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Mean continuity for potentials of functions in Musielak-Orlicz spaces

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Abstract

Our aim in this paper is to show mean continuity in a certain strong sense at points except in a small set for potentials of functions in Musielak-Orlicz spaces.

§ 1. Introduction

For the Riesz potential

$$I_\alpha f(x) := \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy,$$

where $0 < \alpha < N$ and $f \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ $(1 \le p < \infty)$ is assumed to satisfy

$$\int_{\mathbf{R}^N} (1+|x|)^{\alpha-N} |f(x)|\, dx < \infty,$$

the following mean continuity is known (see, e.g., [1], [10] and [14]):

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If $p > 1$, $\alpha p < N$ and $1/p^\sharp = 1/p - \alpha/N$, then

$$\lim_{r \to 0+} \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} |I_\alpha f(x) - I_\alpha f(x_0)|^{p^\sharp} \, dx = 0$$

for $x_0 \in \mathbf{R}^N \setminus E$ with a set E of (α, p)-capacity zero. ($|B(x_0, r)|$ denotes the Lebesgue measure of $B(x_0, r)$.)

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. Mean continuity of Riesz potentials of functions in variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [3] (also, cf. [2] and [4] for mean continuity of functions in variable exponent Sobolev spaces). For Riesz potentials on the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, see [11]. These spaces are special cases of so-called Musielak-Orlicz spaces ([12]).

Our aim in this paper is to show mean continuity in a certain strong sense at points except in a small set for potentials of functions in Musielak-Orlicz spaces as an extension of the above results. Recently, a capacity defined by potentials of functions in Musielak-Orlicz spaces was introduced in [5]. We discuss the size of the exceptional sets using such capacity.

§ 2. Preliminaries

In this paper, we consider a function

$$\Phi(x, t) := t\phi(x, t) : \mathbf{R}^N \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions $(\Phi 1)$ – $(\Phi 4)$:

$(\Phi 1)$ $\phi(\,\cdot\,, t)$ is measurable on $\mathbf{R}^N$ for each $t \geq 0$ and $\phi(x, \,\cdot\,)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;

$(\Phi 2)$ there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x, 1) \leq A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

$(\Phi 3)$ $\phi(x, \cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geq 1$ such that

$$\phi(x, t) \leq A_2 \phi(x, s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 \leq t < s;$$

$(\Phi 4)$ there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x,t) \le \sup_{x \in \mathbf{R}^N} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in \mathbf{R}^N$, then (Φ3) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x,t) := \sup_{0 \le s \le t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) := \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in \mathbf{R}^N$ and $t \ge 0$. Then $\overline{\Phi}(x,t)$ satisfies (Φ1) – (Φ4). Furthermore, $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t) \tag{2.1}$$

for all $x \in \mathbf{R}^N$ and $t \ge 0$.

By (Φ3), we see that

$$\Phi(x,at) \begin{cases} \le A_2 a \Phi(x,t) & \text{if } 0 \le a \le 1 \\ \ge A_2^{-1} a \Phi(x,t) & \text{if } a \ge 1. \end{cases} \tag{2.2}$$

Example 2.1. Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on $\mathbf{R}^N$ such that

(P1) $1 \le p^- := \inf_{x \in \mathbf{R}^N} p(x) \le \sup_{x \in \mathbf{R}^N} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q^- := \inf_{x \in \mathbf{R}^N} q(x) \le \sup_{x \in \mathbf{R}^N} q(x) =: q^+ < \infty$.

Then, $\Phi_{p(\cdot),q(\cdot),a}(x,t) = t^{p(x)}(\log(a+t))^{q(x)}$ $(a \ge e)$ satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p^- > 1$ or $q^- \ge 0$. As a matter of fact, it satisfies (Φ3) if and only if $q(x) \ge 0$ at points x where $p(x) = 1$ and

$$\sup_{x:p(x)>1,q(x)<0} q(x)\log(p(x)-1) < \infty$$

(see section 6: Appendix).

Given $\Phi(x,t)$ as above and an open set G in $\mathbf{R}^N$, the associated Musielak-Orlicz space on G is defined by

$$L^\Phi(G) = \left\{ f \in L^1_{\mathrm{loc}}(G)\,;\, \int_G \Phi\big(y,|f(y)|\big)\,dy < \infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(G)} = \inf\left\{\lambda > 0\,;\ \int_G \overline{\Phi}\big(y, |f(y)|/\lambda\big)\,dy \le 1\right\}$$

(cf. [12]).

Lemma 2.2.

$$(2A_3)^{-1}\int_G \Phi(x,|f(x)|)\,dx \le \|f\|_{L^\Phi(G)} \le 2\left(A_2\int_G \Phi(x,|f(x)|)\,dx\right)^{\sigma}$$

whenever $\|f\|_{L^\Phi(G)} \le 1$, *where* $\sigma = \log 2/\log(2A_3) > 0$.

Proof. Let $f \in L^\Phi(G)$ and suppose $\lambda := \|f\|_{L^\Phi(G)} \le 1$. Then by (2.1),

$$\int_G \Phi(x,|f(x)|)\,dx \le 2A_3\int_G \overline{\Phi}(x,|f(x)|)\,dx \le 2A_3\lambda\int_G \overline{\Phi}(x,|f(x)|/\lambda)\,dx \le 2A_3\lambda.$$

On the other hand, suppose $\lambda^* := \int_G \Phi(x,|f(x)|)\,dx \le A_2^{-1}$. Choose $k \in \mathbf{N}$ such that $(2A_3)^{-k} < A_2\lambda^* \le (2A_3)^{-k+1}$. Then, by (2.1) and ($\Phi$4)

$$\int_G \overline{\Phi}(x, 2^{k-1}|f(x)|)\,dx \le A_2\int_G \Phi(x, 2^{k-1}|f(x)|)\,dx \le A_2(2A_3)^{k-1}\lambda^* \le 1.$$

Hence $\|f\|_{L^\Phi(G)} \le 2^{1-k}$. Since $2^{-k} < (A_2\lambda^*)^\sigma$,

$$\|f\|_{L^\Phi(G)} \le 2\left(A_2\int_G \Phi(x,|f(x)|)\,dx\right)^{\sigma}.$$

□

We shall also consider the following conditions:

(Φ5) for every $\gamma > 0$, there exists a constant $B_\gamma \ge 1$ such that

$$\phi(x,t) \le B_\gamma\phi(y,t)$$

whenever $|x-y| \le \gamma t^{-1/N}$ and $t \ge 1$;

($\Phi 3^*$) $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon_0 > 0$, namely there exists a constant $A_{2,\varepsilon_0} \ge 1$ such that

$$t^{-\varepsilon_0}\phi(x,t) \le A_{2,\varepsilon_0}s^{-\varepsilon_0}\phi(x,s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 < t < s.$$

Example 2.3. Let $\Phi_{p(\cdot),q(\cdot),a}(x,t)$ be as in Example 2.1. It satisfies ($\Phi5$) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)| \le \frac{C_p}{\log(1/|x-y|)} \quad \text{for } |x-y| \le \frac{1}{2}$$

with a constant $C_p \ge 0$,

and

(Q2) $q(\cdot)$ is log-log-Hölder continuous, namely

$$|q(x)-q(y)| \le \frac{C_q}{\log(\log(1/|x-y|))} \quad \text{for } |x-y| \le e^{-2}$$

with a constant $C_q \ge 0$.

It satisfies ($\Phi3^*$) if $p^- > 1$ with $0 < \varepsilon_0 < p^- - 1$.

In this paper, as a kernel function on $\mathbf{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a function $k(r) : (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(k1) $k(r)$ is non-increasing and lower semicontinuous on $(0,\infty)$;

(k2) $\int_0^1 k(r)r^{N-1}\,dr < \infty$;

(k3) there exists a constant $K_1 \ge 1$ such that $k(r) \le K_1 k(r+1)$ for all $r \ge 1$.

By (k2), $k(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$. We set $k(0) = \lim_{r\to 0+} k(r)$.

Let

$$\bar{k}(r) := \frac{N}{r^N}\int_0^r k(\rho)\rho^{N-1}\,d\rho$$

for $r > 0$. Then $k(r) \le \bar{k}(r)$, $\bar{k}(r)$ is non-increasing and

$$\lim_{r\to 0+} r^N \bar{k}(r) = 0. \tag{2.3}$$

For $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = |x|^{\alpha-N}$ and the Bessel kernel g_α of order α are typical examples of $k(x)$ satisfying above conditions.

We define the k-potential of a locally integrable function f on $\mathbf{R}^N$ by

$$k * f(x) = \int_{\mathbf{R}^N} k(x-y)f(y)\,dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^N} k(1+|y|)|f(y)|\,dy < \infty, \tag{2.4}$$

which is equivalent to the condition that $k * |f| \not\equiv \infty$ by the conditions (k2) and (k3) (see [10, Theorem 1.1, Chapter 2]). Note that $k * f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ under this assumption.

Set

$$\Gamma(x, s) := s^{-1}\bar{k}(s^{-1/N})\Phi^{-1}(x, s) \quad (x \in \mathbf{R}^N,\ s > 0),$$

where $\Phi^{-1}(x, s) = \sup\{t > 0; \Phi(x, t) < s\}$.

Here we note:

$$\Gamma(x, \Phi(x, t)) \approx t\Phi(x, t)^{-1}\bar{k}\big(\Phi(x, t)^{-1/N}\big), \tag{2.5}$$

since $\Phi^{-1}\big(x, \Phi(x, t)\big) \approx t$ (cf. [7, Lemma 5.2 (4)]). (For two functions f and g, $f \approx g$ means that there is a constant $C \geq 1$ such that $C^{-1}g \leq f \leq Cg$.)

We shall consider the following condition (Φk):

(Φk) $s \mapsto s^{-\varepsilon_1}\Gamma(x, s)$ is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon_1 > 0$, namely there exists a constant $A_\Gamma \geq 1$ such that

$$s_1^{-\varepsilon_1}\Gamma(x, s_1) \leq A_\Gamma s_2^{-\varepsilon_1}\Gamma(x, s_2)$$

for all $x \in \mathbf{R}^N$ whenever $0 < s_1 < s_2$.

Example 2.4. If k is the Riesz kernel I_α, then $\Phi_{p(\cdot),q(\cdot),a}(x, t)$ in Example 2.1 satisfies (Φk) if $\alpha p^+ < N$.

We consider a function $\Psi(x, t) : \mathbf{R}^N \times [0, \infty) \to [0, \infty)$ satisfying the following conditions:

($\Psi 1$) $\Psi(\cdot, t)$ is measurable on $\mathbf{R}^N$ for each $t \geq 0$ and $\Psi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;

($\Psi 2$) there is a constant $A_4 \geq 1$ such that

$$\Psi(x, at) \leq A_4 a\Psi(x, t)$$

for all $x \in \mathbf{R}^N$, $t > 0$ and $0 \leq a \leq 1$;

($\Psi\Phi k$) there exists a constant $A_5 \geq 1$ such that

$$\Psi\big(x, \Gamma(x, s)\big) \leq A_5 s$$

for all $x \in \mathbf{R}^N$ and $s > 0$.

Note: ($\Psi 2$) implies that $\Psi(x, \cdot)$ is uniformly almost increasing on $[0, \infty)$; if we assume (Φk), then $\Gamma(x, t) \to \infty$ uniformly as $t \to \infty$, and hence ($\Psi\Phi k$) implies that $\Psi(\cdot, t)$ is bounded on $\mathbf{R}^N$ for every $t > 0$.

Example 2.5. For $\Phi_{p(\cdot),q(\cdot),a}(x,t)$ in Example 2.1 and the Riesz kernel I_α $(0 < \alpha < N)$, if $\alpha p^+ < N$, then

$$\Gamma(x,s) \approx s^{1/p^\sharp(x)}[\log(e+s)]^{-q(x)/p(x)}$$

with

$$\frac{1}{p^\sharp(x)} := \frac{1}{p(x)} - \frac{\alpha}{N},$$

so that we may take

$$\Psi(x,t) = t^{p^\sharp(x)}(\log(e+t))^{p^\sharp(x)q(x)/p(x)}.$$

We know the following result (see [6, Corollary 6.3]; also cf. [7, Corollary 6.5]; note that condition $(\Psi\Phi k)$ given there is essentially the same as the above one, in view of (2.5)).

Lemma 2.6. *Suppose $\Phi(x,t)$ satisfies $(\Phi3^*)$, $(\Phi5)$ and (Φk); $\Psi(x,t)$ satisfies $(\Psi1)$, $(\Psi2)$ and $(\Psi\Phi k)$. Then there exists a constant $C^* > 0$, such that*

$$\int_{B(0,1)} \Psi(x, k * f(x)/C^*)\, dx \le 1$$

for all $f \ge 0$ satisfying $\|f\|_{L^\Phi(B(0,1))} \le 1$.

§ 3. Mean continuity

In this section, we prove our main theorem, which gives an extension of Meyers [9], Harjulehto-Hästö [4] and the authors [3, Theorem 4.5], [11, Theorem 3.4].

For a measurable function u on $\mathbf{R}^N$, we define the integral mean over a measurable set $E \subset \mathbf{R}^N$ of positive measure by

$$\fint_E u(x)\, dx := \frac{1}{|E|}\int_E u(x)\, dx.$$

Theorem 3.1. *Let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying (2.4) and set*

$$E_1 := \{x \in \mathbf{R}^N : k * f(x) = \infty\},$$
$$E_2 := \Big\{x \in \mathbf{R}^N : \limsup_{r\to 0+} \fint_{B(x,r)} \Phi\left(z, r^N \bar{k}(r) f(z)\right) dz > 0\Big\}.$$

(1) *Suppose $k(r)$ satisfies*

(k4) *there is a constant $K_2 > 0$ such that*

$$k(r/2) \le K_2 k(r) \quad \text{for all } 0 < r \le 1.$$

Then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} |k * f(x) - k * f(x_0)|\, dx = 0 \tag{3.1}$$

for all $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$.

(2) *Besides the assumptions on $k(r)$, $\Phi(x,t)$ and $\Psi(x,t)$ given in Lemma 2.6, assume further that $k(r)$ satisfies*

(k5) *there is a constant $K_3 > 0$ such that*

$$k(rs) \le K_3 \bar{k}(r) k(s) \quad \text{for all } 0 < r \le 1,\ 0 < s \le 1.$$

Then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, |k * f(x) - k * f(x_0)|)\, dx = 0 \tag{3.2}$$

for all $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$.

Note that (k5) implies (k4) with $K_2 = K_3 \bar{k}(1/2)$. The Riesz kernel I_α $(0 < \alpha < N)$ satisfies (k5).

Lemma 3.2. *Let $x_0 \in \mathbf{R}^N$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Phi\left(z, r^N \bar{k}(r) f(z)\right) dz = 0.$$

Then

$$\lim_{r\to 0+} \bar{k}(r) \int_{B(x_0,r)} f(y)\, dy = 0.$$

Proof. For $\varepsilon > 0$ $(\varepsilon \le 1)$, we see from $(\Phi3)$, $(\Phi2)$ and $(\Phi4)$ that

$$\begin{aligned}
\int_{B(x_0,r)} f(y)\, dy &\le \int_{B(x_0,r)} \varepsilon r^{-N} \bar{k}(r)^{-1}\, dy + A_2 \int_{B(x_0,r)} f(y) \frac{\phi(y, \varepsilon^{-1} r^N \bar{k}(r) f(y))}{\phi(y,1)}\, dy \\
&\le \nu_N \varepsilon \bar{k}(r)^{-1} + A_1 A_2 \varepsilon r^{-N} \bar{k}(r)^{-1} \int_{B(x_0,r)} \Phi(y, \varepsilon^{-1} r^N \bar{k}(r) f(y))\, dy \\
&\le \nu_N \varepsilon \bar{k}(r)^{-1} + A(\varepsilon) r^{-N} \bar{k}(r)^{-1} \int_{B(x_0,r)} \Phi(y, r^N \bar{k}(r) f(y))\, dy,
\end{aligned}$$

where $\nu_N = |B(0,1)|$, so that

$$\limsup_{r\to 0+} \bar{k}(r) \int_{B(x_0,r)} f(y)\,dy \le \nu_N \varepsilon.$$

Hence, we have the required result. □

Proof of Theorem 3.1. Let $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$ and write

$$\begin{aligned} k * f(x) - k * f(x_0) &= \int_{B(x_0,2|x-x_0|)} k(x-y)f(y)\,dy \\ &\quad + \int_{\mathbf{R}^N \setminus B(x_0,2|x-x_0|)} k(x-y)f(y)\,dy - k * f(x_0) \\ &= I_1(x) + I_2(x). \end{aligned}$$

(1) If $y \in \mathbf{R}^N \setminus B(x_0, 2|x-x_0|)$, then $|x_0 - y| \le 2|x-y|$. Hence, if $|x_0 - y| \le 1$, then $k(x-y) \le k(|x_0-y|/2) \le K_2 k(x_0-y)$ by (k1) and (k4); if $1 < |x_0 - y| \le 2$, then $|x-y| \ge |x_0-y|/2 > 1/2$, so that $k(x-y) \le k(1/2) \le k(1/2)k(2)^{-1}k(x_0-y)$ by (k1); if $|x_0 - y| > 2$ and $|x - x_0| \le 1$, then $k(x-y) \le k(|x_0-y|-1) \le K_1 k(x_0-y)$ by (k1) and (k3). Thus,

$$k(x-y) \le K' k(x_0-y) \tag{3.3}$$

with $K' = \max\{K_2,\, k(1/2)/k(2),\, K_1\}$, whenever $y \in \mathbf{R}^N \setminus B(x_0, 2|x-x_0|)$ and $|x-x_0| \le 1$.

By (k1), $k(r)$ is continuous a.e. on $(0,\infty)$, so that $k(x-y) \to k(x_0-y)$ as $x \to x_0$ for almost every $y \in \mathbf{R}^N$. Since $k * f(x_0) < \infty$, noting (3.3) we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x\to x_0} I_2(x) = 0. \tag{3.4}$$

Hence

$$\lim_{r\to 0+} \fint_{B(x_0,r)} |I_2(x)|\,dx = 0. \tag{3.5}$$

For I_1, note that

$$0 \le I_1(x) \le \int_{B(x_0,r)} k(x-y)f(y)\,dy = k * f_r(x)$$

for $x \in B(x_0, r/2)$, where $f_r := f\chi_{B(x_0,r)}$ and χ_E is the characteristic function of E. Hence,

$$\begin{aligned} \fint_{B(x_0,r/2)} I_1(x)\,dx &\le \fint_{B(x_0,r/2)} k * f_r(x)\,dx \\ &= \int_{B(x_0,r)} \left(\fint_{B(x_0,r/2)} k(x-y)\,dx \right) f(y)\,dy. \end{aligned}$$

Since

$$\fint_{B(x_0,r/2)} k(x-y)\,dx \le \fint_{B(x_0,r/2)} k(x_0-x)\,dx = \bar{k}(r/2) \le 2^N \bar{k}(r),$$

we have

$$\lim_{r\to 0+} \fint_{B(x_0,r)} I_1(x)\,dx = 0$$

by Lemma 3.2. Thus, together with (3.5), we obtain (3.1).

(2) Since (k5) implies (k4), (3.4) holds under our assumptions. Hence

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, 2|I_2(x)|)\,dx = 0 \tag{3.6}$$

by (Ψ2) and the boundedness of $\Psi(x,1)$.

We will show that

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, 2k * f_r(x))\,dx = 0. \tag{3.7}$$

Let $0 < r \le 1$, $x = x_0 + rz$ with $|z| < 1$. For $y \in B(x_0, r)$, write $y = x_0 + rw$ with $|w| < 1$. If $|z-w| \le 1$, then by (k5) $k(x-y) \le K_3\bar{k}(r)k(z-w)$. If $1 < |z-w| < 2$, then $r < |x-y| < 2r$, so that by (k1), (k5) and (k3)

$$k(x-y) \le k(r) \le K_3\bar{k}(r)k(1) \le K_3K_1\bar{k}(r)k(2) \le K_1K_3\bar{k}(r)k(z-w).$$

Hence

$$k * f_r(x) = \int_{B(x_0,r)} k(x-y)f(y)\,dy \le K_1K_3 \int_{B(0,1)} r^N\bar{k}(r)k(z-w)f(x_0+rw)\,dw$$

if $0 < r \le 1$. Thus, to prove (3.7) it is enough to show

$$\lim_{r\to 0+} \int_{B(0,1)} \Psi(x_0+rz, 2k * g_r(z))\,dz = 0, \tag{3.8}$$

where $g_r(w) = r^N\bar{k}(r)f_r(x_0+rw)$.

Let

$$\Phi_{x_0,r}(x,t) = \Phi(x_0+rx, t) \qquad \text{and} \qquad \Psi_{x_0,r}(x,t) = \Psi(x_0+rx,t).$$

Then, $\Phi_{x_0,r}$ satisfies (Φ1), (Φ2), ($\Phi3^*$), (Φ4) and (Φk) with the same constants A_1, ε_0, A_{2,ε_0}, A_3, ε_1 and A_Γ. Further, it satisfies (Φ5) with the same B_γ whenever $0 < r \le 1$.

As to $\Psi_{x_0,r}$, it satisfies (Ψ1) and (Ψ2) with the same constant A_4. The pair $(\Phi_{x_0,r},\ \Psi_{x_0,r})$ satisfies ($\Psi\Phi k$) with the same constant A_5.

Therefore, by Lemma 2.6, there exists a constant $C^* > 0$ independent of x_0 and $0 < r \le 1$ such that

$$\int_{B(0,1)} \Psi_{x_0,r}\left(z, \frac{k * g_r(z)}{C^*\lambda_r}\right)\,dz \le 1,$$

or

$$\int_{B(0,1)} \Psi\left(x_0 + rz, \frac{k * g_r(z)}{C^* \lambda_r}\right) dz \le 1,$$

where $\lambda_r = \|g_r\|_{L^{\Phi_{x_0,r}}(B(0,1))}$. Then, by ($\Psi$2), we have

$$\int_{B(0,1)} \Psi\big(x_0 + rz, 2k * g_r(z)\big)\, dz \le 2A_4 C^* \lambda_r$$

whenever $2C^*\lambda_r \le 1$. Now, $x_0 \notin E_2$ implies

$$\begin{aligned}\int_{B(0,1)} \Phi_{x_0,r}\big(z, g_r(z)\big)\, dz &= \int_{B(0,1)} \Phi\big(x_0 + rz, r^N \bar{k}(r) f_r(x_0 + rz)\big)\, dz \\ &= |B(0,1)| \fint_{B(x_0,r)} \Phi(x, r^N \bar{k}(r) f(x))\, dx \to 0 \qquad \text{as } r \to 0+\,.\end{aligned}$$

Hence, by Lemma 2.2, $\lambda_r \to 0$ as $r \to 0+$. Thus (3.8), and hence (3.7) holds.

Since

$$\begin{aligned}\Psi(x, |k * f(x) - k * f(x_0)|) &\le A_4 \Psi(x, I_1(x) + |I_2(x)|) \\ &\le A_4^2 \big(\Psi(x, 2I_1(x)) + \Psi(x, 2|I_2(x)|)\big)\end{aligned}$$

by (Ψ2), and

$$\begin{aligned}\fint_{B(x_0,r/2)} \Psi(x, 2I_1(x))\, dx &\le A_4 \fint_{B(x_0,r/2)} \Psi(x, 2k * f_r(x))\, dx \\ &\le 2^N A_4 \fint_{B(x_0,r)} \Psi(x, 2k * f_r(x))\, dx,\end{aligned}$$

(3.2) follows from (3.6) and (3.7). $\square$

§ 4. Mean continuity (II)

Set

$$u_{B(x_0,r)} := \fint_{B(x_0,r)} u(y)\, dy$$

for $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$.

Combining (3.1) and (3.2) in Theorem 3.1, we see that

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, |k * f(x) - (k * f)_{B(x_0,r)}|)\, dx = 0 \tag{4.1}$$

holds for $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$. In this section, we shall show that this holds also for $x_0 \in E_1 \setminus E_2$ under the following additional condition for k:

(k6) there exists a constant $K_4 > 0$ such that

$$k(r) - k(s) \le K_4(s-r)r^{-1}k(r)$$

whenever $0 < r < s$.

The Riesz kernel $I_\alpha(x) = |x|^{\alpha-N}$ $(0 < \alpha < N)$ satisfies this condition.

Note that if k satisfies (k6), then k is continuous and

$$d(-r^{-1}k(r)) \le (1+K_4)r^{-1}k(r)\frac{dr}{r}. \tag{4.2}$$

Theorem 4.1. *Besides the assumptions on $k(r)$, $\Phi(x,t)$ and $\Psi(x,t)$ given in Lemma 2.6, assume further that $k(r)$ satisfies* (k5) *and* (k6). *Let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying* (2.4). *Then* (4.1) *holds for all $x_0 \in \mathbf{R}^N \setminus E_2$, where*

$$E_2 = \left\{ x \in \mathbf{R}^N : \limsup_{r\to 0+} \fint_{B(x,r)} \Phi\left(z, r^N \bar{k}(r) f(z)\right) dz > 0 \right\}.$$

Lemma 4.2. *Let $x_0 \in \mathbf{R}^N$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying* (2.4). *Then*

$$g(t) := k(t) \int_{B(x_0,t)} f(y)\, dy$$

is bounded on $[\delta, \infty)$ for $\delta > 0$.

Proof. It is enough to show that $g(t)$ is bounded on $[1,\infty)$, since $\int_{B(x_0,1)} f(y)\,dy < \infty$ by (2.4).

If $1 \le |x_0 - y| < t$, then $1 + |y| \le m + t$ for an integer m such that $m \ge 1 + |x_0|$. Hence, by (k3), $k(t) \le {K_1}^m k(m+t) \le {K_1}^m k(1+|y|)$. Therefore

$$\begin{aligned} g(t) &\le k(1) \int_{B(x_0,1)} f(y)\,dy + {K_1}^m \int_{B(x_0,t)\setminus B(x_0,1)} k(1+|y|) f(y)\,dy \\ &\le k(1) \int_{B(x_0,1)} f(y)\,dy + {K_1}^m \int_{\mathbf{R}^N} k(1+|y|) f(y)\,dy < \infty \end{aligned}$$

for $t \ge 1$. □

Lemma 4.3. *Let $x_0 \in \mathbf{R}^N$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ satisfying* (2.4) *and*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Phi\left(z, r^N \bar{k}(r) f(z)\right) dz = 0.$$

Then

$$\lim_{r\to 0+} r \int_{2r}^{\infty} t^{-1} k(t) \left(\int_{B(x_0,t)} f(y)\,dy \right) \frac{dt}{t} = 0.$$

Proof. Let $\varepsilon > 0$. Then, by Lemma 3.2 and $k(t) \le \bar{k}(t)$, there exists a constant $0 < \delta \le 1$ such that

$$k(t) \int_{B(x_0,t)} f(y)\, dy \le \varepsilon$$

for all $t \in (0, \delta)$. By the previous lemma, there exists $M > 0$ such that

$$k(t) \int_{B(x_0,t)} f(y)\, dy \le M < \infty$$

for all $t \in [\delta, \infty)$. Hence, for $0 < r \le \delta/2$, we have

$$\int_{2r}^{\infty} t^{-1} k(t) \left(\int_{B(x_0,t)} f(y)\, dy \right) \frac{dt}{t} \le \varepsilon \int_{2r}^{\delta} t^{-1} \frac{dt}{t} + M \int_{\delta}^{\infty} t^{-1} \frac{dt}{t} \le \varepsilon r^{-1} + M\delta^{-1},$$

so that

$$\limsup_{r \to 0+} r \int_{2r}^{\infty} t^{-1} k(t) \left(\int_{B(x_0,t)} f(y)\, dy \right) \frac{dt}{t} \le \varepsilon.$$

Hence, we have the required result. $\square$

Proof of Theorem 4.1. Let $x_0 \in \mathbf{R}^N \setminus E_2$ and let $x \in B(x_0, r)$. Also, let $0 < r \le 1$. Write

$$\begin{aligned}
k * f(x) - (k * f)_{B(x_0,r)} &= \int_{B(x_0,2r)} k(x-y) f(y)\, dy \\
&\quad + \int_{\mathbf{R}^N \setminus B(x_0,2r)} k(x-y) f(y)\, dy - (k * f)_{B(x_0,r)} \\
&= \int_{B(x_0,2r)} k(x-y) f(y)\, dy \\
&\quad + \int_{\mathbf{R}^N \setminus B(x_0,2r)} \left(\fint_{B(x_0,r)} \left(k(x-y) - k(y-z)\right)\, dz \right) f(y)\, dy \\
&\qquad - \int_{B(x_0,2r)} \left(\fint_{B(x_0,r)} k(y-z)\, dz \right) f(y)\, dy \\
&= I_1(x) + I_2(x) - I_3.
\end{aligned}$$

For I_2, let $|x_0 - x| < r$, $|x_0 - z| < r$ and $|x_0 - y| \ge 2r$. Then, by (k6)

$$|k(x-y) - k(z-y)| \le 2K_4 |x-z| |x_0 - y|^{-1} \max\{k(x-y),\, k(z-y)\}.$$

As in the proof of Theorem 3.1, we see that

$$k(x-y) \le K' k(x_0 - y) \qquad \text{and} \qquad k(z-y) \le K' k(x_0 - y)$$

with $K' = \max\{K_3\bar{k}(1/2),\ k(1/2)/k(2),\ K_1\}$. Hence

$$
\begin{aligned}
|I_2(x)| &\le 2K_4K' \left(\fint_{B(x_0,r)} |x-z|\,dz \right) \int_{\mathbf{R}^N\setminus B(x_0,2r)} |x_0-y|^{-1}k(x_0-y)f(y)\,dy \\
&\le Cr \int_{2r}^{\infty} t^{-1}k(t)dF_{x_0}(t),
\end{aligned}
$$

where $F_{x_0}(t) = \int_{B(x_0,t)} f(y)\,dy$. In view of (4.2) and Lemma 4.2, integration by parts yields

$$
\int_{2r}^{\infty} t^{-1}k(t)dF_{x_0}(t) \le C\int_{2r}^{\infty} t^{-1}k(t)F_{x_0}(t)\frac{dt}{t}.
$$

Therefore by Lemma 4.3,

$$
\lim_{r\to 0+} \sup_{x\in B(x_0,r)} |I_2(x)| = 0.
$$

As to I_3, we have by Lemma 3.2

$$
0 \le I_3 \le \bar{k}(r)\int_{B(x_0,2r)} f(y)\,dy \le 2^N\bar{k}(2r)\int_{B(x_0,2r)} f(y)\,dy \to 0
$$

as $r \to 0+$.

Hence, by $(\Psi 2)$

$$
\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, 2|I_2(x) - I_3|)\,dx = 0.
$$

On the other hand, the arguments to obtain (3.7) in the proof of Theorem 3.1 show that

$$
\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, 2I_1(x))\,dx = 0.
$$

Hence again using $(\Psi 2)$ we see that

$$
\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, |k*f(x) - (k*f)_{B(x_0,r)}|)\,dx = 0.
$$

□

§ 5. Size of exceptional sets

First, we introduce a notion of capacity (cf. [5]). For a set $E \subset \mathbf{R}^N$ and an open set $G \subset \mathbf{R}^N$, we define the (k,Φ)-capacity of E relative to G by

$$
C_{k,\Phi}(E;G) = \inf_{f\in S_k(E;G)} \int_G \overline{\Phi}\,(y, f(y))\,dy,
$$

where $S_k(E;G)$ is the family of all nonnegative measurable functions f on $\mathbf{R}^N$ such that f vanishes outside G and $k*f(x) \ge 1$ for every $x \in E$. Here, note that $E \subset G$ is not required.

Lemma 5.1 ([5, Proposition 3.1]). *The set function $C_{k,\Phi}(\,\cdot\,;G)$ is countably subadditive and nondecreasing.*

We say that E is of (k,Φ)-capacity zero, written as $C_{k,\Phi}(E)=0$, if

$$C_{k,\Phi}(E\cap G;G)=0 \qquad \text{for every bounded open set } G.$$

Lemma 5.2 ([5, Proposition 3.3]). *For $E\subset \mathbf{R}^N$, $C_{k,\Phi}(E)=0$ if and only if there exists a nonnegative function $f\in L^\Phi(\mathbf{R}^N)$ such that $k*f\not\equiv\infty$ and*

$$k*f(x)=\infty \qquad \textit{whenever } x\in E.$$

By Lemma 5.2 we have

Proposition 5.3. *If $f\in L^\Phi(\mathbf{R}^N)$, then E_1 in Theorem 3.1 has (k,Φ)-capacity zero .*

To estimate the size of E_2 in Theorem 3.1, we introduce a Hausdorff measure defined by the (variable) measure function

$$h(r;x)=r^N\Phi(x,r^{-N}\bar{k}(r)^{-1})$$

for $x\in\mathbf{R}^N$ and $r>0$.

We define the Hausdorff h-measure of $E\subset\mathbf{R}^N$ by

$$H_h(E)=\inf\left\{\sum_j h(r_j;x_j):\bigcup_j B(x_j,r_j)\supset E,\ 0<r_j<1\right\}.$$

Here we note that

(h1) there exists a constant $A>0$ such that $h(5r;x)\le Ah(r;x)$ for all $x\in\mathbf{R}^N$ and $r>0$;

(h2) $\lim_{r\to 0} r^{-N}\left(\inf_x h(r;x)\right)=\infty$.

We show the following result (cf. Meyers [8, 9]; also cf. [10, Chapter 5, Lemma 8.2]).

Lemma 5.4. *If $f\in L^\Phi(\mathbf{R}^N)$, then $H_h(E_{h,f})=0$, where*

$$E_{h,f}:=\left\{x\in\mathbf{R}^N:\limsup_{r\to 0+}\frac{1}{h(r;x)}\int_{B(x,r)}\Phi(y,|f(y)|)\,dy>0\right\}.$$

Proof. It suffices to show that $H_h(E(a)) = 0$ for each $a > 0$, where

$$E(a) := \left\{ x \in \mathbf{R}^N : \limsup_{r\to 0+} \frac{1}{h(r;x)} \int_{B(x,r)} \Phi(y, |f(y)|)\, dy > a \right\}.$$

For $\varepsilon > 0$, by (h2) we can find $\delta > 0$ ($\delta \le 1$) such that

$$h(r;x) > \varepsilon^{-1} r^N$$

for all $x \in \mathbf{R}^N$ and $0 < r < \delta$. For each $x \in E(a)$, take $B(x, r(x))$ such that $0 < r(x) < \delta$ and

$$\frac{1}{h(r(x);x)} \int_{B(x,r)} \Phi(y, |f(y)|)\, dy > a.$$

By a covering lemma (see, e.g., [1, Theorem 1.4.1]), we can take a disjoint subfamily $\{B(x_j, r(x_j))\}$ such that $E(a) \subset \bigcup_j B(x_j, 5r(x_j))$. Then

$$\begin{aligned} H_h(E(a)) &\le \sum_j h(5r(x_j); x_j) \\ &\le A \sum_j h(r(x_j); x_j) \\ &\le A a^{-1} \int_{\bigcup_j B(x_j, r(x_j))} \Phi(y, |f(y)|)\, dy. \end{aligned}$$

Note here that

$$\begin{aligned} \varepsilon^{-1} \sum_j r(x_j)^N &\le \sum_j h(r(x_j); x_j) \\ &\le a^{-1} \int_{\bigcup_j B(x_j, r(x_j))} \Phi(y, |f(y)|)\, dy, \end{aligned}$$

so that

$$\left| \bigcup_j B(x_j, r(x_j)) \right| \le C a^{-1} \varepsilon \int_{\mathbf{R}^N} \Phi(y, |f(y)|)\, dy.$$

Since $f \in L^\Phi(\mathbf{R}^N)$, by the absolute continuity of integrals we see that $H_h(E(a)) = 0$, as required. $\square$

On the other hand, by [5, Corollary 4.8], we have the following result.

Lemma 5.5. *Suppose $\Phi(x,t)$ satisfies ($\Phi 5$). If $f \in L^\Phi(\mathbf{R}^N)$, then $C_{k,\Phi}(E_{h,f}) =$ 0.*

Here note that the condition

$$\limsup_{r\to 0+} \frac{\sup_{y\in B(x,r)} \Phi(y, r^{-N}\bar{k}(r)^{-1})}{\inf_{y\in B(x,r)} \Phi(y, r^{-N}\bar{k}(r)^{-1})} < \infty \tag{5.1}$$

in [5, Corollary 4.8] is satisfied by ($\Phi5$), since $r^N\bar{k}(r) \le 1$ for small $r > 0$ by (2.3).

Now, we consider a further condition on $\Phi(x,t)$:

($\Phi6$) there exists a constant $A_6 > 0$ such that

$$\Phi(x,s)\,\Phi(x,t) \le A_6\,\Phi(x,st)$$

for all $x \in \mathbf{R}^N$, $s \ge 1$ and $t > 0$.

Example 5.6. Let $\Phi_{p(\cdot),q(\cdot),a}(x,t)$ be as in Example 2.1. It satisfies ($\Phi6$) if and only if $q^+ \le 0$; cf. [11, Proposition 3.7].

Lemma 5.7. *Suppose $\Phi(x,t)$ satisfies ($\Phi5$) and ($\Phi6$). Let f be a nonnegative measurable function on $\mathbf{R}^N$ and let E_2 be as in Theorem 3.1. Then $E_2 \subset E_{h,f}$.*

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^N$ and let $x \in \mathbf{R}^N$. By (2.3), there is $0 < r_1 \le 1$ such that $r_1^N\bar{k}(r_1) \le 1$. If $0 < r \le r_1$ and $y \in B(x,r)$, then by ($\Phi6$) and ($\Phi5$),

$$\Phi\big(y, r^N\bar{k}(r)f(y)\big) \le A_6 B_\gamma \frac{\Phi(y, f(y))}{\Phi(x, r^{-N}\bar{k}(r)^{-1})},$$

where $\gamma = \bar{k}(r_1)^{-1/N}$. Hence $E_2 \subset E_{h,f}$. □

Combining this lemma with Lemmas 5.4 and 5.5, we obtain

Proposition 5.8. *Assume that Φ satisfies ($\Phi5$) and ($\Phi6$). If $f \in L^\Phi(\mathbf{R}^N)$, then E_2 in Theorem 3.1 has Hausdorff h-measure zero, that is, $H_h(E_2) = 0$, and it has (k,Φ)-capacity zero.*

Remark 1. The above definition of the Hausdorff measure is slightly different from the one in [5]. However, noting (5.1), we see that the proof of [5, Theorem 4.10] is valid for H_h and we have the following result:

Suppose $\Phi(x,t)$ satisfies ($\Phi5$). If $H_h(E) = 0$, then $C_{k,\Phi}(E) = 0$.

Applying Theorem 3.1, Proposition 5.3 and Proposition 5.8 to $k = I_\alpha$, we can state:

Corollary 5.9. *Let $0 < \alpha < N$ and let $f \in L^\Phi(\mathbf{R}^N)$ satisfy (2.4) with $k = I_\alpha$. Suppose $\Phi(x,t)$ satisfies ($\Phi3^*$), ($\Phi5$), ($\Phi6$) and*

(ΦI_α) *$s \mapsto s^{-\varepsilon_1-\alpha/N}\Phi^{-1}(x,s)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon_1 > 0$;*

$\Psi(x,t)$ satisfies ($\Psi 1$), ($\Psi 2$) and

($\Psi\Phi I_\alpha$) *there exists a constant $A_5' \geq 1$ such that*

$$\Psi\left(x, s^{-\alpha/N}\Phi^{-1}(x,s)\right) \leq A_5' s$$

for all $x \in \mathbf{R}^N$ and $s > 0$.

Then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, |I_\alpha * f(x) - I_\alpha * f(x_0)|)\, dx = 0$$

holds for all $x_0 \in \mathbf{R}^N \setminus E$ for a set E of (I_α,Φ)-capacity zero.

§ 6. Appendix: uniform almost-increasingness of $t^{p(\xi)}\left(\log(e+t)\right)^{q(\xi)}$

In this section, we give an outline of a proof of the equivalence stated in the last part of Example 2.1.

For a positive function $f(t)$ on $(0,\infty)$, set

$$A[f] := \sup_{t>0,\lambda>1} \frac{f(t)}{f(\lambda t)}.$$

f is almost increasing on $(0,\infty)$ if and only if $A[f] < \infty$. Note that f is non-decreasing on $(0,\infty)$ iff $A[f] = 1$.

A family $\{f_\xi(t)\}_{\xi\in X}$ of positive functions on $(0,\infty)$ is uniformly almost increasing if and only if

$$\sup_{\xi\in X} A[f_\xi] < \infty.$$

For $p \geq 0$ and $q \in \mathbf{R}$, we consider the function

$$F_{p,q}(t) = t^p\left(\log(e+t)\right)^q, \quad t \in [0,\infty).$$

Obviously, if $q \geq 0$, then $F_{p,q}(t)$ is non-decreasing on $(0,\infty)$. If $p = 0$ and $q < 0$, then $F_{0,q}(t)$ is not almost increasing. In case $p > 0$ and $q < 0$, it is easy to see that $F_{p,q}(t)$ is almost increasing. We are interested in the evaluation of $A[F_{p,q}]$ in this case. Since

$$A[F_{p,q}] = A[F_{p/(-q),-1}]^{-q},$$

we will evaluate $A[F_{r,-1}]$ for $r > 0$.

Let $c_0 := \log(e+1)$. We see that

$$\frac{1}{c_0}\log(e+\lambda) \;\le\; \sup_{t>0}\frac{\log(e+\lambda t)}{\log(e+t)} \;\le\; 1+\log\lambda \;\le\; 2\log(e+\lambda)$$

for $\lambda \ge 1$. Hence, letting

$$L(r) := \sup_{\lambda\ge 1}\lambda^{-r}\log(e+\lambda),$$

we have

$$\frac{1}{c_0}L(r) \le A[F_{r,-1}] \le 2L(r) \qquad (r>0). \tag{6.1}$$

Here note that $\sup_{1\le\lambda\le e}\lambda^{-r}\log(e+\lambda) \le 2$,

$$\sup_{\lambda>e}\lambda^{-r}\log(e+\lambda) \le 2\sup_{\lambda>e}\lambda^{-r}\log\lambda \le \frac{2}{er},$$

$L(r) \ge \log(e+1) = c_0$ and

$$L(r) \ge \frac{1}{e}\log\left(e+e^{1/r}\right) > \frac{1}{er},$$

so that

$$\max\left(\frac{1}{er}, c_0\right) \le L(r) \le 2\max\left(\frac{1}{er}, 1\right) \qquad (r>0).$$

Hence, by (6.1),

$$\max\left(\frac{1}{c_0 er}, 1\right) \le A[F_{r,-1}] \le 4\max\left(\frac{1}{er}, 1\right) \quad (r>0).$$

Thus, for $p>0$ and $q<0$,

$$\left[\max\left(\frac{-q}{c_0 ep}, 1\right)\right]^{-q} \le A[F_{p,q}] \le \left[4\max\left(\frac{-q}{p}, 1\right)\right]^{-q}.$$

Note that $e^{-1/e} \le (-q)^{-q} \le \max(1, (-q_0)^{-q_0})$ if $q_0 \le q < 0$. Then from the above inequalities we have:

Proposition 6.1. *Let X be a nonepmty set and let $p(\cdot)$ and $q(\cdot)$ be real valued functions on X such that $p(\xi) \ge 0$ for all $\xi \in X$ and $\inf_{\xi\in X} q(\xi) > -\infty$. Then, the following* (1) *and* (2) *are equivalent to each other:*

(1) *The family $\{F_{p(\xi),q(\xi)}(t)\}_{\xi\in X}$ is uniformly almost increasing on $(0,\infty)$;*

(2) *$q(\xi) \ge 0$ at points $\xi \in X$ where $p(\xi) = 0$, and*

$$\sup_{\xi\in X,\ p(\xi)>0,\ q(\xi)<0} q(\xi)\log p(\xi) < \infty.$$

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