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Schiffer functions on domains in $\mathbb{C}^n$

By

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Abstract

In the previous paper [3] we showed the variation formulas for L_1- and L_0-principal functions with the pole $\mathrm{Re}\,\{\frac{1}{z}\}$ for the moving domains $D(t)(\ni 0)$ in $\mathbb{C}_z$ with $t \in B = \{t \in \mathbb{C} \mid |t| < \rho\}$, and studied variations of the Schiffer span $s(t)$ for $(D(t), 0)$. In this paper we apply them to introduce a Schiffer function $S(\zeta)$ on a domain D in $\mathbb{C}^n$, and show that, if $D(\neq \mathbb{C}^n)$ is a polynomially convex domain in $\mathbb{C}^n$, then $\log S(\zeta)$ is plurisubharmonic on D.

§ 1. Introduction and main result

Let R be a bordered Riemann surface with boundary $\partial R = C_1 + \cdots + C_\nu$ in a larger Riemann surface $\widetilde{R}$, where C_j $(j = 1, \ldots, \nu)$ is a C^ω smooth contour in $\widetilde{R}$. Let $a \in R$ and let $V = \{|z| < r\}$ be a local coordinate of a neighborhood U of a in R such that a corresponds to 0. Among all harmonic functions u on $R \setminus \{a\}$ with singularity $\mathrm{Re}\,\frac{1}{z}$ at a normalized $\lim_{z\to 0}(u(z) - \mathrm{Re}\,\frac{1}{z}) = 0$, we uniquely have two special functions p_i $(i = 1, 0)$ with the following boundary conditions: for each C_j, p_1 satisfies $p_1(z) =$ constant c_j on C_j and $\int_{C_j} \frac{\partial p_1(z)}{\partial n_z} ds_z = 0$ (where $\frac{\partial}{\partial n_z}$ is the outer normal derivative and ds_z is the arc length element at z of C_j), while p_0 satisfies $\frac{\partial p_0(z)}{\partial n_z} = 0$ on C_j. We call $p_i(z)$ the L_i-*principal function* for (R, a) with respect to the local coordinate V, simply, for $(R, 0)$. We consider the coefficient A_{i1} $(i = 0, 1)$ of z for $P_i := p_i(z) + ip_i^*(z) = \frac{1}{z} + A_{i1}z + A_{i2}z^2 + \cdots$ on V, and call $\alpha_i := \mathrm{Re}\,A_{i1}$ the L_i-*constant* for $(R, 0)$. The *Schiffer span* (or analytic span) s for $(R, 0)$ is defined by the difference $s := \alpha_0 - \alpha_1$. It is known that s is positive (see [9] and [8, p.45–46]).

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We shall recall the geometric meaning of the Schiffer span. We assume that D is a bordered planar Riemann surface bounded by C^ω smooth contours $C_1, \ldots, C_\nu$ and $D \ni 0$. Let $\mathcal{F}(D)$ be the set of all univalent functions F on D such that

$$F(z) = \frac{1}{z} + A_1 z + A_2 z^2 + \cdots \qquad \text{at } z = 0. \tag{1.1}$$

Among all $F \in \mathcal{F}(D)$, there are special ones P_1 and P_0 with $\operatorname{Re} P_1(z) = c_j$ (constant) and $\operatorname{Im} P_0(z) = \tilde{c}_j$ (constant) on C_j ($j = 1, \ldots, \nu$), respectively, which are called the *vertical slit* and the *horizontal slit mappings* for $(D, 0)$.

Proposition 1.1 (Schiffer [9]). *Let the notation be as above. Let $\mathcal{E}_F$ be the Euclidean area of the complement of the image $F(D)$ in $\mathbb{P}$. We set $\mathcal{E}(D) = \sup\{\mathcal{E}_F \mid F \in \mathcal{F}(D)\}$ and $M(z) = \frac{P_1(z)+P_0(z)}{2}$. Then*

$$(1)\ M \in \mathcal{F}(D) \quad \textit{and} \quad (2)\ \mathcal{E}(D) = \mathcal{E}_M = \frac{\pi}{2} s.$$

We call $M(z)$ *the maximizing function of the Schiffer span* s for $(D, 0)$.

The purpose of this paper is to introduce Schiffer functions of several complex variables based on Schiffer spans of one complex variable and to give a certain significant property of Schiffer functions.

Definition 1.2. Let D be a domain in $\mathbb{C}^n$ ($n \geq 2$) and let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be an arbitrarily fixed point in D. Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ with $\|a\| = 1$ and let T_a be a mapping from $\mathbb{C}_\tau$ to $\mathbb{C}^n$ by the rule $z_j = a_j \tau + \zeta_j$ ($j = 1, \ldots, n$). We set $L_a = T_a(\mathbb{C})$, which is a complex line passing through ζ in $\mathbb{C}^n$. We identify $D \cap L_a$ with its pre-image $T_a^{-1}(D \cap L_a)$ in $\mathbb{C}_\tau$. Let $\hat{T}_a^{-1}(D \cap L_a)$ be the connected component of $T_a^{-1}(D \cap L_a)$ containing $\tau = 0$. Then we can consider the Schiffer span $s_a(\zeta)$ for $(\hat{T}_a^{-1}(D \cap L_a), 0)$. We define the *Schiffer function* $S(\zeta)$ on D in $\mathbb{C}^n$ by

$$S(\zeta) = \sup_{\|a\|=1} \{s_a(\zeta)\}, \quad \zeta \in D. \tag{1.2}$$

Thus $S(\zeta)$ is a real-valued and non-negative function on D, which is invariant under the paralell translations and the unitary transformations in $\mathbb{C}^n$. The following is our main theorem.

Theorem 1.3. *Let $D\,(\neq \mathbb{C}^n)$ be a polynomially convex domain in $\mathbb{C}^n$ and let $S(\zeta)$ be the Schiffer function on D. Then*

(i) $S(\zeta) > 0$ *on* D.

(ii) $\log S(\zeta)$ *is plurisubharmonic on* D.

(iii) *If* $\zeta_0 \in \partial D$, *then* $\lim_{D\ni\zeta\to\zeta_0} S(\zeta) = +\infty$.

§ 2. Variation formula for the Schiffer span

Here we summarize in short the concerning contents in [1], [2], [3], and [4] to prove Theorem 1.3.

Let $B = \{t \in \mathbb{C}_t : |t| < \rho\}$. We consider a variation of domains:

$$\mathcal{D} : t \in B \to D(t) \subset \mathbb{C}_z.$$

We identify the variation $\mathcal{D}$ with the subset $\cup_{t\in B}(t, D(t))$ of $B \times \mathbb{C}_z$, and write $\mathcal{D} = \cup_{t\in B}(t, D(t))$ and $\partial\mathcal{D} = \cup_{t\in B}(t, \partial D(t))$. When each $D(t)$, $t \in B$ is a domain bounded by C^ω smooth contours $C_j(t)$ $(j = 1, \ldots, \nu)$ in $\mathbb{C}_z$ and each $C_j(t)$ varies C^ω smoothly with $t \in B$, we say that $\mathcal{D}$ is a *smooth variation.* We assume that $\mathcal{D}$ contains $B \times \{0\}$. Then each $D(t)$, $t \in B$ carries the L_i-principal function $p_i(t, z)$ $(i = 0, 1)$ for $(D(t), 0)$. Namely, for $t \in B$, $p_i(t, z)$ is harmonic on $D(t) \setminus \{0\}$ and continuous on $\overline{D(t)}$ such that

$$p_i(t, z) = \mathrm{Re}\left\{\frac{1}{z} + \sum_{n=1}^{\infty} A_{in}(t) z^n\right\} \quad \text{at } z = 0 \quad (i = 0, 1),$$

and satisfy the following boundary condition (L_1) and (L_0), respectively: for $j = 1, \ldots, \nu$,

$$(L_1) \quad p_1(t, z) = c_j(t) \text{ (constant) on } C_j(t) \quad \text{and} \quad \int_{C_j(t)} \frac{\partial p_1(t, z)}{\partial n_z} ds_z = 0;$$

$$(L_0) \quad \frac{\partial p_0(t, z)}{\partial n_z} = 0 \quad \text{on } C_j(t).$$

Then we have the Schiffer span $s(t) = \mathrm{Re}\{A_{01}(t)\} - \mathrm{Re}\{A_{11}(t)\}$ for $(D(t), 0)$.

Lemma 2.1 (Variation formulas in [3]). *Let the notation be as above. Then we have*

$$\frac{\partial^2 \mathrm{Re}\{A_{11}(t)\}}{\partial t \partial \bar{t}} = -\frac{1}{\pi}\int_{\partial D(t)} k_2(t, z)\Big|\frac{\partial p_1(t, z)}{\partial z}\Big|^2 ds_z - \frac{4}{\pi}\iint_{D(t)}\Big|\frac{\partial^2 p_1(t, z)}{\partial \bar{t}\partial z}\Big|^2 dxdy,$$

$$\frac{\partial^2 \mathrm{Re}\{A_{01}(t)\}}{\partial t \partial \bar{t}} = \frac{1}{\pi}\int_{\partial D(t)} k_2(t, z)\Big|\frac{\partial p_0(t, z)}{\partial z}\Big|^2 ds_z + \frac{4}{\pi}\iint_{D(t)}\Big|\frac{\partial^2 p_0(t, z)}{\partial \bar{t}\partial z}\Big|^2 dxdy.$$

Here

$$k_2(t, z) = \Big(\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\Big|\frac{\partial\varphi}{\partial z}\Big|^2 - 2\mathrm{Re}\Big\{\frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}}\Big\} + \Big|\frac{\partial\varphi}{\partial t}\Big|^2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\Big)\Big|\frac{\partial\varphi}{\partial z}\Big|^{-3}$$

on $\partial\mathcal{D}$, *where* $\varphi(t, z)$ *is a* C^2 *defining function of* $\partial\mathcal{D}$.

Note that the *Levi curvature* $k_2(t,z)$ for $\partial\mathcal{D}$ (see [5, (1.3)] and [6, (7)]) does not depend on the choice of defining functions $\varphi(t,z)$ of $\partial\mathcal{D}$.

Remark 1. Let each $R(t)$, $t\in B$ be a bordered Riemann surface over $\mathbb{C}_z$ with a finite number of branch points $\zeta_k(t)$ $(k=1,\dots,m)$ such that $R(t)$ is planar and $\partial R(t)$ moves smoothly with $t\in B$. If $\zeta_k(t)$ is a holomorphic function on B and $\zeta_k(t)\neq\zeta_l(t)$ $(k\neq l)$ and if each $R(t)$ contains 0 with $\zeta_k(t)\neq 0$ $(k=1,\dots,m)$, then Lemma 2.1 holds by the standard use of the immersion theorem in [7].

To proceed our argument to the variation of open planar Riemann surfaces which may have infinitely many ideal boundary components, we recall the definition of the vertical slit and the horizontal slit mappings and the Schiffer span.

Let R be such an open planar Riemann surface. Let $a\in R$ and let $V:=\{|z|<r\}$ be a local coordinate of a neighborhood U of a in R such that a corresponds to $z=0$. We choose a canonical exhaustion $\{R_n\}_{n=1,2,\dots}$ of R such that $a\in R_1$, $R_n\Subset R_{n+1}$, $R=\cup_{n=1}^{\infty}R_n$, and $R\setminus R_n$ has no relatively compact connected component in R. Then each R_n $(n=1,2,\dots)$ carries the vertical slit mapping $P_1^n(z)$ such that

$$P_1^n(z)=\frac{1}{z}+A_{11}^n z+A_{12}^n z^2+\cdots \quad \text{at}\ \ z=0.$$

Since the Dirichlet integral $\iint_{R_n}[(\frac{\partial(P_1^n-P_1^m)}{\partial x})^2+(\frac{\partial(P_1^n-P_1^m)}{\partial y})^2]dxdy\to 0$ as $m\geq n\to\infty$, $P_1^n(z)$ converges a certain univalent function $P_1(z)$ on R uniformly on any compact set in R, so that

$$P_1(z)=\frac{1}{z}+A_{11}z+A_{12}z^2+\cdots \quad \text{at}\ \ z=0.$$

Further, $P_1(z)$ does not depend on the choice of the canonical exhaustion $\{R_n\}_n$ of R, so that $P_1(z)$ is uniquely determined by R, a and V. We call $P_1(z)$ the vertical slit mapping and $\operatorname{Re}A_{11}$ the L_1-constant for (R,a) with respect to the local coordinate V, simply for $(R,0)$. Similarly, we define the horizontal slit mapping $P_0(z)$ and the L_0-constant $\operatorname{Re}A_{01}$ for $(R,0)$, where

$$P_0(z)=\frac{1}{z}+A_{01}z+A_{02}z^2+\cdots \quad \text{at}\ \ z=0.$$

We call $s:=\operatorname{Re}B_1-\operatorname{Re}A_1$ the Schiffer span for $(R,0)$. If we denote by s_n (>0) the Schiffer span for $(R_n,0)$, then $s_n\searrow s$ as $n\to\infty$.

Let $\pi:\widetilde{\mathcal{R}}\to B$ be a holomorphic family such that $\widetilde{\mathcal{R}}$ is a complex 2-dimensional manifold, π is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto a disk B in $\mathbb{C}_t$, and each fiber $\widetilde{R}(t)=\pi^{-1}(t)$, $t\in B$ is irreducible and non-singular in $\widetilde{\mathcal{R}}$. We set $\widetilde{\mathcal{R}}=\cup_{t\in B}(t,\widetilde{R}(t))$. Let $\mathcal{R}=\cup_{t\in B}(t,R(t))$ be a subdomain in $\widetilde{\mathcal{R}}$ such that $\widetilde{R}(t)\Supset R(t)\neq\emptyset$ for $t\in B$,

$R(t)$ is a bordered planar Riemann surface in $\widetilde{R}(t)$, and $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t)$ $(j = 1, \dots, \nu)$, where ν is independent of $t \in B$. We regard the complex manifold $\mathcal{R}$ as a variation of Riemann surfaces $R(t)$ with parameter $t \in B$:

$$\mathcal{R} : t \in B \ \rightarrow \ R(t) \Subset \widetilde{R}(t). \tag{2.1}$$

We denote by $\Gamma(B, \mathcal{R})$ the set of all holomorphic sections of $\mathcal{R}$ over B. Let $\sigma \in \Gamma(B, \mathcal{R})$ and let $U_\sigma := B \times \{|z| < r_\sigma\}$ be a π-local coordinate of a neighborhood V_σ of σ in $\mathcal{R}$ such that σ corrresponds to $B \times \{0\}$. Let $t \in B$ be fixed. Then each $R(t)$ admits the Schiffer span $s(t)$ for $(R(t), 0)$. Under these notations we showed the following:

Lemma 2.2 (Fundamental lemma in [3]). *Let $\mathcal{R} : t \in B \rightarrow R(t)$ be a smooth variation as (2.1). Assume that $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is a two-dimensional Stein manifold. Then we have*

1. *the L_1-constant* $\mathrm{Re}\,\{A_{11}(t)\}$ *for* $(R(t), 0)$ *is superharmonic on B;*
2. *the L_0-constant* $\mathrm{Re}\,\{A_{01}(t)\}$ *for* $(R(t), 0)$ *is subharmonic on B;*
3. *the Schiffer span $s(t)$ for* $(R(t), 0)$ *is logarithmically subharmonic on B* (*i.e.,* $\log s(t)$ *is subharmonic on B*).

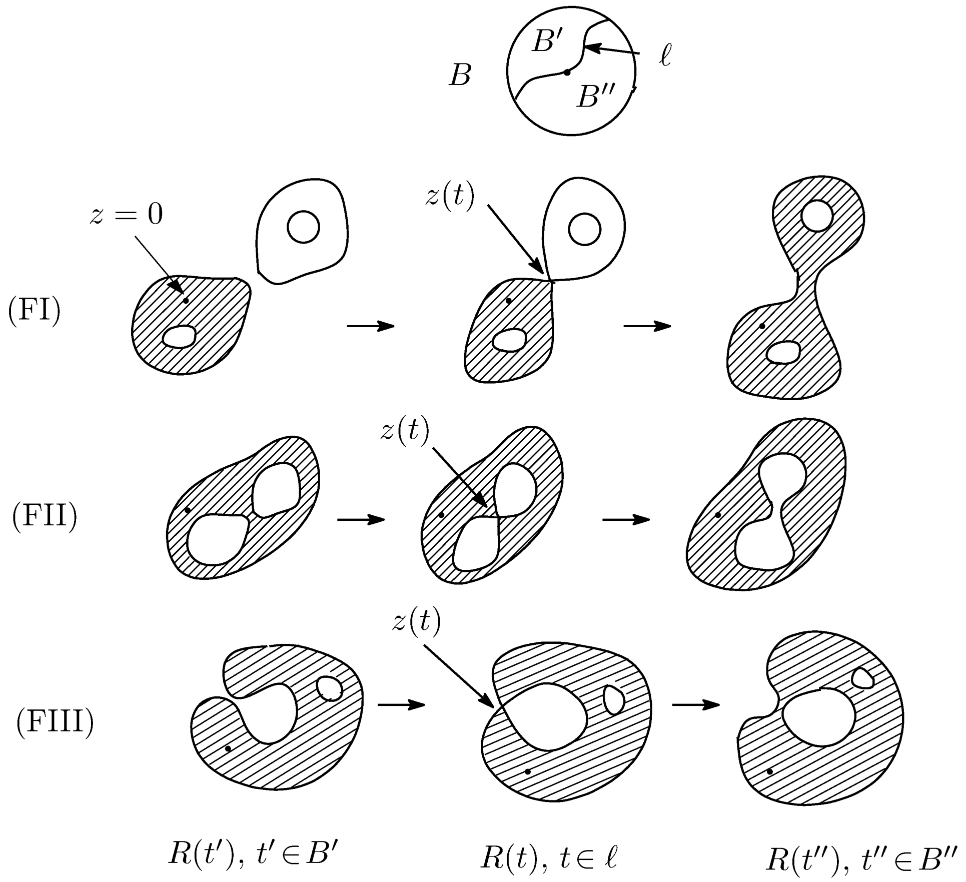
In [1] and [4], we showed the variation formulas of the same type for L_1- and L_0-principal functions with *two logarithmic poles* for the smooth variation $\mathcal{R} : t \in B \rightarrow R(t)$, respectively. Then we applied them to a simultaneous uniformization problem of Schottky coverings of compact Riemann surfaces and to the variation of *harmonic spans* (see [8, p.133] for the definition). We note that the harmonic span $h(t)$ is subharmonic on B (see [4, Theorem 4.1]), but the Schiffer span $s(t)$ is moreover logarithmically subharmonic.

In [2] and [4] we studied certain *non-smooth variation* $\mathcal{R} : t \in B \rightarrow R(t)$ under pseudoconvexity. Precisely, we consider the variation of domains $\widetilde{\mathcal{R}} : t \in B \rightarrow \widetilde{R}(t)$ such that $\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \widetilde{R}(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ with real 3-dimensional C^ω smooth boundary $\partial\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \partial\widetilde{R}(t))$ over $B \times \mathbb{C}_z$.

We assume that

(I) there exists arcs l in B such that $\widetilde{R}(t)$ for $t \in B \setminus \ell$ consists of finite number of domains with C^ω smooth boundary but $\widetilde{R}(t)$ for $t \in \ell$ consists of piecewise smooth boundary domains with one singular point $z(t)$;

(II) there exists a constant section $B \times \{0\}$ $(\in \Gamma(B, \widetilde{\mathcal{R}}))$. Let $R(t)$ be a connected domain of $\widetilde{R}(t)$ for $t \in B$ which contains 0, so that $\mathcal{R} = \cup_{t\in B}(t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}$, $\partial R(t)$, $t \in \ell$ is not smooth, and the variation $t \in B \rightarrow \partial R(t)$ is not C^ω smooth with $t \in B$ like the figures below.



We distinguished the singular point $z(t)$ from the following two cases **(c1)** and **(c2)** (to be precise, see [2, p.3–4] and [4, p.47]):
(c1) only one boundary point of $R(t)$ lies over the singular point $z(t)$ (like figure (FI));
(c2) two boundary points of $R(t)$ lie over the singular point $z(t)$ (like figures (FII) and (FIII)).

Under the above situation, we extended the variation of *harmonic spans* h to the following:

Lemma 2.3 (Theorem 1.3 in [2]). *Assume that there exists a constant section* $B \times \{1\}$ *as well as* $B \times \{0\}$. *Then we have*

1. *for the variation* $\mathcal{R}$ *of case* **(c1)** *such that each* $R(t), t \in B$ *is planar, the harmonic span* $h(t)$ *for* $(R(t), 0, 1)$ *is* C^1 *subharmonic on* B*;*
2. *for the variation* $\mathcal{R}$ *of case* **(c2)***, there exists a counterexample such that* $h(t)$ *is neither of class* C^1 *nor subharmonic on* B.

By Lemma 2.2 and the similar consideration to Lemma 2.3, we can show the next:

Lemma 2.4. *Let* $\widetilde{\mathcal{R}} = \cup_{t \in B}(t, \widetilde{R}(t))$ *be an unramified pseudoconvex domain over* $B \times \mathbb{C}_z$ *with the above assumptions* (**I**) *and* (**II**). *We assume that each* $\widetilde{R}(t)$, $t \in B$ *is* *planar. If $\mathcal{R}$ defined by $\widetilde{\mathcal{R}}$ is of case* **(c1)**, *then the Schiffer span $s(t)$ for $(R(t),0)$ is C^1 logarithmically subharmonic on B.*

We note that, if each $\widetilde{R}(t)$, $t \in B$ consists of simply connected domains, then $\mathcal{R}$ is of case (**c1**). (We use it for Theorem 1.3.)

Let $\mathcal{R} : t \in B \to R(t)$ satisfy the following conditions:

(1) $\mathcal{R}$ is a ramified pseudoconvex domain over $B \times \mathbb{C}_z$ with disjoint branch surfaces ζ_k $(k = 1, 2, \ldots)$;

(2) each $R(t)$, $t \in B$ is a planar Riemann surface sheeted over $\mathbb{C}_z$;

(3) there exist exhausted pseudoconvex domains $\mathcal{R}_n = \cup_{t\in B}(t, R_n(t))$ such that $\mathcal{R}_n \to \mathcal{R}$ as $n \to \infty$, $0 \in R_n(t) \Subset R_{n+1}(t)$ for $t \in B$, and each $\mathcal{R}_n$ $(n = 1, 2, \ldots)$ is of case **(c1)**. Precisely, for a fixed $t_0 \in B$, we find a small disk $B_0 = \{|t - t_0| < \rho\} \Subset B$ such that the restriction $\mathcal{R}_n|_{B_0}$ of $\mathcal{R}_n$ over B_0 is (fiber precerving) biholomorphic to an unramified domain $\mathcal{D}_n$ over $B_0 \times \mathbb{C}_w$ of case **(c1)**.

Let $\xi \in \Gamma(\mathcal{R}, B)$ such that $\xi \neq \zeta_k$ $(k = 1, 2, \ldots)$ and write $\xi : t \in B \to \xi(t)$. We set $E_\xi := \{t \in B : \xi(t) = \zeta_k(t) \text{ for some } k\}$, so that E_ξ is an isolated set in B. We simply put $B' = B \setminus E_\xi$. Since $\xi(t)$ is an ordiary point of $R(t)$ for $t \in B'$, we can consider the vertical slit mapping $P_1^\xi(t,z)$ and the horizontal slit mapping $P_0^\xi(t,z)$ for each $(R(t), \xi(t))$, $t \in B'$ such that

$$P_i^\xi = \frac{1}{z - \xi(t)} + \sum_{n=1}^{\infty} A_{in}^\xi(t)(z - \xi(t))^n \quad \text{at } z = \xi(t) \quad (i = 0, 1).$$

Let $s_\xi(t) = \operatorname{Re}\{A_{01}^\xi(t)\} - \operatorname{Re}\{A_{11}^\xi(t)\}$ be the Schiffer span for $(R(t), \xi(t))$.

Corollary 2.5.

(i) $\operatorname{Re}\{A_{11}^\xi(t)\}$ *for $(R(t), \xi(t))$ is superharmonic on B';*

(ii) $\operatorname{Re}\{A_{01}^\xi(t)\}$ *for $(R(t), \xi(t))$ is subharmonic on B';*

(iii) $\log s_\xi(t)$ *for $(R(t), \xi(t))$ is subharmonic on B'.*

Proof. For $t \in B'$, we set $\widetilde{R}(t) := \{z - \xi(t) \in \mathbb{C}_w : z \in R(t)\}$ and $\widetilde{\mathcal{R}} = \cup_{t\in B'}(t, \widetilde{R}(t))$. Then $\widetilde{\mathcal{R}}$ is a pseudoconvex domain over $B' \times \mathbb{C}_w$ (as well as $\mathcal{R}$ over $B \times \mathbb{C}_z$). We have the vertical slit mapping $\widetilde{P}_1(t,w)$ for $(\widetilde{R}(t), 0)$ such that $\widetilde{P}_1(t,w) = \frac{1}{w} + \sum_{n=1}^{\infty} \widetilde{A}_n(t) w^n$ at $w = 0$. By Lemma 2.2, $\operatorname{Re}\{\widetilde{A}_{11}(t)\}$ is superharmonic on B'. It is clear that $P_1^\xi(t,z) = \widetilde{P}_1(t,w)$ for $w = z - \xi(t)$, so that $\widetilde{A}_{11}(t) = A_{11}^\xi(t)$ and $\operatorname{Re}\{\widetilde{A}_{11}(t)\} = \operatorname{Re}\{A_{11}^\xi(t)\}$. Thus $\operatorname{Re}\{A_{11}^\xi(t)\}$ is superharmonic on B'. Similarly, we have the assertion (ii) and (iii). $\square$

Remark 2. By use of Proposition 4.1 in §4, we see that, *if $R(t)$ for any $t \in B$ is not of class O_{AD}, then at any $t_0 \in E_\xi$ it holds* $\lim_{B' \ni t \to t_0} S_\xi(t) = +\infty$.

§ 3. Proof of Theorem 1.3

To prove Theorem 1.3, we prepare the following lemmas.

Lemma 3.1. *Let D be a simply connected domain in $\mathbb{C}_z$ and $D \neq \mathbb{C}_z$. For $\zeta \in D$, we denote by $s(\zeta)$ the Schiffer span for (D, ζ) and by $d(\zeta)$ the distance from ζ to ∂D. Then we have*

$$s(\zeta) \geq \frac{1}{8d(\zeta)^2} > 0.$$

Proof. By the Riemann's mapping theorem, there exists a holomorphic function $w = f(z)$ on D such that $f(\zeta) = 0$, $f'(\zeta) = 1$, and $\Delta := f(D) = \{|w| < r(\zeta)\} \subset \mathbb{C}_w$. Let $\widetilde{s}$ be the Schiffer span for $(\Delta, 0)$. From Proposition 4.1 we have $s(\zeta) = |f'(\zeta)|^2 \widetilde{s} = \widetilde{s}$. By Proposition 4.2 we see that $\widetilde{s} = \frac{2}{r(\zeta)^2}$. Thus $s(\zeta) = \frac{2}{r(\zeta)^2}$. It follows from the Koebe's $\frac{1}{4}$-theorem for the univalent functions that we see $d(\zeta) \geq \frac{r(\zeta)}{4}$, which implies $s(\zeta) \geq \frac{1}{8d(\zeta)^2} > 0$. □

Lemma 3.2. *Let D be a domain in $\mathbb{C}_z$ and $D \neq \mathbb{C}_z$. For $\zeta \in D$ we denote by $s(\zeta)$ the Schiffer span for (D, ζ). Then $\log s(\zeta)$ is a subharmonic function on D.*

Proof. For a fixed $\zeta \in D$, we set $D(\zeta) = \{w = z - \zeta \in \mathbb{C}_w : z \in D\} \subset \mathbb{C}_w$ and consider the following variation of domains in $D \times \mathbb{C}_w$:

$$\mathcal{D} : \zeta \in D \to D(\zeta) \subset \mathbb{C}_w.$$

Since $\mathcal{D}$ is pseudoconvex in $D \times \mathbb{C}_w$, it follows from Lemma 2.2 that the Schiffer span $\widetilde{s}(\zeta)$ for $(D(\zeta), 0)$ is logarithmically subharmonic on D. Since $\widetilde{s}(\zeta) = s(\zeta)$ for $\zeta \in D$ by the geometrical meaning of the Schiffer span or Corollary 2.5, we prove the lemma. □

Let D be a domain in $\mathbb{C}^n$ and let $\zeta = (\zeta_1, \ldots, \zeta_n) \in D$. Let $D(\zeta_1, \ldots, \zeta_{n-1})$ denote the section of D over the complex line $z_j = \zeta_j$ $(j = 1, \ldots, n-1)$ in $\mathbb{C}^n$, i.e., $D(\zeta_1, \ldots, \zeta_{n-1}) = \{z_n \in \mathbb{C}_{z_n} \mid (\zeta_1, \ldots, \zeta_{n-1}, z_n) \in D\} \subset \mathbb{C}_{z_n}$. Let $S_n(\zeta)$ denote the Schiffer span for $(D(\zeta_1, \ldots, \zeta_{n-1}), \zeta_n)$ in $\mathbb{C}_{z_n}$. Then $S_n(\zeta)$ is a positive-valued function on D, which we call the *Schiffer function on D with respect to z_n*.

Lemma 3.3. *Let D $(\neq \mathbb{C}^n)$ be a polynomially convex domain in $\mathbb{C}^n$ and let $S_n(\zeta)$ be the Schiffer function on D with respect to z_n. Then $\log S_n(\zeta)$ is a plurisubharmonic function on D.*

Proof. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in D$ be fixed. We must show that the restriction of $S_n(\zeta)$ to any complex line L passing through ζ in D is logarithmically subharmonic. Precisely, let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ with $\|c\| = 1$ and let T_c be a mapping from $\mathbb{C}_\tau$ to $\mathbb{C}^n$ by the rule $z_j = c_j\tau + \zeta_j$ $(j = 1, \ldots, n)$. We set $L_c = T_c(\mathbb{C}_\tau)$, which is a complex line passing through ζ in $\mathbb{C}^n$, and set $\Delta = T_c^{-1}(D \cap L_c) \subset \mathbb{C}_\tau$ (which depends on ζ and c). Then we show

$$s_n(\tau) := S_n(c_1\tau + \zeta_1, \cdots, c_n\tau + \zeta_n)$$

is logarithmically subharmonic on Δ.

[Case 1] $c = (0, \ldots, 0, 1)$.
We simply write $D_0 := D(\zeta_1, \ldots, \zeta_{n-1}) \subset \mathbb{C}_{z_n}$. By the definition of the Schiffer span $s_n(\tau)$, it is equal to the Schiffer span $\widetilde{s}_n(\tau)$ for $(D_0, \zeta_n + \tau)$. Since Lemma 3.2 implies that $\log \widetilde{s}_n(\tau)$, and hence $\log s_n(\tau)$ is subharmonic for $\tau \in \Delta$.

[Case 2] $c = (c_1, \ldots, c_n) \neq (0, \ldots, 0, 1)$ *with* $\|c\| = 1$.
We set $z_j(\tau) = c_j\tau + \zeta_j$ $(j = 1, \ldots, n)$, and for $\tau \in \Delta$ we set

$$\widetilde{D}_n(\tau) := D(z_1(\tau), \ldots, z_{n-1}(\tau)) = \{z_n \in \mathbb{C}_{z_n} \mid (z_1(\tau), \ldots, z_{n-1}(\tau), z_n) \in D\} \subset \mathbb{C}_{z_n},$$

so that $\widetilde{D}_n(\tau) \ni z_n(\tau)$. Since D is a polynomially convex domain in $\mathbb{C}^n$, and hence so is $\widetilde{D}_n(\tau)$ in $\mathbb{C}_{z_n}$, namely, $\widetilde{D}_n(\tau)$ consists of simply connected domains. For simplicity, we use the same notation $\widetilde{D}_n(\tau)$ for the connected component of $\widetilde{D}_n(\tau)$ containing $z_n(\tau)$. For each $\tau \in \Delta$ we denote by $\widetilde{s}_n(\tau)$ the Schiffer span for $(\widetilde{D}_n(\tau), z_n(\tau))$. Since $\widetilde{\mathcal{D}}_n := \cup_{\tau\in\Delta}(\tau, \widetilde{D}_n(\tau))$ satisfies the conditions (1),(2),(3) in §2, it follows from (iii) in Corollary 2.5 that $\log \widetilde{s}_n(\tau)$ is subharmonic on Δ. Since $s_n(\tau) = \widetilde{s}_n(\tau)$ for $\tau \in \Delta$ by the definitions, we complete the proof of the lemma. □

Proof of Theorem 1.3 (i). Since $D \neq \mathbb{C}^n$, we find $\zeta_0 \notin D$. Let $\zeta \in D$ and L be a complex line passing through ζ and ζ_0. Since the connected component D_L of $D \cap L\,(\neq L)$ containing ζ is simply connected, the Schiffer span $s_L(\zeta)$ for (D_L, ζ) is positive by Lemma 3.1. From the definition of $S(\zeta)$, we have $S(\zeta) \geq s_L(\zeta) > 0$, which is desired. □

Proof of Theorem 1.3 (ii). Letting $z = (z_1, \ldots, z_n)$ denote the original coordinates in $\mathbb{C}^n$, we fix a unitary matrix U and form the coordinate transformation $w = (w_1, \ldots, w_n) = (z_1, \ldots, z_n)U$ of $\mathbb{C}^n$. For $\zeta \in D$ we consider the Schiffer function $S_n^U(\zeta)$ on D with respect to w_n. By Lemma 3.3, $\log S_n^U(\zeta)$ is plurisubharmonic on D. Thus,

$$\mathbf{S}(\zeta) := \sup_U \{S_n^U(\zeta)\} \quad \text{for } \zeta \in D \tag{3.1}$$

is logarithmically plurisubharmonic on D. By the definition of $S(\zeta)$ defined in (1.2), we have $\mathbf{S}(\zeta) = S(\zeta)$ for $\zeta \in D$, which proves (ii). $\square$

Proof of Theorem 1.3 (iii). Let $\zeta_0 \in \partial D$ and let $\{\zeta_n\}_n$ be a sequence of points in D such that $\zeta_n \to \zeta_0$ $(n \to \infty)$. Let d_n be the Euclidean distance between ζ_n and ∂D. For each ζ_n we find a point $\zeta_n' \in \partial D$ such that $d_n = \|\zeta_n - \zeta_n'\|$ and a complex line L_n passing through ζ_n and ζ_n'. Let $s_{L_n}(\zeta_n)$ be the Schiffer span for $(L_n \cap D, \zeta)$ in the complex plane L_n. Since $L_n \cap D$ is simply connected, Lemma 3.1 implies $s_{L_n}(\zeta_n) \geq \frac{1}{8d_n^2} \to \infty$ $(n \to \infty)$. Since $S(\zeta_n) \geq s_{L_n}(\zeta_n)$, we have (iii). $\square$

§ 4. Example of Theorem 1.3

Proposition 4.1. *Let D be a domain in $\mathbb{C}_z$ and $\zeta \in D$. Let $s(\zeta)$ be the Schiffer span for (D, ζ). For every univalent function $f : D \to \widetilde{D}$ it holds that*

$$\widetilde{s}(f(\zeta)) = |f'(\zeta)|^{-2} s(\zeta).$$

where $\widetilde{s}(f(\zeta))$ is the Schiffer span for $(\widetilde{D}, f(\zeta))$.

Proof. Let $\mathcal{F}(D)$ be the set of all univalent functions on D with (1.1), and let $M(\zeta)$ be the maximizing function of the Schiffer span $s(\zeta)$ for (D, ζ). Let $f : D \to \widetilde{D}$ be a univalent function. Putting $\xi = f(\zeta)$, we similarly have $\widetilde{\mathcal{F}}(\widetilde{D})$, $\widetilde{s}(\xi)$, $\widetilde{M}(w) \in \widetilde{\mathcal{F}}(\widetilde{D})$. For the maximizing function $\widetilde{M}(w)$ of the Schiffer span $\widetilde{s}(\xi)$ for $(\widetilde{D}, \xi)$, if we set $\hat{M}(z) := \widetilde{M}(f(z))$, then $\hat{M}$ is the univalent function on D, the area of $\mathbb{P} \setminus \hat{M}(D)$ is equal to the area of $\mathbb{P} \setminus \widetilde{M}(\widetilde{D})$, namely $\frac{\pi}{2}\widetilde{s}(\xi)$, and

$$\begin{aligned}
\hat{M}(z) &= (f(z) - \xi)^{-1} + \sum_{n=1}^{\infty} A_n (f(z) - \xi)^n \\
&= \{f'(\zeta)(z - \zeta)(1 + o(z - \zeta))\}^{-1} + o(z - \zeta) \\
&= \{f'(\zeta)\}^{-1}\{(z - \zeta)^{-1} + 0 + o(z - \zeta)\}
\end{aligned}$$

at $z = \zeta$. Hence $f'(\zeta)\hat{M}(z) \in \mathcal{F}(D)$. The area of $\mathbb{P} \setminus (f'(\zeta)\hat{M}(D))$ is equal to $|f'(\zeta)|^2 \widetilde{s}(\xi)$. Therefore $|f'(\zeta)|^2 \widetilde{s}(\xi) \leq s(\zeta)$. Similarly, we consider the mapping $z = f^{-1}(w) : \widetilde{D} \to D$ and then we have $|(f^{-1})'(\xi)|^2 s(\zeta) \leq \widetilde{s}(\xi)$, namely $|f'(\zeta)|^{-2} s(\zeta) \leq \widetilde{s}(\xi)$. Thus we see that $\widetilde{s}(f(\zeta)) = |f'(\zeta)|^{-2} s(\zeta)$. $\square$

Proposition 4.2. *Let $D = \{|z| < r\} \subset \mathbb{C}$ and let $\zeta \in D$. Then the Schiffer span $s(\zeta)$ for (D, ζ) is*

$$s(\zeta) = \frac{2}{r^2 \{1 - \left(\frac{|\zeta|}{r}\right)^2\}^2}.$$

Proof. In [3] we calculate the Schiffer span $\widetilde{s}$ for $(D,0)$ is $\frac{2}{r^2}$. Combing with Proposition 4.1, we have the assertion. □

We give an example of Theorem 1.3.

Example 4.3. Let $0 < c_j < +\infty$ $(j = 1,2)$ be fixed and consider the ellipsoid $D = \{(z_1, z_2) \in \mathbb{C}^2 \mid |\frac{z_1}{c_1}|^2 + |\frac{z_2}{c_2}|^2 < 1\}$ (which is a polynomially convex domain). Let $\zeta = (\zeta_1, \zeta_2) \in D$ be fixed. For $a = (a_1, a_2) \in \mathbb{C}^2$ with $\|a\| = 1$, let T_a be a mapping from $\mathbb{C}_\tau$ to $\mathbb{C}^2$ by the rule $z_j = a_j\tau + \zeta_j$ $(j = 1,2)$. We identify $D \cap L_a$ in $\mathbb{C}^2$ with its pre-image $T_a^{-1}(D \cap L_a) =: \Delta_a$ containing $\tau = 0$ in $\mathbb{C}_\tau$. We consider the Schiffer span $s_a(\zeta)$ for $(\Delta_a, 0)$, and the Schiffer function $S(\zeta) = \sup_{\|a\|=1}\{s_a(\zeta)\}$ on D. We put $A_j := a_j/c_j$, $Z_j := \zeta_j/c_j$ $(j = 1,2)$, and have

$$\begin{aligned}\Delta_a &= \{|\frac{a_1\tau + \zeta_1}{c_1}|^2 + |\frac{a_2\tau + \zeta_2}{c_2}|^2 < 1\} \\ &= \{|\tau + \frac{\overline{A_1}Z_1 + \overline{A_2}Z_2}{|A_1|^2 + |A_2|^2}|^2 < \frac{|A_1|^2 + |A_2|^2 - |A_1Z_2 - A_2Z_1|^2}{(|A_1|^2 + |A_2|^2)^2}\},\end{aligned}$$

Proposition 4.2 implies

$$s_a(\zeta) = \frac{2(|A_1|^2 + |A_2|^2 - |A_1Z_2 - A_2Z_1|^2)}{(1 - |Z_1|^2 - |Z_2|^2)^2}. \tag{4.1}$$

We write $\zeta_j = |\zeta_j|e^{i\theta_j}$ $(j = 1,2)$, and move the direction $a_1 = (\sin\Theta)e^{i\varphi_1}$, $a_2 = (\cos\Theta)e^{i\varphi_2}$ $(0 \le \Theta \le \frac{\pi}{2}, \varphi_1, \varphi_2 : \text{free})$ under $|a_1|^2 + |a_2|^2 = 1$. The numerator in the right-hand side of (4.1) is written into

$$\begin{aligned}&\frac{2}{c_1^2c_2^2}\big(|a_1|^2c_2^2 + |a_2|^2c_1^2 - |a_1\zeta_2 - a_2\zeta_1|^2\big) \\ =&\frac{2}{c_1^2c_2^2}\Big[(c_1^2 - |\zeta_1|^2)\frac{1+\cos 2\Theta}{2} + (c_2^2 - |\zeta_2|^2)\frac{1-\cos 2\Theta}{2} \\ &\qquad + |\zeta_1\zeta_2|(\sin 2\Theta)\mathrm{Re}\,\{e^{i(\varphi_1-\varphi_2+\theta_2-\theta_1)}\}\Big] \\ \le&\frac{1}{c_1^2c_2^2}\Big[c_1^2 + c_2^2 - (|\zeta_1|^2 + |\zeta_2|^2) + \sqrt{\{c_1^2 - c_2^2 - (|\zeta_1|^2 - |\zeta_2|^2)\}^2 + 4|\zeta_1\zeta_2|^2}\Big]\end{aligned}$$

When $\varphi_j = \theta_j$ $(j = 1,2)$, namely, $\arg a_j = \arg\zeta_j$ and $(|a_1|, |a_2|) = (\sin\Theta_\zeta, \cos\Theta_\zeta)$, where $\Theta_\zeta = \frac{1}{2}\tan^{-1}\frac{2|\zeta_1\zeta_2|}{c_1^2 - c_2^2 - (|\zeta_1|^2 - |\zeta_2|^2)}$ with $0 \le \Theta_\zeta \le \pi/2$, the above $\le$ reduces to $=$, so that we have, for $a_1 := (\sin\Theta_\zeta)e^{i\arg\zeta_1}$, $a_2 := (\cos\Theta_\zeta)e^{i\arg\zeta_2}$,

$$S(\zeta) = s_a(\zeta) = \frac{c_1^2 + c_2^2 - (|\zeta_1|^2 + |\zeta_2|^2) + \sqrt{\{c_1^2 - c_2^2 - (|\zeta_1|^2 - |\zeta_2|^2)\}^2 + 4|\zeta_1\zeta_2|^2}}{c_1^2c_2^2(1 - |\zeta_1/c_1|^2 - |\zeta_2/c_2|^2)^2} > 0,$$

and $\lim_{\zeta\to\partial D} S(\zeta) = \infty$. Let us prove that $S(\zeta)$ is logarithmically plurisubharmonic on D by the same idea for (3.1) as follows: Since $S(\zeta) = \sup_{\|a\|=1}\{s_a(\zeta)\}$, it suffrices to show that, for arbitrarily fixed $\|a\| = 1$, the function $s_a(\zeta)$ for ζ is logarithmic plurisubharmonic on D. Such $s_a(\zeta)$ is of the form (4.1). Precisely, it is enough to show: *Let* $\widetilde{D} = \{|Z_1|^2 + |Z_2|^2 < 1)\}$ *in* $\mathbb{C}^2$ *and let* $A = (A_1, A_2) \in \mathbb{C}^2 \setminus \{O\}$ *be fixed. Then (4.1) is logarithmically plurisubharmonic for* $Z \in \widetilde{D}$.

We set $A = (A_1, A_2)$, $\tilde{A} = A/\|A\| = (\tilde{A}_1, \tilde{A}_2)$, so that $\|\tilde{A}\| = 1$ and

$$s_a(\zeta) = 2\,\|A\|^2\ \frac{1 - |\tilde{A}_1 Z_2 - \tilde{A}_2 Z_1|^2}{(1 - \|Z\|^2)^2}$$

It is easy to show that the right hand side is logarithmically plurisubharmonic for Z on $\widetilde{D}$, and hence so is $s_a(\zeta)$ on D. We thus prove the assertion.

Remark 3. In case $c_1 = c_2 = 1$, i.e., $D = \{|z_1|^2 + |z_2|^2 < 1\}$, we have

$$S(\zeta) = \frac{2}{\{1 - (|\zeta_1|^2 + |\zeta_2|^2)\}} \quad (> 0 \ \text{ on } D),$$

which is equal to $s_a(\zeta)$ for $a = \zeta/\|\zeta\|$. Thus, $\log S(\zeta)$ certainly is a strictly plurisubharmonic exhaustion function on D.

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