma,n)}] = \tilde{\lambda}_{(\gamma,n)}\, p_{(\gamma,n)}, \tag{3.14}$$

where $\tilde{\lambda}_{(\gamma,n)} = n[(n-1)A''/2 + B' + 2(A\eta_\gamma)']$.

In the following lemma, we present all possible forms of η_γ.

Lemma 3.3. *Let* $\operatorname*{Res}_{\zeta} = \operatorname*{Res}_{z=\zeta}[(B(z) - A'(z))/(A(z)(z-x))]$ *be a residue at the point* $\zeta \in \mathbb{C} \cup \{\infty\}$*. For each* $\mathcal{L} \in \mathbb{L}$*, corresponding to the choice of the closed curve* γ*, there exist at most two or four types of function* $\eta_\gamma(x)$*.*

- *For any pair of nonzero polynomial* A *of degree at most 2, and linear polynomial* B*,*
  (i) $\eta_{\mathrm{I}} = 0$*,* (ii) $\eta_{\mathrm{II}}(x) = -\operatorname*{Res}_{x} = (A'(x) - B(x))/A(x)$*.*

Only the following two cases admit additional two types of η:

- *For any pair of polynomials A, B such that $A = a_0(x-a_1)(x-a_2) \neq 0$ with $a_1 \neq a_2$, $\deg B = 1$, and $B - A'$ has no common root with A,*
  (iii) $\eta_{\mathrm{III}} = \operatorname*{Res}_{a_1}$, (iv) $\eta_{\mathrm{IV}} = \operatorname*{Res}_{a_2}$.
- *For any pair of polynomials A, B such that $A = a_0(x - a_1) \neq 0$, $\deg B = 1$, and $B - A'$ has no common root with A, then*
  (iii)' $\eta_{\mathrm{III}} = \operatorname*{Res}_{a_1}$, (iv)' $\eta_{\mathrm{IV}} = \operatorname*{Res}_{\infty}$.

Proof of Proposition 1. First we show that the equation (3.9) for ξ is satisfied with (3.8). Let

$$\tilde{\mathcal{L}} = \xi^{-1}\,\mathcal{L}\,\xi - \mu\,. \tag{3.15}$$

From (3.8), $\tilde{\mathcal{L}}$ can be expressed in the following form:

$$\tilde{\mathcal{L}} = \tilde{A}(x)\partial^2 + \tilde{B}(x)\partial, \tag{3.16}$$

where $\tilde{A}(x)$ and $\tilde{B}(x)$ are polynomials of $\deg \tilde{A}(x) \leq 2$ and $\deg \tilde{B}(x) = 1$, respectively. Comparing (3.15) and (3.16) we obtain

$$\tilde{A} = A, \quad \tilde{B} = B + 2A\xi'/\xi, \quad \mathcal{L}[\xi] = \mu\xi, \tag{3.17}$$

which lead us to find a Riccati type equation with respect to ξ'/ξ:

$$A\left((\xi'/\xi)' + (\xi'/\xi)^2\right) + B\xi'/\xi = \mu. \tag{3.18}$$

The second equation in (3.17) shows that ξ'/ξ is the rational function such that

$$\xi'/\xi = (\tilde{B} - B)/2A = \tilde{q}/q, \tag{3.19}$$

where numerator $\tilde{q}$ and denominator q are both polynomials and their degrees are

$$\deg(\tilde{q}) \leq 1, \quad 0 \leq \deg(q) \leq 2. \tag{3.20}$$

Let a_0, a_1, a_2 be points in $\mathbb{C}$ such that $a_1 \neq a_2$ and $a_0 \neq 0$ and let b_1 be the nonzero leading coefficient of B. We assume that $A(x)$ takes one of the four possible forms without loss of generality: $a_0, a_0(x - a_1), a_0(x - a_1)^2, a_0(x - a_1)(x - a_2)$. Let us introduce

$$h = \frac{B - A'}{A} + \frac{\xi'}{\xi}. \tag{3.21}$$

In the case when $\tilde{B} = B$, it is easy to see from (3.7) that $h = (B - A')/A$. If $\tilde{B} \neq B$, by using (3.17)–(3.20), we can rewrite h in several forms:

$$h = \frac{B - A'}{A} + \frac{\tilde{B} - B}{2A} = \frac{2\mu - (\tilde{B} - B)'}{\tilde{B} - B} = \frac{\mu - (A\xi'/\xi)'}{A\xi'/\xi}. \tag{3.22}$$

Taking into account the degrees of polynomials A, B and $\tilde{B}$, we can read (3.22) that the degree of denominator polynomial of h is less than or equal to 1, and the possible pole must be taken from the zero of A, necessarily, the function h must be proportional to one of the following functions: 0, 1, $1/(x - a_1)$ and $1/(x - a_2)$. We consider all cases to solve (3.18) for ξ'/ξ and unknown spectral parameter μ under the conditions (3.7).

(1) $\tilde{B} = B$.
Putting $\tilde{B} = B$ to (3.7), we have $\xi'/\xi = 0$ which leads to $\mu = 0$. This case corresponds to the case (i) in Lemma 3.3.

(2) $\tilde{B} \neq B$ and $h = 0$.
In this case, from the definition of h, it is obvious that $\xi'/\xi = (A' - B)/A$, which corresponds to the case (ii) in Lemma 3.3.

(3) $\tilde{B} \neq B$ and $h = \kappa \neq 0$.
Putting $h = \kappa$ to (3.21) and (3.22), it is easy to see that $\xi'/\xi = \kappa + (A' - B)/A$ and that $\tilde{B} - B$ is some nonzero constant, denoted by $2\hat{\kappa}$, that is $\tilde{B} - B = 2\hat{\kappa} \neq 0$, from which $\xi'/\xi = \hat{\kappa}/A$ follows. These two expressions of ξ'/ξ lead us to

$$\hat{\kappa} = \kappa A + A' - B,$$

which shows that possible degree of A is only one ($\kappa(\neq 0), \hat{\kappa} \in \mathbb{C}, \deg B = 1$). Thus we take A as $a_0(x - a_1)$ here. If the polynomial $A' - B$ has a common root with the polynomial A, then $\frac{\tilde{B}-B}{2A} = \frac{\hat{\kappa}}{a_0(x-a_1)} = \kappa - \frac{b_1}{a_0}$ for any $x \in \mathbb{C}\backslash\{a_1\}$ which can be satisfied only in the case when $\hat{\kappa} = 0$, but this contradicts the assumption that $\tilde{B} \neq B$. Therefore, if ξ'/ξ exists in this case, A must be a linear polynomial which has no common root with $A' - B$. As a consequence we obtain

$$\begin{aligned} \frac{\xi'(x)}{\xi(x)} &= \frac{\hat{\kappa}}{a_0(x - a_1)} = \frac{1}{2\pi i} \int_{\mathrm{III}'} \frac{-\hat{\kappa} dz}{a_0(z - a_1)(z - x)} \\ &= \frac{1}{2\pi i} \int_{\mathrm{III}'} \left[-\kappa + \frac{B(z) - A(z)'}{A(z)} \right] \frac{dz}{z - x} = \eta_{\mathrm{III}'}(x) \end{aligned}$$

where III' is a closed curve which encloses a_1, but not x and ∞, which corresponds to the case (iii)' in Lemma 3.3.

(4) $\tilde{B} \neq B$ and $h = \kappa/(x - a_2) \neq 0$.
From the assumption on A, the point a_2 is appeared only in the case when $A =$

$a_0(x-a_1)(x-a_2)$ with $a_1 \neq a_2, a_0 \neq 0$. Putting $h = \kappa/(x-a_2)$ and $A = a_0(x-a_1)(x-a_2)$ to (3.21) and (3.22), it is easy to see that $A\xi'/\xi = \kappa a_0(x-a_1) + A' - B$ and that $\tilde{B} - B$ is a linear polynomial, from which we can see that the degree of polynomial $A\xi'/\xi$ is just one. Then we introduce a constant by $\hat{\kappa} = (A\xi'/\xi)'$ which is not equal to μ, otherwise κ becomes 0. Thus we have $A\xi'/\xi = \hat{\kappa}(x-a_2)$ which can be rewritten as $\xi'/\xi = \hat{\kappa}/(a_0(x-a_1))$. These two expressions of ξ'/ξ leads us to $\hat{\kappa}(x-a_2) = \kappa a_0(x-a_1) + A' - B$. As a consequence we obtain

$$\begin{aligned}\frac{\xi'(x)}{\xi(x)} &= \frac{\hat{\kappa}}{a_0(x-a_1)} = \frac{1}{2\pi i}\int_{\mathrm{III}} \frac{-\hat{\kappa}dz}{a_0(z-a_1)(z-x)} \\ &= \frac{1}{2\pi i}\int_{\mathrm{III}} \left[\frac{-\kappa}{z-a_2} + \frac{B(z)-A(z)'}{A(z)}\right]\frac{dz}{z-x} = \eta_{\mathrm{III}}(x),\end{aligned}$$

where III is a closed curve which encloses a_1, but not x and a_2. This is the case (iii) in Lemma 3.3.

(5) $\tilde{B} \neq B$ and $h = \kappa/(x-a_1) \neq 0$.
From (3.22), if $A\xi'/\xi$ has a factor $x-a_1$, A is also required to have a factor $x-a_1$. Putting $h = \kappa/(x-a_1)$ to (3.22), it is easy to see that $\deg(\tilde{B}-B) = \deg(A\xi'/\xi) = 1$ and $A\xi'/\xi = (x-a_1)(\mu - (A\xi'/\xi)')/\kappa = \hat{\kappa}(x-a_1)$ with some nonzero constant $\hat{\kappa}$. Hence we have only three cases as follows:

$$(A, \xi'/\xi) \in \left\{(a_0(x-a_1), \hat{\kappa}), (a_0(x-a_1)^2, \hat{\kappa}/(x-a_1)), (a_0(x-a_1)(x-a_2), \hat{\kappa}/(x-a_2))\right\}.$$

– $(A, \xi'/\xi) = (a_0(x-a_1)(x-a_2), \hat{\kappa}/(x-a_2))$

$$\begin{aligned}\frac{\xi'(x)}{\xi(x)} &= \frac{\hat{\kappa}}{x-a_2} = \frac{1}{2\pi i}\int_{\mathrm{IV}} \frac{-\hat{\kappa}dz}{(z-a_2)(z-x)} \\ &= \frac{1}{2\pi i}\int_{\mathrm{IV}} \left[\frac{-\kappa}{z-a_1} + \frac{B(z)-A'(z)}{A(z)}\right]\frac{dz}{z-x} = \eta_{\mathrm{IV}}(x),\end{aligned}$$

where we have used $\xi'/\xi = \hat{\kappa}/(x-a_2) = \kappa/(x-a_1) + (A'-B)/A$ which follows from (3.21), and IV is a closed curve which encloses a_2, but not x and a_1. This is the case (iv) in Lemma 3.3.

– $(A, \xi'/\xi) = (a_0(x-a_1), \hat{\kappa})$

$$\frac{\xi'(x)}{\xi(x)} = \hat{\kappa} = \frac{1}{2\pi i}\int_{\mathrm{IV}'} \frac{-\hat{\kappa}dz}{z-x} = \frac{1}{2\pi i}\int_{\mathrm{IV}'} \left[\frac{-\kappa}{z-a_1} + \frac{B(z)-A'(z)}{A(z)}\right]\frac{dz}{z-x} = \eta_{\mathrm{IV}'}(x),$$

where we have used $\xi'/\xi = \hat{\kappa} = \kappa/(x-a_1) + (A'-B)/A$ which follows from (3.21), and IV$'$ is a closed curve which encloses ∞, but not x and a_1. This is the case (iv)' in Lemma 3.3.

– $(A, \xi'/\xi) = (a_0(x-a_1)^2, \hat{\kappa}/(x-a_1))$
Putting $h = \kappa/(x-a_1), A = a_0(x-a_1)^2, \xi'/\xi = \hat{\kappa}/(x-a_1)$ and $B = b_0(x-b_1)$ to (3.21), and equating coefficients of x^n on both sides, we obtain $B = (\kappa - \hat{\kappa} + 2)a_0(x-a_1)$. This is the case when $(A,B) = (a_0(x-a_1)^2, b_0(x-a_1))$, in which, ξ'/ξ is allowed to contain an arbitrary constant, $\hat{\kappa}$. (The case $\hat{\kappa} = 0$ is considered in the case (1))

Conversely, if (3.9) holds, then it follows (3.8) from Lemma 3.2. □

For later use, we introduce polynomials $A_\gamma(x)$ and $A_{\gamma,\tilde{\gamma}}(x)$ for $\gamma, \tilde{\gamma} \in \mathbb{G}_{A,B}$ such that

$$\frac{A'_\gamma(x)}{A_\gamma(x)} = \frac{1}{2\pi i}\int_\gamma \frac{A'(z)}{A(z)}\frac{dz}{x-z}, \quad \frac{A'_{\gamma,\tilde{\gamma}}(x)}{A_{\gamma,\tilde{\gamma}}(x)} = \frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{A'_\gamma(z)}{A_\gamma(z)}\frac{dz}{x-z}. \tag{3.23}$$

Note that A_γ is a factor of A and $A_{\gamma,\tilde{\gamma}}$ is the greatest common divisor of A_γ and $A_{\tilde{\gamma}}$, which are uniquely determined up to the scaling constants.

For a given $\gamma \in \mathbb{G}_{A,B}$, there exists $\gamma^* \in \mathbb{G}_{A,B}$ such that $\xi_{\gamma^*} = \kappa(w\xi_\gamma)^{-1}$ with some nonzero constant κ. Here and hereafter we take integration constants so as to satisfy $A_{\mathrm{I}} = A_{\mathrm{I},\gamma} = A_{\gamma,\rho}/A_{\rho,\gamma} = A_{\gamma,\gamma^*} = A_\gamma A_{\gamma^*}/A = A_{\gamma,\rho}A_{\gamma^*,\rho}/A_\rho = \xi_\gamma\xi_{\gamma^*}w = 1$. Clearly $\xi_{\gamma^{**}} = \xi_\gamma$ holds. Hence γ^* can be considered as the dual of γ. Correspondingly, we say that ξ_{γ^*} is the dual of ξ_γ. For the elements of $\mathbb{G}_{A,B}$ defined before Lemma 3.3, one can find that II is the dual of I, i.e. II $=$ I* and, if exists, IV is the one of III, i.e. IV $=$ III*. Employing these normalizations, we obtain following corollary.

Corollary 3.4. *It holds that for $\gamma \in \mathbb{G}_{A,B}$,*

$$\xi_\gamma(x)\xi_{\gamma^*}(x)w(x) = 1.$$

In particular, $\xi_{\mathrm{I}}(x) = 1, \xi_{\mathrm{II}}(x) = w(x)^{-1}, \xi_{\mathrm{I}}(x)\,\xi_{\mathrm{II}}\,w(x) = 1, \xi_{\mathrm{III}}(x)\,\xi_{\mathrm{IV}}(x)\,w(x) = 1$. Analogously,

$$A_\gamma(x)A_{\gamma^*}(x) = A(x), \quad A_{\gamma,\rho}(x)A_{\gamma,\rho^*}(x) = A_\gamma(x), \quad A_{\mathrm{I}}(x) = 1, \quad A_{\mathrm{II}}(x) = A(x).$$

We give several examples of ξ_γ, A_γ and $A_{\gamma,\rho}$ under some normalized constant factor:

- Jacobi polynomials case: Let $A = 1-x^2, B = b-a-(a+b+2)x$ and $a_1 = 1, a_2 = -1$. Then

$$\begin{aligned}
&\xi_{\mathrm{I}} = 1, \quad \xi_{\mathrm{II}} = (1-x)^{-a}(1+x)^{-b}, \quad \xi_{\mathrm{III}} = (1-x)^{-a}, \quad \xi_{\mathrm{IV}} = (1+x)^{-b},\\
&A_{\mathrm{I}} = 1, \quad A_{\mathrm{II}} = 1-x^2, \quad A_{\mathrm{III}} = 1-x \quad A_{\mathrm{IV}} = 1+x,\\
&A_{\mathrm{I,I}} = A_{\mathrm{I,II}} = A_{\mathrm{I,III}} = A_{\mathrm{I,IV}} = A_{\mathrm{III,IV}} = 1, \quad A_{\mathrm{II,II}} = 1-x^2,\\
&A_{\mathrm{II,III}} = A_{\mathrm{III,II}} = A_{\mathrm{III,III}} = 1-x, \quad A_{\mathrm{II,IV}} = A_{\mathrm{IV,II}} = A_{\mathrm{IV,IV}} = 1+x.
\end{aligned}$$

- Laguerre polynomials case: Let $A = x, B = -x + a + 1$ and $a_1 = 0, a_2 = \infty$. Then

$$\xi_{\rm I} = 1, \quad \xi_{\rm II} = e^x x^{-a}, \quad \xi_{\rm III} = x^{-a}, \quad \xi_{\rm IV} = e^x,$$
$$A_{\rm I} = 1, \quad A_{\rm II} = x, \quad A_{\rm III} = x, \quad A_{\rm IV} = 1,$$
$$A_{\rm II,II} = A_{\rm II,III} A_{\rm III,II} = A_{\rm III,III} = x, \quad A_{\rm I,I} = \cdots = A_{\rm I,IV} = A_{\rm IV,I} = \cdots = A_{\rm IV,IV} = 1.$$

- Hermite polynomials case: Let $A = 1, B = -2x$, then

$$\xi_{\rm I} = 1, \quad \xi_{\rm II} = e^{x^2}, \quad A_{\rm I} = A_{\rm II} = 1, \quad A_{\rm I,I} = A_{\rm I,II} = A_{\rm II,I} = A_{\rm II,II} = 1.$$

§ 4. Orthogonal polynomials with jump in degree

In this section we give a method to construct orthogonal polynomials with jump in degree by using the results of §2 and §3. First we consider the Darboux transformation of the quasi-polynomial eigenfunction $\phi_\alpha(x)$ for $\alpha \in \mathbb{I}$. Let us fix a operator $\mathcal{L} = A\partial^2 + B\partial \in \mathbb{L}$. Then we can determine all possible sequences of quasi-polynomial eigenfunctions. Set $\mathbb{I} = \mathbb{G}_{A,B} \times \mathbb{Z}_{\geq 0}$. For simplicity, we assume that for each $\alpha = (\gamma, n) \in \mathbb{I}$, a quasi-polynomial eigenfunction $\phi_\alpha(x) = \xi_\gamma(x) p_\alpha$ is uniquely determined up to the multiplier constant and these quasi-polynomial eigenfunctions are mutually independent. Note that, if $\gamma = {\rm I}$, then $\phi_{({\rm I},n)}(x; A, B)$ for $n \in \mathbb{Z}_{\geq 0}$ is a polynomial of degree n, since $\xi_{\rm I} = 1$.

Now we apply the Darboux transformation (2.14) to the system of quasi-polynomial eigenfunctions of $\mathcal{L}$. By taking the Darboux parameter as $d = (\rho, j) \in \mathbb{I}$, then the seed function $\chi^{(1)}(x)$ is given by $\phi_d(x)$. For our purpose we choose the normalization factor as $r^{(1)} = \chi^{(1)} A_\rho/\xi_\rho$, and we obtain Darboux transformed eigenfunctions from (2.14):

$$\phi_\alpha^{(1)} = \mathcal{T}_d^{(1)}[\phi_\alpha] = \begin{cases} \xi_\rho^{-1} A_\rho W(\chi^{(1)}, \phi_\alpha) & \text{if } \alpha \neq d \\ A_\rho (Aw\xi_\rho)^{-1} = \xi_{\rho^*} A_{\rho^*}^{-1} = \xi_{\gamma^*} A_{\gamma^*}^{-1} & \text{otherwise} \end{cases}, \tag{4.1}$$

where $\chi^{(1)} = \phi_d$ and $W(f, g) = fg' - f'g$. If $W(\chi^{(1)}, \phi_\alpha) \neq 0$, this transformed eigenfunction can be divided into the gauge factor part $\xi_\alpha^{(1)}$ and the polynomial factor part $P_{\alpha,d}$ as $\phi_\alpha^{(1)}(x) = \xi_\alpha^{(1)}(x) P_{\alpha,d}(x)$ where

$$\xi_\alpha^{(1)} = \frac{A_{\gamma,\rho}}{A_{\gamma,\rho^*}} \xi_\gamma, \tag{4.2}$$

$$P_{\alpha,d} = \frac{A_\rho A_\gamma}{A_{\rho,\gamma}^2 \xi_\rho \xi_\gamma} \begin{vmatrix} \phi_d & \phi_\alpha \\ \phi_d' & \phi_\alpha' \end{vmatrix} = A_{\gamma,\rho^*} A_{\gamma^*_,\rho} \begin{vmatrix} p_d & p_\alpha \\ \eta_\rho p_d + p_d' & \eta_\gamma p_\alpha + p_\alpha' \end{vmatrix}, \tag{4.3}$$

where we have used that the rational function

$$\eta_\gamma - \eta_\rho = \frac{1}{2\pi i} \int_{\gamma - \rho} \frac{B(z) - A'(z)}{A(z)} \frac{dz}{z - x}$$

multiplied by $A_{\gamma,\rho^*}A_{\gamma^*,\rho} = A_\rho A_\gamma/(A_{\gamma,\rho})^2$ is a polynomial in x. From this expression and $\xi_{\mathrm{I}} = A_{\mathrm{I}} = A_{\mathrm{I},\rho} = 1$, one can easy to see that $\phi^{(1)}_{(\mathrm{I},n)}$ is a polynomial in x, if $(\mathrm{I}, n) \neq d$.

Proposition 4.1. *The Darboux transformed eigenfunction $\phi^{(1)}_\alpha(x)$ of $\mathcal{L}^{(1)}$ for $\alpha = (\gamma, n) \in \mathbb{G}_{A,B} \times \mathbb{Z}_{\geq 0}$ is the following type of quasi-polynomial:*

$$\phi^{(1)}_\alpha(x) = \tilde{\xi}^{(1)}_\alpha(x) P_{\alpha,d}(x), \tag{4.4}$$

whose gauge factor $\tilde{\xi}^{(1)}_\alpha$ is given in terms of the original gauge factor $\xi_\gamma(x)$ or $\xi_{\gamma^*}(x)$ with shifted parameters:

$$\tilde{\xi}^{(1)}_\alpha(x) = \begin{cases} \xi^{(1)}_\gamma(x) \text{ if } \alpha \neq d \\ \xi^{(1)}_{\gamma^*}(x) \text{ otherwise} \end{cases},$$

where $\xi^{(1)}_\gamma$ and $\xi^{(1)}_{\gamma^*}$ are respectively given by $\xi^{(1)}_\gamma = \xi(x; A, \hat{B}^{(1)}, \gamma)$ and $\xi^{(1)}_{\gamma^*} = \xi(x; A, \hat{B}^{(1)}, \gamma^*)$. Here the linear polynomial $\hat{B}^{(1)}$ can be presented as,

$$\hat{B}^{(1)}(x) = B(x) + W(A_\rho(x), A_{\rho^*}(x)).$$

Proof. From (3.23) and (4.2), we obtain

$$\frac{(\xi^{(1)}_\gamma(x))'}{\xi^{(1)}_\gamma(x)} = \frac{\xi'_\gamma(x)}{\xi_\gamma(x)} + \frac{A'_{\gamma,\rho}(x)}{A_{\gamma,\rho}} - \frac{A'_{\gamma,\rho^*}(x)}{A_{\gamma,\rho^*}} = \frac{1}{2\pi i}\int_\gamma \Bigg(\frac{B(z) - A'(z)}{A(z)} + \frac{A'_{\rho^*}(x)}{A_{\rho^*}} - \frac{A'_\rho(x)}{A_\rho}\Bigg)\frac{dz}{z-x} = \frac{1}{2\pi i}\int_\gamma \frac{\hat{B}^{(1)}(z) - A'(z)}{A(z)}\frac{dz}{z-x}.$$

Recall that the degree of polynomial A_ρ is less than or equal to the degree of A and the degree of A_{ρ^*} is given by $\deg(A) - \deg(A_\rho)$. Then $W(A_\rho, A_{\rho^*})$ must be a linear polynomial or a constant. Clearly $\hat{B}^{(1)}(z)$ and $B(z)$ are both linear polynomials in z. Thus we have $\xi^{(1)}_\gamma = \xi(x; A, \hat{B}^{(1)}, \gamma)$. It follows that $\xi^{(1)}_\gamma$ can be expressed by $\xi_\gamma = \xi(x; A, B, \gamma)$ with shifted parameters. □

Here we explicitly present the transformed Sturm-Liouville operator. By construction, $\phi^{(1)}_\alpha$ satisfies $\mathcal{L}^{(1)}[\phi^{(1)}_\alpha] = \lambda_\alpha \phi^{(1)}_\alpha$, which can be rewritten as

$$(\mathcal{D}_{w^{(1)}\phi^{(1)}_d} A \mathcal{D}_{1/\phi^{(1)}_d})[\phi^{(1)}_\alpha] = (\lambda_\alpha - \lambda_d)\phi^{(1)}_\alpha$$

or, equivalently,

$$(\mathcal{D}_{w^{(1)}} A \mathcal{D})[\phi^{(1)}_\alpha]/\phi^{(1)}_\alpha - (\mathcal{D}_{w^{(1)}} A \mathcal{D})[\phi^{(1)}_d]/\phi^{(1)}_d = \lambda_\alpha - \lambda_d,$$

for $\alpha, d \in \mathbb{I}$. In addition the coefficients of

$$\mathcal{L}^{(1)} = \mathcal{D}_{w^{(1)}} A \mathcal{D} + C^{(1)} = A^{(1)}\partial^2 + B^{(1)}\partial + C^{(1)},$$

where $w^{(1)} = Aw/\left(r^{(1)}\right)^2 = (A_{\rho^*}/A_\rho)\left(\xi_\rho/\chi^{(1)}\right)^2 w$, are given by

$$A^{(1)}(x) = A(x), \quad B^{(1)}(x) = B(x) + A(x)\left(\log\left(\left(\frac{\xi_\rho(x)}{\chi^{(1)}(x)}\right)^2 \frac{A_{\rho^*}(x)}{A_\rho(x)}\right)\right)',$$
$$C^{(1)}(x) = \lambda^{(1)} - (\mathcal{D}_{w^{(1)}}\, A\,\mathcal{D})[\phi_d^{(1)}(x)]/\phi_d^{(1)}(x).$$

It is easy to calculate the degree of polynomial $P_{\alpha,d}(x)$ as follows,

$$\deg\left(P_{\alpha,d}(x)\right) \le n + j + \epsilon(\gamma,\rho), \tag{4.5}$$

where

$$\epsilon(\gamma,\rho) = \begin{cases} -1 \text{ if } \gamma = \rho \\ \ \ 1 \ \text{ if } \gamma = \rho^* \\ \ \ 0 \ \text{ otherwise} \end{cases} \text{for } \gamma,\rho \in \mathbb{G}_{A,B}. \tag{4.6}$$

As a consequence we can divide $\mathbb{I} = \mathbb{G}_{A,B} \times \mathbb{Z}_{\ge 0}$ according to the corresponding gauge factor. Let us define

$$X_\gamma^{(0)} = \{(\gamma, n) \mid n \in \mathbb{Z}_{\ge 0}\}, \quad X_\gamma^{(1)} = \left\{(\gamma', n) \in \mathbb{I} \mid \tilde{\xi}^{(1)}_{\gamma',n} = \xi_\gamma^{(1)}\right\},$$

for $\gamma \in \mathbb{G}_{A,B}$. From the proposition 4.1, it follows that

$$X_\gamma^{(1)} = \begin{cases} X_\gamma^{(0)} \backslash \{d = (\gamma, j)\} & \text{if } \gamma = \rho \\ X_\gamma^{(0)} \cup \{d = (\gamma^*, j)\} & \text{if } \gamma = \rho^* \\ X_\gamma^{(0)} & \text{otherwise} \end{cases}. \tag{4.7}$$

If $\alpha \in X_{\mathrm{I}}^{(1)}$, then $\tilde{\xi}_\alpha^{(1)} = \xi_I^{(1)} = 1$. Hence we obtain a sequence of polynomial eigenfunctions with jump in degree:

$$\boldsymbol{P}^{(1)}_{X_{\mathrm{I}}} = \{P_{\alpha,d}(x) \mid \alpha \in X_{\mathrm{I}}^{(1)}\},$$

such that $\mathcal{L}^{(1)}[P_{\alpha,d}] = \lambda_\alpha\, P_{\alpha,d}$.

It is well-known that the Darboux transformation can be classified into three states: delete-state, add-state and isospectral-state. In the following, we will look into the Darboux transformed polynomial eigenfunctions, $\boldsymbol{P}^{(1)}_{X_{\mathrm{I}}}$, and give the list of degrees of polynomial eigenfunctions

$$\deg(X_{\mathrm{I}}^{(1)}) = \{\deg(P_{\alpha,d}) \mid \alpha \in X_{\mathrm{I}}^{(1)}\}$$

in each case. GKM have already discussed these situations in their several papers for explicit examples. In contrast to their approach, our method provides a systematic way in construction of all these cases.

- Delete-state: $d = (\mathrm{I}, j)$.

$$\deg({X_{\mathrm{I}}}^{(1)}) = \{j-1, j, \ldots, 2j-2, 2j, 2j+1, \ldots\}$$

- Add-state: $d = (\mathrm{II}, j)$. We obtain several definite orthogonal polynomial systems with jump in degree as in the case which has been discussed by DEK[5].

$$\deg({X_{\mathrm{I}}}^{(1)}) = \{0, j+1, j+2, j+3, \ldots\}$$

  Here the constant eigenfunction is coming from the Darboux transformation of $\phi_d = \phi_{(\mathrm{II},j)}$, that is, $\phi_d^{(1)} \propto 1$, and the $(j+1+n)$th degree polynomial eigenfunction is given by $\phi_{(\mathrm{I},n)}^{(1)}$.

- Isospectral-state: $d = (\mathrm{III}, j)$ or (IV, j). The exceptional orthogonal polynomials introduced by GKM[6] are classified into this case.

$$\deg({X_{\mathrm{I}}}^{(1)}) = \{j, j+1, j+2, j+3, \ldots\}$$

To be a system of positive definite orthogonal polynomials on the real line, it is required to hold additional conditions, which are discussed in the following subsection.

§ 4.1. List of the orthogonal polynomials with jump(s) in degree

Here we will present the list of Darboux transformed polynomial eigenfunctions which has a positive finite weight function $w(x)$ on the real line. We consider the cases that all the moments are finite, that is, for all n, the integral

$$\int_{x_1}^{x_2} x^n w(x) dx$$

exists. We divide the cases with the leading coefficient $A(x)$ of the second order derivative term of $\mathcal{L} \in \mathbb{L}$, which is a polynomial of degree at most two:

1) $\deg A(x) = 2$ and $A(x)$ has two distinct real zeros. (Jacobi polynomials)

2) $\deg A(x) = 2$ and $A(x)$ has a double root. (Bessel polynomials)

3) $A(x) = a_0(x-a)^2$ and $B(x) = b_0(x-a)$ with nonzero constants a_0, b_0. (Power functions)

4) $\deg A(x) = 1$. (Laguerre polynomials)

5) $\deg A(x) = 0$. (Hermite polynomials)

In order to obtain the positive weight function $w^{(1)}(x)$, the following conditions are required to be fulfilled:

- There exist two distinct points a, b on the real line such that $w(x)A(x)Q(x)\big|_{x=a} = w(x)A(x)Q(x)\big|_{x=b} = 0$ for any polynomial $Q(x)$. The interval (a, b) is either finite or infinite.

- The zero of $\chi^{(1)}$ in the interval (a, b) causes the singularity of $w^{(1)}$. All zeros of $\chi^{(1)}$ lie outside the interval (a, b).

- The function $w(x)$ is non-negative in the interval (a, b): $w(x) \geq 0$ for $x \in (a, b)$.

In the following we only present the case which has a positive weight obtained from the Darboux transformation of $\mathcal{L} = A\partial^2 + B\partial$. In each case we give $\xi_\rho, w^{(1)}$ and $\boldsymbol{P}^{(1)}_{X_{\mathrm{I}}}$. Here and hereafter we may denote $(\gamma, n) \in \mathbb{G}_{A,B} \times \mathbb{Z}_{\geq 0}$ by n_γ.

Jacobi polynomials case Let $A(x) = 1 - x^2, B(x) = b - a - (a + b + 2)x$ and $J_n^{(a,b)}(x) = P_{n_{\mathrm{I}}}(x; A, B)$. For $a, b > -1$, we have

$$\int_{-1}^{1} w(x; a, b) J_n^{(a,b)}(x) J_m^{(a,b)}(x) dx = h_n^{(a,b)} \delta_{n,m},$$

where $w(x; a, b) = (1 - x)^a (1 + x)^b$ and $h_n^{(a,b)} = \frac{2^{a+b+1}\Gamma(n+a+1)\Gamma(n+b+1)}{(2n+a+b+1)n!\Gamma(n+a+b+1)}$.

- Case I. $\boldsymbol{d} = (\mathrm{I}, j)$: $\xi_{\mathrm{I}}(x) = 1$, $w^{(1)} = w(x; a + 1, b + 1)/(J_j^{(a,b)}(x))^2$. Orthogonality for $j = 0, 1, 2$ and $m, n \in \mathbb{Z}_{\geq 0}\backslash\{j\}$ under the following choice of parameters a, b:

  $$\int_{-1}^{1} \frac{w(x; a + 1, b + 1)}{(J_j^{(a,b)}(x))^2} P_{n_{\mathrm{I}},j_{\mathrm{I}}}(x; A, B) P_{m_{\mathrm{I}},j_{\mathrm{I}}}(x; A, B) dx = h_{j,n}^{(a,b)} \delta_{n,m}.$$

  When $j = 0$, $P_{n_{\mathrm{I}},0_{\mathrm{I}}}$ returns to the original Jacobi polynomial with shifted parameters: $P_{n_{\mathrm{I}},0_{\mathrm{I}}} = J_{n-1}^{(a+1,b+1)}(x)$.

  When $j = 1$ and $\{-2 < a < -1, -1 < b\}$ or $\{-1 < a, -2 < b < -1\}$ with $a+b \neq -2$, then $\{P_{n_{\mathrm{I}},1_{\mathrm{I}}}\}_{n=0,2,3,4,\ldots}$ is an orthogonal system: the ground state eigenfunction is $P_{0_{\mathrm{I}},1_{\mathrm{I}}} = 1$, and the next eigenfunction is given by the quadratic polynomial $P_{2_{\mathrm{I}},1_{\mathrm{I}}}$ which has 1 zero in the orthogonal interval $(-1, 1)$.

  When $j = 2$ and $-2 < a, b < -1$ with $a + b \neq -3$, then $\{P_{n_{\mathrm{I}},2_{\mathrm{I}}}\}_{n=0,1,3,4,\ldots}$ is an orthogonal system: the ground state eigenfunction is given by the quadratic polynomial $P_{1_{\mathrm{I}},2_{\mathrm{I}}}(x)$ which has no zero in the orthogonal interval $(-1, 1)$. The second eigenfunction is given by the linear polynomial $P_{0_{\mathrm{I}},2_{\mathrm{I}}}(x)$ which has 1 zero $\in (-1, 1)$. The third one is given by the quartic polynomial $P_{3_{\mathrm{I}},2_{\mathrm{I}}}(x)$ which has 2 zeros $\in (-1, 1)$, and so on. In this case, we have following inequality with respect to the value of spectral parameters:

  $$\lambda_{1_{\mathrm{I}}} = -(a + b + 2) > \lambda_{0_{\mathrm{I}}} = 0 > \lambda_{3_{\mathrm{I}}} = -3(a + b + 4) > \lambda_{4_{\mathrm{I}}} > \cdots,$$

where the largest value of spectral parameter is given by $\lambda_{1_{\rm I}}$ which corresponds to the ground state.

- Case II. $d_1 = (\mathrm{II}, j)$: $\xi_{\mathrm{II}}(x) = (1-x)^{-a}(1+x)^{-b}$, $w^{(1)} = w(x; a-1, b-1)/(J_j^{(-a,-b)}(x))^2$. Orthogonality relations for $n, m \in \mathbb{Z}_{\geq 0}$ and
$(a, b) \in \displaystyle\bigcup_{\substack{1\leq\mu_1+\mu_2\leq j \\ \mu_1,\mu_2\in\mathbb{N}\cup\{0\},\ \mu_1+\mu_2+j\in 2\mathbb{Z} \\ \mu_1 \text{ or } \mu_2\in 2\mathbb{Z}+1}} ((j-\mu_1, j+1-\mu_1), (j-\mu_2, j+1-\mu_2))$ or

$$\begin{cases} (a,b) \text{ or } (b,a) \in \left(\cup_{\mu=0}^{(j-1)/2}(2\mu, 2\mu+1), (j,\infty)\right) & \text{if } j \text{ is odd} \\ (a,b) \text{ or } (b,a) \in \left(\cup_{\mu=0}^{j/2-1}(2\mu, 2\mu+1) \cup (j,\infty), (j,\infty)\right) & \text{if } j \text{ is even} \end{cases}$$

with $\prod_{\mu=1}^{j}(a+b-j-\mu) \neq 0$:

$$\int_{-1}^{1} \frac{w(x; a-1, b-1)}{(J_j^{(-a,-b)}(x))^2} \phi_n(x)\phi_m(x)dx = h_{j,n}^{(a,b)}\delta_{n,m},$$

where

$$\phi_n(x) = \begin{cases} P_{j_{\mathrm{II}}, j_{\mathrm{II}}}(x; A, B) \sim 1 & n = 0 \\ P_{n-1_{\mathrm{I}}, j_{\mathrm{II}}}(x; A, B) & n = 1, 2, \ldots \end{cases}.$$

- Case III (Exceptional Jacobi polynomial, J2 family). $d = (\mathrm{III}, j)$: III encloses 1. $\xi_{\mathrm{III}}(x) = (1-x)^{-a}$, $w^{(1)} = w(x; a-1, b+1)/(J_j^{(-a,b)}(x))^2$. The positive weight $w^{(1)}$ is given in $j-1 < a < j$, $-2 < b < -1$ or $j < a$, $-1 < b$, with $\prod_{\mu=1}^{j}(a-b-j-\mu) \neq 0$.

- Case IV (Exceptional Jacobi polynomial, J1 family). $d = (\mathrm{IV}, j)$: IV encloses -1. $\xi_{\mathrm{IV}}(x) = (1+x)^{-b}$, $w^{(1)} = w(x; a+1, b-1)/(J_j^{(a,-b)}(x))^2$. The positive weight $w^{(1)}$ is given in $-2 < a < -1$, $j-1 < b < j$ or $-1 < a$, $j < b$, with $\prod_{\mu=1}^{j}(b-a-j-\mu) \neq 0$.

Laguerre polynomials case Let $A(x) = x$, $B(x) = -x + a + 1$ and $L_n^{(a)}(x) = P_n(x; A, B)$. For $a > -1$, $\int_0^\infty w(x; a)L_n^{(a)}(x)L_m^{(a)}(x)dx = h_n^{(a)}\delta_{n,m}$, where $w(x; a) = e^{-x}x^a$ and $h_n^{(a)} = \Gamma(n + a + 1)/n!$.

- Case I. $d = (\mathrm{I}, j)$: $\xi_{\mathrm{I}}(x) = 1$, $w^{(1)} = w(x; a+1)/(L_j^{(a)}(x))^2$. One can find the positive weight only at $j = 0, 1$. When $j = 0$, $P_{n_{\mathrm{I}}, 0_{\mathrm{I}}}$ is the original Laguerre polynomial with shifted parameter: $P_{n_{\mathrm{I}}, 0_{\mathrm{I}}} = L_{n-1}^{(a+1)}$. When $j = 1$, we obtain

$$\int_0^\infty \frac{w(x; a+1)}{(x-a-1)^2} P_{n_{\mathrm{I}}, 1_{\mathrm{I}}}(x)P_{m_{\mathrm{I}}, 1_{\mathrm{I}}}(x)dx = h_{1,n}^{(a)}\delta_{n,m},$$

for $m, n \in \mathbb{Z}_{\geq 0}\backslash\{1\}$ and $-2 < a < -1$.

- Case II. $d = (\mathrm{II}, j)$: $\xi_{\mathrm{II}}(x) = e^x x^{-a}$, $w^{(1)} = w(x; a-1)/(L_j^{(-a)}(-x))^2$. Orthogonality for $j, m, n \in \mathbb{Z}_{\geq 0}$ and $\begin{cases} a \in (0,1) \cup (2,3) \cup \cdots \cup (j-1, j) \text{ if } j \text{ is odd} \\ a \in (0,1) \cup (2,3) \cup \cdots \cup (j, \infty) \quad\ \text{if } j \text{ is even} \end{cases}$,

$$\int_0^\infty \frac{w(x; a-1)}{(L_j^{(-a)}(-x))^2} \phi_n(x) \phi_m(x) dx = h_{j,n}^{(a)} \delta_{n,m},$$

where

$$\phi_n(x) = \begin{cases} P_{j_{\mathrm{II}}, j_{\mathrm{II}}}(x; A, B) \sim 1 \; n = 0 \\ P_{n-1_{\mathrm{I}}, j_{\mathrm{II}}}(x; A, B) \quad n = 1, 2, \ldots \end{cases}.$$

- Case III (Exceptional Laguerre polynomial, L2 family). $d = (\mathrm{III}, j)$: III encloses 0. $\xi_{\mathrm{III}}(x) = x^{-a}$, $w^{(1)} = w(x; a-1)/(L_j^{(-a)}(x))^2$. Orthogonality for $j, m, n \in \mathbb{Z}_{\geq 0}$ and $a > j$, $\displaystyle\int_0^\infty \frac{w(x; a-1)}{(L_j^{(-a)}(x))^2} P_{n_{\mathrm{I}}, j_{\mathrm{III}}}(x) P_{m_{\mathrm{I}}, j_{\mathrm{III}}}(x) dx = h_{j,n}^{(a)} \delta_{n,m}$.

- Case IV (Exceptional Laguerre polynomial, L1 family). $d = (\mathrm{IV}, j)$: IV encloses ∞. $\xi_{\mathrm{IV}}(x) = e^x$, $w^{(1)} = w(x; a+1)/(L_j^{(a)}(-x))^2$. Orthogonality for $j, m, n \in \mathbb{Z}_{\geq 0}$ and $a > -1$, $\displaystyle\int_0^\infty \frac{w(x; a+1)}{(L_j^{(a)}(-x))^2} P_{n_{\mathrm{I}}, j_{\mathrm{IV}}}(x) P_{m_{\mathrm{I}}, j_{\mathrm{IV}}}(x) dx = h_{j,n}^{(a)} \delta_{n,m}$.

Hermite polynomials case Let $A(x) = 1, B(x) = -2x$ and $H_n(x) = P_n(x; A, B)$. For $n, m \in \mathbb{Z}_{\geq 0}$ $\int_0^\infty w(x) H_n(x) H_m(x) dx = h_n r_{n,m}$, where $w(x) = e^{-x^2}$ and $h_n = 2^n n! \sqrt{\pi}$.

- Case I. $d = (\mathrm{I}, \mathrm{j})$: $\xi_{\mathrm{I}}(x) = 1$, $w^{(1)} = w(x)/(H_j(x))^2$. Only $j = 0$ case exists: $P_{0_{\mathrm{I}}, n_{\mathrm{I}}}$ is the ordinary Hermite polynomial $H_{n-1}(x)$.

- Case II. $d = (\mathrm{II}, j)$: $\xi_{\mathrm{II}}(x) = e^{x^2}$, $w^{(1)} = w(x)/(H_j(ix))^2$. Orthogonality for $m, n \in \mathbb{Z}_{\geq 0}$ and $j = 2k \in 2\mathbb{Z}_{\geq 0}$,

$$\int_{-\infty}^\infty \frac{w(x)}{(H_{2k}(ix))^2} \phi_n(x) \phi_m(x) dx = h_{2k,n} \delta_{n,m},$$

where

$$\phi_n(x) = \begin{cases} P_{j_{\mathrm{II}}, j_{\mathrm{II}}}(x; A, B) = 1 \; n = 0 \\ P_{n-1_{\mathrm{I}}, 2k_{\mathrm{II}}}(x) \qquad\quad n = 1, 2, \ldots \end{cases}.$$

§ 5. Application to the superintegrable Hamiltonians

All known superintegrable Hamiltonians are closely related to the classical orthogonal polynomials. It is quite natural to expect that new superintegrable Hamiltonian can be constructed from the exceptional orthogonal polynomials. The superintegrable system admits more integrals of motion than degrees of freedom. The energy values can be calculated algebraically and the wave functions can be written in terms of the classical orthogonal polynomials multiplied by the ground state.

In this section, we review the superintegrable Hamiltonian, which is constructed from the exceptional Jacobi polynomials [14]. Let us consider the following Hamiltonian given in polar coordinates:

$$H_k = -\frac{1}{2}\Delta + \frac{1}{2}\omega^2 r^2 + \frac{k^2}{2r^2}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2(k\phi)} + \frac{\beta^2-\frac{1}{4}}{\cos^2(k\phi)} + \frac{4\,(1+b\cos(2k\phi))}{(b+\cos(2k\phi))^2}\right) \tag{5.1}$$

where $b = (\beta+\alpha)/(\beta-\alpha)$, which can be considered as a generalization of the Tremblay-Turbiner-Winternitz system, known as a superintegrable system,

$$H^{TTW} = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + \omega^2 r^2 + \frac{k^2}{r^2}\left(\frac{a}{\cos^2(k\theta)} + \frac{b}{\sin^2(k\theta)}\right).$$

By using the separation of variables in polar coordinates $\Psi = \Phi(\phi)R(r)$, the Schrödinger equation associated with this Hamiltonian, $H_k\Psi - E\Psi = 0$, can be rewritten into the radial part equation

$$\left(-\frac{1}{2r}\partial_r r\partial_r + \frac{1}{2}\omega r^2 + \frac{k^2A^2}{2r^2} - E\right)R(r) = 0, \tag{5.2}$$

and the angular part equation

$$\left(-\frac{1}{k^2}\partial_\phi^2 + \frac{\alpha^2-\frac{1}{4}}{\sin^2(k\phi)} + \frac{\beta^2-\frac{1}{4}}{\cos^2(k\phi)} + \frac{4\,(1+b\cos(2k\phi))}{(b+\cos(2k\phi))^2} - A^2\right)\Phi(\phi) = 0. \tag{5.3}$$

Here we note that the radial equation is exactly that of a two-dimensional oscillator and the angular part is a deformation of a Darboux-Poschl-Teller potential[13]. Both parts can be solved as follows.

Radial part Let $R(r) = Y_m^{(A_n)}(y), y = \omega r^2$. Then

$$\left(y\partial_y^2 + (1+kA_n-y)\partial_y + \frac{E}{4\omega}\right)Y_m^{(A_n)}(y) = 0, \tag{5.4}$$

whose solutions are given in terms of Laguerre polynomials

$$Y_m^{(A_n)}(y) = G_y L_m^{(A_n)}(y),$$

where $G_y = y^{A_n/2}e^{-y/2}$.

Angular part Let $\Phi(\phi) = X_n(x), x = \cos(2k\phi)$. Then we obtain

$$\left(G_x T^{\alpha,\beta} G_x^{-1} - A^2\right) X_n(x) = 0, \tag{5.5}$$

where $G_x = (1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}/(x-b)$ and

$$T^{\alpha,\beta} = 4(x^2-1)\partial_x^2 + \frac{4(\beta-\alpha)(1-bx)}{b-x}\left((x+b)\partial_x - 1\right) + (\alpha+\beta+1)^2. \tag{5.6}$$

Since the eigenfunctions of the operator (5.6) are given by the exceptional Jacobi polynomials, the solutions of (5.5) can be presented as $X_n(x) = G_x P_{n_{\mathrm{I}},1_{\mathrm{IV}}}(x; 1-x^2,)$. Thus corresponding eigenvalues are given by $A^2 \equiv A_n^2 = (2n-1+\alpha+\beta)^2\,(n \geq 1)$. To show the superintegrability of this example, we employed the method developed by Kalnins, Kress and Miller, which make use of ladder operators for the wavefunctions to construct additional integrals of motion. The key to the method is to utilize ladder operators, which transform the wave functions but leave the energy fixed. Please refer to [14] for the detail discussions.

This method of constructing Hamiltonians and their integrals of motion can be extended in a straightforward manner to other families of exceptional polynomials. Additionally, other families of Hamiltonians, say separable in Cartesian coordinates, can be obtained in a similar way from the Sturm-Liouville equations for other exceptional polynomials, e.g. extensions of the singular harmonic oscillator via exceptional Laguerre polynomials[9, 10].

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