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Toroidal compactifications of Shimura varieties of PEL type and its applications

By

Naoki Imai * and Yoichi Mieda **

Abstract

In this survey paper, we explain a theory of toroidal compactifications of Shimura varieties of PEL type after Lan. We also explain its application to cohomology of the Shimura varieties.

Introduction

A Shimura variety of PEL type is a moduli space of abelian varieties with additional structures. The cohomology of Shimura varieties of PEL type are important, because they should realize the Langlands correspondences. It is often useful to consider compactifications of Shimura varieties for studying cohomology of the Shimura varieties. A toroidal compactification of a Shimura variety of PEL type is a compactification determined by data of a cone decomposition. It depends on the choice of the cone decomposition data, and does not have moduli interpretation. However, there is a degenerating family of abelian varieties on the toroidal compactification. The purpose of this paper is to explain a statement of the main theorem of [Lan1] on toroidal compactifications of Shimura varieties of PEL type. In short, toroidal compactifications of Shimura varieties of PEL type over a ring of integers outside some bad primes are constructed by algebraic methods in [Lan1].

Here, we briefly recall some related results. Toroidal compactifications of Shimura varieties over the field of complex numbers was obtained by Ash-Mumford-Rapoport-Tai in [AMRT]. Algebraic construction of toroidal compactifications of Siegel modular

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*Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.
e-mail: `naoki@ms.u-tokyo.ac.jp`
**Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan.
e-mail: `mieda@math.kyoto-u.ac.jp`

varieties over a ring of integers outside bad primes was obtained by Faltings-Chai in [FC]. Larsen constructed toroidal compactifications of Picard modular varieties over a ring of integers outside bad primes in a way based on methods in [FC] (cf. [Lar]). There is also an unpublished preprint [Fuj] by Fujiwara on toroidal compactifications of Shimura varieties of PEL type. The construction of toroidal compactifications in [Lan1] is based on that in [FC], but there are a lot of technical difficulties for generalization, which is resolved in [Lan1]. A coincidence over the the field of complex numbers of analytic construction in [AMRT] and algebraic construction in [Lan1] is proved in [Lan2].

We explain the contents of this paper. In Section 1, we recall the definition of Shimura varieties of PEL type. In Section 2, we define a notion of a cusp label, which is an index set of some stratification of a toroidal compactification. In Section 3, we recall a general notion of a cone decomposition. In Section 4, we define a degeneration data in the principal level case, which is used to construct a degeneration family of abelian varieties. In Section 5, we define cone decomposition data, which determine a toroidal compactification. In Section 6, we state a theorem on the existence of toroidal compactifications. In Section 7, we mention some application of the theory of toroidal compactifications to the cohomology of Shimura varieties of PEL type.

Notation

Let $\Box$ be a set of prime numbers. We say an integer is prime to $\Box$ if it is not divisible by any prime number in $\Box$. For a positive integer n, we write $\Box \nmid n$ if n is prime to $\Box$. For a commutative ring R, we write $R_{(\Box)}$ for the localization of R at the multiplicative subset of $\mathbb{Z}$ generated by the non-zero integers prime to $\Box$. We write $\widehat{\mathbb{Z}}^{\Box}$ and $\mathbb{A}^{\Box}$ for the integral adèles and adèles away from $\Box$ respectively.

§ 1. Shimura varieties of PEL-type

Let B be a finite-dimensional semisimple algebra over $\mathbb{Q}$ with positive involution $\star$. Here, positivity of $\star$ means that $\mathrm{Tr}_{B/\mathbb{Q}}(bb^\star) > 0$ for any non-zero $b \in B$. We write F for the center of B. Let $\mathcal{O}$ be an order in B that is stable by $\star$.

Definition 1.1. Let $t = [B : \mathbb{Q}]$. The discriminant $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ is the ideal of $\mathbb{Z}$ generated by the set of elements

$$\{\det\bigl((\mathrm{Tr}_{B/\mathbb{Q}}(x_i x_j))_{1\le i,j\le t}\bigr) \mid x_1, \dots, x_t \in \mathcal{O}\}.$$

We put $\mathbb{Z}(1) = \mathrm{Ker}(\exp\colon \mathbb{C} \to \mathbb{C}^\times)$. For $z \in \mathbb{C}$, the complex conjugate of z is denoted by z^{c}.

Definition 1.2. A PEL-type $\mathcal{O}$-lattice is a triple $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ where

1. L is an $\mathcal{O}$-module that is finite free over $\mathbb{Z}$,
2. $\langle\,\cdot\,,\,\cdot\,\rangle\colon L \times L \to \mathbb{Z}(1)$ is a $\mathbb{Z}$-bilinear alternating pairing such that $\langle bx, y\rangle = \langle x, b^\star y\rangle$ for $x, y \in L$ and $b \in \mathcal{O}$,
3. $h\colon \mathbb{C} \to \mathrm{End}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}}(L \otimes_{\mathbb{Z}} \mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism such that $\langle h(z)x, y\rangle = \langle x, h(z^{\mathrm{c}})y\rangle$ for $x, y \in L \otimes_{\mathbb{Z}} \mathbb{R}$ and $z \in \mathbb{C}$,
4. the $\mathbb{R}$-bilinear pairing
$$\frac{1}{2\pi\sqrt{-1}}\langle\,\cdot\,, h(\sqrt{-1})\,\cdot\,\rangle\colon (L \otimes_{\mathbb{Z}} \mathbb{R}) \times (L \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$$
is symmetric and positive definite for any choice of $\sqrt{-1} \in \mathbb{C}$.

Let $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ be a PEL-type $\mathcal{O}$-lattice. We define an algebraic group scheme $G = G_{(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)}$ over $\mathbb{Z}$ by
$$G(R) = \{(g, r) \in GL_{\mathcal{O}\otimes_{\mathbb{Z}}R}(L \otimes_{\mathbb{Z}} R) \times \mathbf{G}_{\mathrm{m}}(R) \mid \langle gx, gy\rangle = r\langle x, y\rangle \text{ for all } x, y \in L\}$$
for a commutative $\mathbb{Z}$-algebra R. We put $I_{\mathrm{bad}} = 2$ if $B \otimes_{F,\tau} \mathbb{R} \simeq M_k(\mathbb{H})$ for some $\mathbb{Q}$-algebra homomorphism $\tau\colon F \to \mathbb{R}$ and some positive integer k. Otherwise, we put $I_{\mathrm{bad}} = 1$. Then G is smooth over $\mathbb{Z}_{(p)}$ if $p \nmid I_{\mathrm{bad}}\,\mathrm{Disc}$ by [Lan1, Corollary 1.2.3.12].

Let $L \otimes_{\mathbb{Z}} \mathbb{C} = V_0 \oplus V_0^{\mathrm{c}}$ be the unique decomposition such that $h(z)$ acts by $1 \otimes z$ on V_0 and by $1 \otimes z^{\mathrm{c}}$ on V_0^{c} for $z \in \mathbb{C}$.

Definition 1.3. The reflex field F_0 of $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ is the fixed field of $\mathbb{C}$ by the elements $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ such that V_0 and $V_0 \otimes_{\mathbb{C},\sigma} \mathbb{C}$ are isomorphic as $(B \otimes_{\mathbb{Q}} \mathbb{C})$-modules.

Remark 1.4. We have $F_0 = \mathbb{Q}(\mathrm{Tr}_{\mathbb{C}}(b; V_0) \mid b \in B)$ by [Lan1, Corollary 1.2.5.6]. Therefore F_0 is a finite extension of $\mathbb{Q}$.

Definition 1.5. Let S be a scheme, and let $\mathcal{M}$ be any locally free $\mathcal{O}_S$-module of finite rank on which $\mathcal{O}$ acts by morphisms of $\mathcal{O}_S$-modules. We take a $\mathbb{Z}$-basis $\{\alpha_1, \dots, \alpha_t\}$ of $\mathcal{O}$. Let $\{\alpha_1^\vee, \dots, \alpha_t^\vee\}$ be the $\mathbb{Z}$-basis of $\mathcal{O}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}, \mathbb{Z})$ dual to $\{\alpha_1, \dots, \alpha_t\}$. Then we have an isomorphism $\mathcal{O}_S[X_1^{\pm 1}, \dots, X_t^{\pm 1}] \simeq \mathcal{O}_S[\mathcal{O}^\vee]$ of $\mathcal{O}_S$-algebra defined by sending X_i to $\alpha_i^\vee$ for $1 \le i \le t$. We put
$$\det{}^{\alpha_1,\dots,\alpha_t}_{\mathcal{O}|\mathcal{M}}(X_1, \dots, X_t) = \det{}_{\mathcal{O}_S}(X_1\alpha_1 + \cdots + X_t\alpha_t|\mathcal{M}) \in \mathcal{O}_S(S)[X_1, \dots, X_t].$$
We define $\det_{\mathcal{O}|\mathcal{M}}$ as the element of $\mathcal{O}_S(S)[\mathcal{O}^\vee]$ corresponding to $\det^{\alpha_1,\dots,\alpha_t}_{\mathcal{O}|\mathcal{M}}(X_1, \dots, X_t)$ under the isomorphism $\mathcal{O}_S(S)[X_1^{\pm 1}, \dots, X_t^{\pm 1}] \simeq \mathcal{O}_S(S)[\mathcal{O}^\vee]$. This element $\det_{\mathcal{O}|\mathcal{M}}$ does not depend on the choice of $\{\alpha_1, \dots, \alpha_t\}$.

If $S = \mathrm{Spec}\, R$ and $M = \mathcal{M}(R)$, then we write $\det_{\mathcal{O}|M} \in R[\mathcal{O}^\vee]$ for $\det_{\mathcal{O}|\mathcal{M}} \in \mathcal{O}_S(S)[\mathcal{O}^\vee]$.

Lemma 1.6. *[Lan1, Lemma 1.2.5.10] The element* $\det_{\mathcal{O}|V_0}$ *of* $\mathbb{C}[\mathcal{O}^\vee]$ *is in the subset* $\mathcal{O}_{F_0}[\mathcal{O}^\vee] \subset \mathbb{C}[\mathcal{O}^\vee]$.

Definition 1.7. Let A be an abelian scheme over a scheme S. We write $\mathcal{P}_A$ for the Poincaré invertible sheaf of A, which is an invertible sheaf on $A \times_S A^\vee$. A polarization of A is a homomorphism $\lambda_A \colon A \to A^\vee$ such that

1. the composite $A \xrightarrow{\sim} (A^\vee)^\vee \xrightarrow{\lambda_A^\vee} A^\vee$ coincides with λ_A,

2. the invertible sheaf $(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ over A is relatively ample over S.

We put

$$L^\sharp = \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle x, y\rangle \in \mathbb{Z}(1) \text{ for all } y \in L\}.$$

We fix a set $\square$ of prime numbers such that $\square \nmid I_{\mathrm{bad}}[L^\sharp : L]\,\mathrm{Disc}$.

Definition 1.8. Let A be an abelian scheme over a scheme S. A prime-to-$\square$ polarization of A is a polarization $\lambda_A \colon A \to A^\vee$ of A such that the rank of $\ker(\lambda_A)$ is prime to $\square$.

Definition 1.9. Let S be a scheme over $\mathrm{Spec}\,\mathcal{O}_{F_0,(\square)}$. A triple over S is a triple (A, λ_A, i_A) where

1. A is an abelian scheme over S,

2. $\lambda_A \colon A \to A^\vee$ is a prime-to-$\square$ polarization of A,

3. $i_A \colon \mathcal{O} \to \mathrm{End}_S(A)$ is a ring homomorphism such that $i_A(b)^\vee \circ \lambda_A = \lambda_A \circ i_A(b^\star)$ for $b \in \mathcal{O}$.

Definition 1.10. Let (A, λ_A, i_A) be a triple over an $\mathcal{O}_{F_0,(\square)}$-scheme S. We say that $\underline{\mathrm{Lie}}_{A/S}$ satisfies the determinant condition defined by $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ if $\det_{\mathcal{O}|\underline{\mathrm{Lie}}_{A/S}}$ agrees with the image of $\det_{\mathcal{O}|V_0}$ under the structural homomorphism $\mathcal{O}_{F_0,(\square)} \to \mathcal{O}_S(S)$, where we consider the action of $\mathcal{O}$ on $\underline{\mathrm{Lie}}_{A/S}$ induced by i_A.

Definition 1.11. Let (A, λ_A, i_A) be a triple over an $\mathcal{O}_{F_0,(\square)}$-scheme S, and let n be a positive integer prime to $\square$. An $\mathcal{O}$-equivariant symplectic isomorphism from $(L/nL)_S$ to $A[n]$ consists of the following data:

1. An $\mathcal{O}$-equivariant isomorphism $\alpha_n \colon (L/nL)_S \xrightarrow{\sim} A[n]$ of group schemes over S.

2. An isomorphism $\nu_n \colon ((\mathbb{Z}/n\mathbb{Z})(1))_S \xrightarrow{\sim} \boldsymbol{\mu}_{n,S}$ of group schemes over S such that the

diagram

$$
\begin{tikzcd}
(L/nL)_S \times_S (L/nL)_S \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\alpha_n\times\alpha_n"', "\wr"] & ((\mathbb{Z}/n\mathbb{Z})(1))_S \arrow[d, dash, "\nu_n"', "\wr"] \\
A[n] \times_S A[n] \arrow[r, "\mathrm{e}^{\lambda_A}"] & \boldsymbol{\mu}_{n.S}
\end{tikzcd}
$$

is commutative, where e^{λ_A} is the λ_A-Weil pairing.

We often write $(\alpha_n, \nu_n)\colon (L/nL)_S \xrightarrow{\sim} A[n]$ for such a symplectic isomorphism.

Definition 1.12. Let (A, λ_A, i_A) be a triple over an $\mathcal{O}_{F_0,(\square)}$-scheme S, and let n be a positive integer prime to $\square$. A principal level n structure of (A, λ_A, i_A) of type $(L\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}^{\square}, \langle\,\cdot\,,\,\cdot\,\rangle)$ is an $\mathcal{O}$-equivariant symplectic isomorphism $(\alpha_n, \nu_n)\colon (L/nL)_S \xrightarrow{\sim} A[n]$ that is symplectic-liftable in the following sense:

There exist a tower $(S_{m'} \twoheadrightarrow S_m)_{n|m, m|m', \square\nmid m'}$ of finite étale surjections with $S_n = S$ and an $\mathcal{O}$-equivariant symplectic isomorphism $(\alpha_{m,S_m}, \nu_{m,S_m})\colon (L/mL)_{S_m} \xrightarrow{\sim} A[m]_{S_m}$ over each S_m such that the pullback of $(\alpha_{l,S_l}, \nu_{l,S_l})$ to S_m is the reduction mod l of $(\alpha_{m,S_m}, \nu_{m,S_m})$ for each l satisfying $n|l$ and $l|m$.

Definition 1.13. Let n be a positive integer prime to $\square$. The moduli problem M_n is defined by the category fibered in groupoids over $(\mathrm{Sch}/\mathcal{O}_{F_0,(\square)})$ whose fiber over an $\mathcal{O}_{F_0,(\square)}$-scheme S is the groupoid $\mathsf{M}_n(S)$ described as follows:
The objects of $\mathsf{M}_n(S)$ are tuples $(A, \lambda_A, i_A, (\alpha_n, \nu_n))$ where

1. (A, λ_A, i_A) is a triple over S,
2. $\underline{\mathrm{Lie}}_{A/S}$ satisfies the determinant condition defined by $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$,
3. (α_n, ν_n) is a principal level n structure of (A, λ_A, i_A) of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \langle\,\cdot\,,\,\cdot\,\rangle)$.

The isomorphisms $(A, \lambda_A, i_A, (\alpha_n, \nu_n)) \sim (A', \lambda_{A'}, i_{A'}, (\alpha'_n, \nu'_n))$ in M_n are given by $\mathcal{O}$-equivariant isomorphisms $f\colon A \xrightarrow{\sim} A'$ as abelian schemes over S such that

1. $\lambda_A = f^{\vee} \circ \lambda_{A'} \circ f$,
2. $f|_{A[n]}\colon A[n] \xrightarrow{\sim} A'[n]$ satisfies $\alpha'_n = (f|_{A[n]}) \circ \alpha_n$.

For a commutative $\widehat{\mathbb{Z}}^{\square}$-algebra R, we put $G^{\mathrm{ess}}(R) = \mathrm{Im}(G(\widehat{\mathbb{Z}}^{\square}) \to G(R))$.

Definition 1.14. Let (A, λ_A, i_A) be a triple over an $\mathcal{O}_{F_0,(\square)}$-scheme S, and let n be a positive integer prime to $\square$. Let H_n be a subgroup of $G^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$. By an H_n-orbit of étale locally defined level n structures for (A, λ_A, i_A), we mean a subscheme α_{H_n} of

$$\underline{\mathrm{Isom}}_S\big((L/nL)_S, A[n]\big) \times_S \underline{\mathrm{Isom}}_S\big(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S}\big)$$

over S that becomes a reduced closed subscheme defined by some H_n-orbit of level n structures after base change to some finite étale covering of S.

For a positive integer n prime to $\square$, we put $\mathcal{U}^{\square}(n) = \mathrm{Ker}(G(\widehat{\mathbb{Z}}^{\square}) \to G(\widehat{\mathbb{Z}}^{\square}/n\widehat{\mathbb{Z}}^{\square}))$. For an open compact subgroup $\mathcal{H}$ of $G(\widehat{\mathbb{Z}}^{\square})$ and a positive integer n such that $\square \nmid n$ and $\mathcal{U}^{\square}(n) \subset \mathcal{H}$, we put $H_n = \mathcal{H}/(\mathcal{U}^{\square}(n)) \subset G^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$.

Lemma 1.15. *Let (A, λ_A, i_A) be a triple over an $\mathcal{O}_{F_0,(\square)}$-scheme S, and let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\square})$. Let n and m be positive integers prime to $\square$ such that $n|m$ and $\mathcal{U}^{\square}(n) \subset \mathcal{H}$. Then there is a canonical bijection from the set of H_m-orbits of étale locally defined level m structures for (A, λ_A, i_A) to the set of H_n-orbits of étale locally defined level m structures for (A, λ_A, i_A), which is induced by taking the reduction mod n of level m structures étale locally.*

Proof. This follows from the symplectic-liftability condition in Definition 1.12. $\square$

Definition 1.16. Let (A, λ_A, i_A) be a triple over an $\mathcal{O}_{F_0,(\square)}$-scheme S, and let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\square})$. Then a level $\mathcal{H}$ structure of (A, λ_A, i_A) of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \langle\,\cdot\,,\,\cdot\,\rangle)$ is a collection $\alpha_{\mathcal{H}} = \{\alpha_{H_n}\}$ labeled by positive integers n such that $\square \nmid n$ and $\mathcal{U}^{\square}(n) \subset \mathcal{H}$ where

1. α_{H_n} is an H_n-orbit of étale locally defined level n structures for any index n,
2. the map induced by the reduction mod n sends α_{H_m} to α_{H_n} for any indices n and m such that $n|m$.

Remark 1.17. The collection $\alpha_{\mathcal{H}} = \{\alpha_{H_n}\}$ in Definition 1.16 is determined by any element α_{H_n} in it by Lemma 1.15.

Definition 1.18. Let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\square})$. The moduli problem $\mathsf{M}_{\mathcal{H}}$ is defined by the category fibered in groupoids over $(\mathrm{Sch}/\mathcal{O}_{F_0,(\square)})$ whose fiber over an $\mathcal{O}_{F_0,(\square)}$-scheme S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows:
The objects of $\mathsf{M}_{\mathcal{H}}(S)$ are tuples $(A, \lambda_A, i_A, \alpha_{\mathcal{H}})$ where

1. (A, λ_A, i_A) is a triple over S,
2. $\underline{\mathrm{Lie}}_{A/S}$ satisfies the determinant condition defined by $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$,
3. $\alpha_{\mathcal{H}}$ is a level $\mathcal{H}$ structure of (A, λ_A, i_A) of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \langle\,\cdot\,,\,\cdot\,\rangle)$.

The isomorphisms $(A, \lambda_A, i_A, \alpha_{\mathcal{H}}) \sim (A', \lambda_{A'}, i_{A'}, \alpha'_{\mathcal{H}})$ in $\mathsf{M}_{\mathcal{H}}(S)$ are given by $\mathcal{O}$-equivariant isomorphisms $f\colon A \xrightarrow{\sim} A'$ as abelian schemes over S such that

1. $\lambda_A = f^{\vee} \circ \lambda_{A'} \circ f$,

2. α_{H_n} is the pullback of α'_{H_n} under the morphism $(f|_{A[n]})_* \times \mathrm{Id}$ for any positive integer n prime to $\Box$ satisfying $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, where

$$(f|_{A[n]})_*\colon \underline{\mathrm{Isom}}_S\big((L/nL)_S, A[n]\big) \to \underline{\mathrm{Isom}}_S\big((L/nL)_S, A'[n]\big)$$

is the morphism induced by $f|_{A[n]}\colon A[n] \to A'[n]$, and Id is the identity morphism on $\underline{\mathrm{Isom}}_S\big(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S}\big)$.

Definition 1.19. Let $g = (g_p)_{p\notin\Box} \in G(\widehat{\mathbb{Z}}^{\Box})$ with $g_p \in G(\mathbb{Z}_p) \subset GL_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p}(L \otimes_{\mathbb{Z}} \mathbb{Z}_p) \times \mathbf{G}_{\mathrm{m}}(\mathbb{Z}_p)$. For each $p \notin \Box$, let Γ_{g_p} be the subgroup of $\overline{\mathbb{Q}}_p^{\times}$ generated by the eigenvalues of the action of g_p on $(L \otimes_{\mathbb{Z}} \mathbb{Z}_p) \oplus \mathbb{Z}_p$. Let $(\overline{\mathbb{Q}}^{\times} \cap \Gamma_{g_p})_{\mathrm{tors}}$ be the subgroup of $\overline{\mathbb{Q}}^{\times}$ consisting of the torsion elements of $\overline{\mathbb{Q}}^{\times} \cap \Gamma_{g_p}$ for an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, which is independent of the choice of the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. We say that $g = (g_p)$ is neat if $\bigcap_{p\notin\Box}(\overline{\mathbb{Q}}^{\times} \cap \Gamma_{g_p})_{\mathrm{tors}} = \{1\}$. We say that an open compact subgroup $\mathcal{H}$ of $G(\widehat{\mathbb{Z}}^{\Box})$ is neat if all elements in $\mathcal{H}$ are neat.

Remark 1.20. If $\mathcal{H} \subset \mathcal{U}^{\Box}(n)$ for some positive integer n such that $\Box \nmid n$ and $n \geq 3$, then $\mathcal{H}$ is neat, because no nontrivial root of unity can be congruent to 1 mod n if $n \geq 3$.

Theorem 1.21. *[Lan1, Theorem 1.4.1.11 and Corollary 7.2.3.10] Let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\Box})$. Then the moduli problem $\mathsf{M}_{\mathcal{H}}$ is a smooth separated algebraic stack of finite type over $\mathcal{O}_{F_0,(\Box)}$. If $\mathcal{H}$ is neat, then $\mathsf{M}_{\mathcal{H}}$ is representable by a smooth quasi-projective scheme over $\mathcal{O}_{F_0,(\Box)}$.*

§ 2. Cusp labels

Definition 2.1. A left $\mathcal{O}$-module M is called an $\mathcal{O}$-lattice if it is finitely generated free $\mathbb{Z}$-module.

Definition 2.2. Let R be a commutative ring. An $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-module is called integrable if it is isomorphic to $M \otimes_{\mathbb{Z}} R$ for some $\mathcal{O}$-lattice M.

Convention. For a commutative ring R, all filtrations on $L\otimes_{\mathbb{Z}}R$ we shall consider will be increasing filtrations $\mathsf{Z} = \{\mathsf{Z}_{-i}\}$ of $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-submodules on $L \otimes_{\mathbb{Z}} R$ indexed by integers $-3 \leq -i \leq 0$ such that $\mathsf{Z}_0 = M$ and $\mathsf{Z}_{-3} = \{0\}$.

Definition 2.3. Let R be a commutative ring, and let $\mathsf{Z} = \{\mathsf{Z}_{-i}\}$ be a filtration on $L \otimes_{\mathbb{Z}} R$. We put $\mathrm{Gr}^{\mathsf{Z}}_{-i} = \mathsf{Z}_{-i}/\mathsf{Z}_{-i-1}$ for $0 \leq i \leq 2$, and $\mathrm{Gr}^{\mathsf{Z}} = \oplus_{0\leq i\leq 2}\,\mathrm{Gr}^{\mathsf{Z}}_{-i}$.

1. We say that Z is integrable if $\mathrm{Gr}^{\mathsf{Z}}_{-i}$ is integrable as an $(\mathcal{O}\otimes_{\mathbb{Z}}R)$-module for $0 \leq i \leq 2$.

2. We say that $\mathtt{Z}$ is split if there exists some isomorphism $\mathrm{Gr}^{\mathtt{Z}} \simeq M$ as $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-modules.

3. We say that $\mathtt{Z}$ is admissible if it is integrable and split.

4. We say that $\mathtt{Z}$ is symplectic if $\mathtt{Z}_{-2}$ and $\mathtt{Z}_{-1}$ are annihilators of each other under the pairing induced from $\langle\,\cdot\,,\,\cdot\,\rangle$.

If $\mathtt{Z}$ is symplectic, let $\langle\,\cdot\,,\,\cdot\,\rangle_{11} \colon \mathrm{Gr}^{\mathtt{Z}}_{-1} \times \mathrm{Gr}^{\mathtt{Z}}_{-1} \to R(1)$ be the pairing induced from $\langle\,\cdot\,,\,\cdot\,\rangle$.

Definition 2.4. Let R be a commutative ring, and let M be an integrable $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-module. An integrable $\mathcal{O} \otimes_{\mathbb{Z}} R$-submodule M' of M is called admissible if the filtration $0 \subset M' \subset M$ is admissible. A surjection $M \to M''$ of integrable $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-modules is called admissible if the kernel is admissible.

Definition 2.5. Let $B \simeq \prod_{i\in I} B_i$ be the decomposition of B into simple factors. Let M be a finite B-module. Then we have a decomposition $M \simeq \bigoplus_{i\in I} M_i^{\oplus m_i}$ according to the decomposition $B \simeq \prod_{i\in I} B_i$, where M_i is the unique simple left B_i-module for $i \in I$. We call $(m_i)_{i\in I}$ the B-multi-rank of M.

Definition 2.6. Let R be a Noetherian commutative flat $\mathbb{Z}$-algebra, and let M be an integrable $(\mathcal{O} \otimes_{\mathbb{Z}} R)$-module, which is isomorphic to $M' \otimes_{\mathbb{Z}} R$ for some $\mathcal{O}$-lattice M' by definition. Then the $\mathcal{O}$-multi-rank of M is defined to be the B-multi-rank of the $M' \otimes_{\mathbb{Z}} \mathbb{Q}$. The $\mathcal{O}$-multi-rank of M does not depend on the choice of M' by [Lan1, Lemma 1.2.1.24].

Definition 2.7. The $\mathcal{O}$-multi-rank of a symplectic admissible filtration $\mathtt{Z}$ on $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}$ is the $\mathcal{O}$-multi-rank of $\mathtt{Z}_{-2}$ as an integral $(\mathcal{O} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box})$-module.

Definition 2.8. We say that a symplectic admissible filtration $\mathtt{Z}$ on $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}$ is fully symplectic with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ if there is a symplectic admissible filtration $\mathtt{Z}_{\mathbb{A}^{\Box}} = \{\mathtt{Z}_{-i,\mathbb{A}^{\Box}}\}$ on $L \otimes_{\mathbb{Z}} \mathbb{A}^{\Box}$ such that $\mathtt{Z}_{-i,\mathbb{A}^{\Box}} \cap (L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}) = \mathtt{Z}_{-i}$ in $L \otimes_{\mathbb{Z}} \mathbb{A}^{\Box}$ for all i.

Definition 2.9. For a positive integer n prime to $\Box$, a fully symplectic-liftable admissible filtration $\mathtt{Z}_n$ of L/nL with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ is the reduction mod n of some fully symplectic admissible filtration $\mathtt{Z}$ on $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}$ with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ with the information of $\mathcal{O}$-multi-rank of $\mathtt{Z}$.

Definition 2.10. For a fully symplectic admissible filtration $\mathtt{Z}$ on $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}$ with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$, a torus argument Φ for $\mathtt{Z}$ is a tuple $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$, where

1. X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multi-rank, and $\phi \colon Y \hookrightarrow X$ is an $\mathcal{O}$-linear embedding,

2. $\varphi_{-2}\colon \mathrm{Gr}^{\mathtt{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \widehat{\mathbb{Z}}^{\square}(1))$ and $\varphi_0\colon \mathrm{Gr}^{\mathtt{Z}}_0 \xrightarrow{\sim} Y \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}$ are $\mathcal{O}$-linear isomorphisms such that the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{20}\colon \mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_0 \to \widehat{\mathbb{Z}}^{\square}(1)$ induced from $\langle\,\cdot\,,\,\cdot\,\rangle$ is the pullback by $(\varphi_{-2}, \varphi_0)$ of the pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\phi}\colon \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \widehat{\mathbb{Z}}^{\square}(1)) \times (Y \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}) \to \widehat{\mathbb{Z}}^{\square}(1)$$

defined by $\langle f, y\rangle_{\phi} = f(\phi(y))$ for $f \in \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}^{\square}, \widehat{\mathbb{Z}}^{\square}(1))$ and $y \in Y\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}^{\square}$. Here we consider the $\mathcal{O}$-action on $\mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \widehat{\mathbb{Z}}^{\square}(1))$ defined by $(bf)(\chi) = f(b^{\star}\chi)$ for $b \in \mathcal{O}$, $f \in \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \widehat{\mathbb{Z}}^{\square}(1))$ and $\chi \in X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}$.

Definition 2.11. Let n be a positive integer prime to $\square$, and let $\mathtt{Z}_n$ be a fully symplectic-liftable admissible filtration on L/nL with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$. Then a torus argument Φ_n at level n for $\mathtt{Z}_n$ is a tuple $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, where

1. X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multi-rank, and $\phi\colon Y \hookrightarrow X$ is an $\mathcal{O}$-linear embedding,

2. $\varphi_{-2,n}\colon \mathrm{Gr}^{\mathtt{Z}_n}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X/nX, \mathbb{Z}/n\mathbb{Z}(1))$ and $\varphi_{0,n}\colon \mathrm{Gr}^{\mathtt{Z}_n}_0 \xrightarrow{\sim} Y/nY$ are $\mathcal{O}$-linear isomorphisms that are reduction mod n of an $\mathcal{O}$-linear isomorphisms $\varphi_{-2}\colon \mathrm{Gr}^{\mathtt{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\square}}(X \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}, \widehat{\mathbb{Z}}^{\square}(1))$ and $\varphi_0\colon \mathrm{Gr}^{\mathtt{Z}}_0 \xrightarrow{\sim} Y \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}$ respectively such that $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ form a torus argument.

Two torus arguments $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ and $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$ at level n are equivalent if and only if there are $\mathcal{O}$-equivariant isomorphisms $\gamma_X : X' \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y'$ such that $\phi = \gamma_X \phi' \gamma_Y$, $\varphi'_{-2,n} = {}^t\gamma_X \varphi_{-2,n}$ and $\varphi'_{0,n} = \gamma_Y \varphi_{0,n}$.

Definition 2.12. For a positive integer n prime to $\square$, a splitting $\delta_n\colon \mathrm{Gr}^{\mathtt{Z}_n} \xrightarrow{\sim} L/nL$ for a fully symplectic-liftable admissible filtration $\mathtt{Z}_n$ is called liftable if it is the reduction mod n of some splitting $\hat{\delta}\colon \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\square}$ for a fully symplectic admissible filtration $\mathtt{Z}$ lifting $\mathtt{Z}_n$.

Definition 2.13. Let n be a positive integer prime to $\square$. A principal cusp label at level n for a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ is an equivalence class of triples $(\mathtt{Z}_n, \Phi_n, \delta_n)$ that consist of the following data:

1. $\mathtt{Z}_n$ is a fully symplectic-liftable admissible filtration on L/nL with respect to the pairing $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$.

2. Φ_n is a torus argument at level n for $\mathtt{Z}_n$.

3. $\delta_n\colon \mathrm{Gr}^{\mathtt{Z}_n} \xrightarrow{\sim} L/nL$ is a liftable splitting.

Two triples $(\mathtt{Z}_n, \Phi_n, \delta_n)$ and $(\mathtt{Z}'_n, \Phi'_n, \delta'_n)$ are equivalent if and only if $\mathtt{Z}_n = \mathtt{Z}'_n$ and Φ_n and Φ'_n are equivalent.

Definition 2.14. Let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\Box})$. Then a collection of orbits of fully symplectic-liftable admissible filtrations with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ for $\mathcal{H}$ is a collection $\mathtt{Z}_{\mathcal{H}} = \{\mathtt{Z}_{H_n}\}$ indexed by positive integers n such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$ where

1. $\mathtt{Z}_{H_n}$ is an H_n-orbit of fully symplectic-liftable admissible filtrations on L/nL with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ for any index n,
2. the map induced by the reduction mod n sends the H_m-orbit $\mathtt{Z}_{H_m}$ to the H_n-orbit $\mathtt{Z}_{H_n}$ for any indices n and m such that $n|m$.

For $g_n \in G^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$, a positive integer n prime to $\Box$ and a fully symplectic-liftable admissible filtration $\mathtt{Z}_n$ of L/nL with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$, let $\mathrm{Gr}_i(g_n)\colon \mathrm{Gr}_i^{g_n^{-1}\mathtt{Z}_n} \to \mathrm{Gr}_i^{\mathtt{Z}_n}$ for $-2 \leq i \leq 0$ and $\mathrm{Gr}(g_n)\colon \mathrm{Gr}^{g_n^{-1}\mathtt{Z}_n} \to \mathrm{Gr}^{\mathtt{Z}_n}$ denote the homomorphisms induced by g_n.

Definition 2.15. Let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\Box})$, and let $\mathtt{Z}_{\mathcal{H}} = \{\mathtt{Z}_{H_n}\}$ be a collection of orbits of fully symplectic-liftable admissible filtrations with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ for $\mathcal{H}$. Then a torus argument $\Phi_{\mathcal{H}}$ at level $\mathcal{H}$ for $\mathtt{Z}_{\mathcal{H}}$ is a collection $\Phi_{\mathcal{H}} = \{\Phi_{H_n}\}$ indexed by positive integers n such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$ satisfying the following:

1. For any index n, there is an element $\mathtt{Z}_n$ in $\mathtt{Z}_{H_n}$ and a torus argument $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ at level n for $\mathtt{Z}_n$ such that Φ_{H_n} is the H_n-orbit of Φ_n, where $(g_n, r_n) \in H_n$ sends Φ_n to the torus argument $\bigl(X, Y, \phi, r_n^{-1}(\varphi_{-2,n} \circ \mathrm{Gr}_{-2}(g_n)), \varphi_{0,n} \circ \mathrm{Gr}_0(g_n)\bigr)$ at level n for $g_n^{-1}\mathtt{Z}_n$.
2. The map induced by the reduction mod n sends the H_m-orbit Φ_{H_m} to the H_n-orbit Φ_{H_n} for any indices n and m such that $n|m$.

Two torus arguments $\Phi_{\mathcal{H}} = \{\Phi_{H_n}\}$ and $\Phi'_{\mathcal{H}} = \{\Phi'_{H_n}\}$ at level $\mathcal{H}$ are equivalent if and only if there is some index n such that Φ_{H_n} contains some torus argument that is equivalent to some torus argument in Φ'_{H_n}.

For a torus argument $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, let $\Gamma_{\Phi_{\mathcal{H}}}$ be the group of the pairs $(\gamma_X, \gamma_Y) \in GL_{\mathcal{O}}(X) \times GL_{\mathcal{O}}(Y)$ such that $\phi = \gamma_X \phi \gamma_Y$, $\varphi_{-2,\mathcal{H}} = {}^t\gamma_X \varphi_{-2,\mathcal{H}}$ and $\varphi_{0,\mathcal{H}} = \gamma_Y \varphi_{0,\mathcal{H}}$, where the last two equalities are equalities as collections of orbits.

Definition 2.16. Let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\Box})$. A cusp label at level $\mathcal{H}$ for a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ is an equivalence class of triples $(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ that consist of the following data:

1. $\mathtt{Z}_{\mathcal{H}} = \{\mathtt{Z}_{H_n}\}$ is a collection of orbits of fully symplectic-liftable admissible filtrations with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ for $\mathcal{H}$.

2. $\Phi_{\mathcal{H}}$ is a torus argument at level $\mathcal{H}$ for $\mathtt{Z}_{\mathcal{H}}$.

3. $\delta_{\mathcal{H}} = \{\delta_{H_n}\}$ is a collection indexed by positive integers n prime to $\Box$ satisfying $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, where δ_{H_n} is the H_n-orbit of a liftable splitting $\delta_n \colon \mathrm{Gr}^{\mathtt{Z}_n} \xrightarrow{\sim} L/nL$ for a representative $\mathtt{Z}_n$ of the H_n-orbit $\mathtt{Z}_{H_n}$, where $(g_n, r_n) \in H_n$ sends δ_n to $g_n^{-1} \circ \delta_n \circ \mathrm{Gr}(g_n) \colon \mathrm{Gr}^{g_n^{-1}\mathtt{Z}_n} \to L/nL$.

Two triples $(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\mathtt{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent if and only if $\mathtt{Z}_{\mathcal{H}} = \mathtt{Z}'_{\mathcal{H}}$ and $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ are equivalent.

Remark 2.17. By the definition of the equivalence in Definition 2.16, for a cusp label $(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, only the existence of $\delta_{\mathcal{H}}$ is important.

Convention. We shall often suppress $\mathtt{Z}_{\mathcal{H}}$ from the notation $(\mathtt{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, with the understanding that the data $\Phi_{\mathcal{H}}$ and $\delta_{\mathcal{H}}$ require an implicit choice of $\mathtt{Z}_{\mathcal{H}}$.

Lemma 2.18. *Let $\mathtt{Z}_{\mathcal{H}}$ be a collection of orbits of fully symplectic-liftable admissible filtrations for $\mathcal{H}$, and let $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ be a torus argument at level $\mathcal{H}$ for $\mathtt{Z}_{\mathcal{H}}$. Let X' and Y' be $\mathcal{O}$-lattices of the same $\mathcal{O}$-multi-rank, and let $\phi' \colon Y' \hookrightarrow X'$ be an $\mathcal{O}$-linear embedding, Let $s_X \colon X \to X'$ and $s_Y \colon Y \to Y'$ be admissible surjections such that $s_X\phi = \phi' s_Y$. Then these naturally induce a collection $\mathtt{Z}'_{\mathcal{H}}$ of orbits of fully symplectic-liftable admissible filtrations for $\mathcal{H}$ and a pair $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ such that $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ is a torus argument at level $\mathcal{H}$ for $\mathtt{Z}'_{\mathcal{H}}$.*

Proof. Let n be a positive integer such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. We take representatives $\mathtt{Z}_n$ and $(\varphi_{-2,n}, \varphi_{0,n})$ of H_n-orbits $\mathtt{Z}_{H_n}$ and $(\varphi_{-2,H_n}, \varphi_{0,H_n})$, where $\mathtt{Z}_{\mathcal{H}} = \{\mathtt{Z}_{H_n}\}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}) = \{(\varphi_{-2,H_n}, \varphi_{0,H_n})\}$. Let $\mathtt{Z}'_{-2,n}$ be the inverse image of the image of

$$s_X^* \colon \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1)) \hookrightarrow \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1)); f \mapsto f \circ s_X$$

under $\mathtt{Z}_{-2,n} = \mathrm{Gr}_{-2}^{\mathtt{Z}_n} \xrightarrow{\varphi_{-2,n}} \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$. Then $\varphi_{-2,n}$ induces an isomorphism $\varphi'_{-2,n} \colon \mathrm{Gr}_{-2}^{\mathtt{Z}'_n} = \mathtt{Z}'_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1))$. Let $s_{Y,n} \colon Y/nY \to Y'/nY'$ be the surjection induced by s_Y. We define $\mathtt{Z}'_{-1,n}$ to be the kernel of the composite

$$\mathtt{Z}_{0,n} \twoheadrightarrow \mathrm{Gr}_0^{\mathtt{Z}_n} \xrightarrow{\varphi_{0,n}} Y/nY \xrightarrow{s_{Y,n}} Y'/nY'.$$

This composite induces an isomorphism $\varphi'_{0,n} \colon \mathrm{Gr}_0^{\mathtt{Z}'_n} \xrightarrow{\sim} Y'/nY'$. Then we define $\mathtt{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ as the collection of H_n-orbits of $\mathtt{Z}'$ and $(\varphi'_{-2,n}, \varphi'_{0,n})$ for the positive integers n such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. $\Box$

Definition 2.19. Let $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be representatives of cusp labels at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$. A surjection $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \to (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ is a pair of admissible surjections $s_X \colon X \to X'$ and $s_Y \colon Y \to Y'$ satisfying the following:

1. We have $s_X \phi = \phi' s_Y$.
2. $\mathsf{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are induced from $\mathsf{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ by s_X and s_Y as in Lemma 2.18.

§ 3. Cone decompositions

Let H be a group of multiplicative type of finite type over a scheme S. We put $\underline{\mathbf{X}}(H) = \underline{\mathrm{Hom}}_S(H, \mathbf{G}_{\mathrm{m},S})$ and $\underline{\mathbf{X}}(H)^\vee = \underline{\mathrm{Hom}}_S(\mathbf{G}_{\mathrm{m},S}, H)$. Then $\underline{\mathbf{X}}(H)$ is an étale sheaf of finitely generated commutative groups, and $\underline{\mathbf{X}}(H)^\vee$ is an étale sheaf of finitely generated free commutative groups.

In the sequel, we assume that H is split and consider $\underline{\mathbf{X}}(H)$ and $\underline{\mathbf{X}}(H)^\vee$ as abelian groups. We put $\underline{\mathbf{X}}(H)^\vee_R = \underline{\mathbf{X}}(H)^\vee \otimes_{\mathbb{Z}} R$ for a commutative ring R.

Definition 3.1. A subset of $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ is called a cone if it is invariant under the natural multiplicative action of $\mathbb{R}^\times_{>0}$. A cone in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ is nondegenerate if its closure does not contain any nonzero $\mathbb{R}$-vector subspace of $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$.

Definition 3.2. A rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ is a cone in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ of the form $\sigma = \mathbb{R}_{>0} v_1 + \cdots + \mathbb{R}_{>0} v_n$ with $v_1, \ldots, v_n \in \underline{\mathbf{X}}(H)^\vee_{\mathbb{Q}}$.

Definition 3.3. A rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ is smooth with respect to the integral structure given by $\underline{\mathbf{X}}(H)^\vee$ if we have $\sigma = \mathbb{R}_{>0} v_1 + \cdots + \mathbb{R}_{>0} v_n$ with $v_1, \ldots, v_n$ forming a part of a $\mathbb{Z}$-basis of $\underline{\mathbf{X}}(H)^\vee$.

For a rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, let $\overline{\sigma}$ be the closure of σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$.

Definition 3.4. Let σ be a rational polyhedral cone in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$. A supporting hyperplane P of σ is a hyperplane in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ such that σ does not overlap with both sides of P. A face of σ is a rational polyhedral cone τ such that $\overline{\tau} = \overline{\sigma} \cap P$ for some supporting hyperplane P of σ.

Let $\langle\,\cdot\,,\,\cdot\,\rangle \colon \underline{\mathbf{X}}(H) \times \underline{\mathbf{X}}(H)^\vee_{\mathbb{R}} \to \mathbb{R}$ be the pairing defined by scalar extension of the natural pairing $\underline{\mathbf{X}}(H) \times \underline{\mathbf{X}}(H)^\vee \to \mathbb{Z}$. For a rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, we put

$$\sigma^\vee = \{x \in \underline{\mathbf{X}}(H) \mid \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\},$$
$$\sigma_0^\vee = \{x \in \underline{\mathbf{X}}(H) \mid \langle x, y\rangle > 0 \text{ for all } y \in \sigma\}.$$

Let Γ be any group acting on $\underline{\mathbf{X}}(H)$. It induces actions on H and $\underline{\mathbf{X}}(H)^\vee$.

Definition 3.5. Let C be a cone in $\underline{\mathbf{X}}(H)^{\vee}_{\mathbb{R}}$. A Γ-admissible rational polyhedral cone decomposition of C is a collection $\Sigma = \{\sigma_j\}_{j\in J}$ of nondegenerate rational polyhedral cones with some index set J satisfying the following:

1. C is the disjoint union of all the σ_j's in Σ. For each $j \in J$, the closure of σ_j in C is a disjoint union of σ_k's with $k \in J$.
2. Σ is invariant under the action of Γ on $\underline{\mathbf{X}}(H)^{\vee}_{\mathbb{R}}$, in the sense that Γ permutes the cones in Σ. Under this action, the set Σ/Γ of Γ-orbits is finite.

A Γ-admissible smooth rational polyhedral cone decomposition of C is a Γ-admissible rational polyhedral cone decomposition $\{\sigma_j\}_{j\in J}$ of C in which every σ_j is smooth.

Let $\mathcal{M}$ be an H-torsor over an S-scheme Z. Then $\mathcal{M}$ is relatively affine over Z, and the H-action on $\mathscr{O}_{\mathcal{M}}$ gives a decomposition $\mathscr{O}_{\mathcal{M}} = \bigoplus_{\chi\in\underline{\mathbf{X}}(H)} \mathscr{O}_{\mathcal{M},\chi}$, where $\mathscr{O}_{\mathcal{M},\chi}$ is the invertible sheaf of χ-eigenspaces under H-action, together with isomorphisms

$$\mathscr{O}_{\mathcal{M},\chi} \otimes_{\mathscr{O}_Z} \mathscr{O}_{\mathcal{M},\chi'} \xrightarrow{\sim} \mathscr{O}_{\mathcal{M},\chi+\chi'} \tag{3.1}$$

giving the $\mathscr{O}_Z$-algebra structure of $\mathscr{O}_{\mathcal{M}}$.

Definition 3.6. Let σ be a nondegenerate rational polyhedral cone in $\underline{\mathbf{X}}(H)^{\vee}_{\mathbb{R}}$. We put $\mathcal{M}(\sigma) = \underline{\mathrm{Spec}}_{\mathscr{O}_Z}(\bigoplus_{\chi\in\sigma^{\vee}} \mathscr{O}_{\mathcal{M},\chi})$ over Z, where $\bigoplus_{\chi\in\sigma^{\vee}} \mathscr{O}_{\mathcal{M},\chi}$ has the structure of an $\mathscr{O}_Z$-algebra given by the isomorphisms (3.1). The σ-stratum $\mathcal{M}_{\sigma}$ of $\mathcal{M}(\sigma)$ is the closed subscheme of $\mathcal{M}(\sigma)$ defined by the ideal sheaf $\bigoplus_{\chi\in\sigma_0^{\vee}} \mathscr{O}_{\mathcal{M},\chi} \subset \bigoplus_{\chi\in\sigma^{\vee}} \mathscr{O}_{\mathcal{M},\chi}$.

Remark 3.7. More generally, if Z is an algebraic stack, we can construct $\mathcal{M}(\sigma)$ and $\mathcal{M}_{\sigma}$ by [LM, Proposition 14.2.4].

§ 4. Degeneration data with principal level structure

We consider the following condition for a PEL-type $\mathcal{O}$-lattice.

Condition 4.1. Let $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ be a PEL-type $\mathcal{O}$-lattice. The action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$.

Lemma 4.2. *[Lan1, Lemma 5.2.7.5] Let n be a positive integer prime to $\Box$, and let $\mathtt{Z}_n$ be a fully symplectic-liftable admissible filtration on L/nL with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$. We take a fully symplectic lifting $\mathtt{Z}$ of $\mathtt{Z}_n$ and its extension $\mathtt{Z}_{\mathbb{A}^{\Box}}$ to $L \otimes_{\mathbb{Z}} \mathbb{A}^{\Box}$ as in Definition 2.8. Let $\mathtt{Z}_{\mathbb{R}}$ be the filtration on $L \otimes_{\mathbb{Z}} \mathbb{R}$ induced from $L \otimes_{\mathbb{Z}} \mathbb{A}^{\Box}$. We put $\mathtt{Z}_{-2,h(\mathbb{C})} = h(\mathbb{C})\mathtt{Z}_{-2,\mathbb{R}}$. Let $\mathtt{Z}^{\perp}_{-2,h(\mathbb{C})}$ be the orthogonal complement of $\mathtt{Z}_{-2,h(\mathbb{C})}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$ in $L \otimes_{\mathbb{Z}} \mathbb{R}$. Let $h_{-1} \colon \mathbb{C} \to \mathrm{End}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}}(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}})$ be the homomorphism induced from the restriction of h to $\mathtt{Z}^{\perp}_{-2,h(\mathbb{C})}$ by the natural isomorphism $\mathtt{Z}^{\perp}_{-2,h(\mathbb{C})} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}$. Then there is a PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$ such that*

1. *Condition 4.1 holds for* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$,

2. $[(L^{\mathsf{Z}_n})^\sharp : L^{\mathsf{Z}_n}]$ *is prime to* $\Box$,

3. *there are isomorphisms* $(L^{\mathsf{Z}_n} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}) \xrightarrow{\sim} (\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$ *and* $(L^{\mathsf{Z}_n} \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n}) \xrightarrow{\sim} (\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$.

Then the moduli problem $\mathsf{M}^{\mathsf{Z}_n}_n$ *over* $\operatorname{Spec}\mathcal{O}_{F_0,(\Box)}$ *defined by* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ *as in Definition 1.13 does not depend on the choice of* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ *up to isomorphisms.*

In the rest of this section, let $(\mathsf{Z}_n, \Phi_n, \delta_n)$ be a representative of a principal cusp label at level n for a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$, where $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$. The liftable splitting $\delta_n\colon \mathrm{Gr}^{\mathsf{Z}_n} \xrightarrow{\sim} L/nL$ defines two pairings

$$\langle\,\cdot\,,\,\cdot\,\rangle_{00,n}\colon \mathrm{Gr}^{\mathsf{Z}_n}_0 \times \mathrm{Gr}^{\mathsf{Z}_n}_0 \to (\mathbb{Z}/n\mathbb{Z})(1), \quad \langle\,\cdot\,,\,\cdot\,\rangle_{10,n}\colon \mathrm{Gr}^{\mathsf{Z}_n}_{-1} \times \mathrm{Gr}^{\mathsf{Z}_n}_0 \to (\mathbb{Z}/n\mathbb{Z})(1)$$

induced from $\langle\,\cdot\,,\,\cdot\,\rangle$ on L.

Definition 4.3. Let S be an $\mathcal{O}_{F_0,(\Box)}$-scheme, and let $(A, \lambda_A, i_A, (\alpha_n, \nu_n)) \in \mathsf{M}^{\mathsf{Z}_n}_n(S)$. Let $\varphi_{-1,n}\colon \mathrm{Gr}^{\mathsf{Z}_n}_{-1} \xrightarrow{\sim} A[n]$ be the composite of the isomorphism $\mathrm{Gr}^{\mathsf{Z}_n}_{-1} \simeq L^{\mathsf{Z}_n}/nL^{\mathsf{Z}_n}$ and $\alpha_n\colon L^{\mathsf{Z}_n}/nL^{\mathsf{Z}_n} \xrightarrow{\sim} A[n]$. We put $\nu(\varphi_{-1,n}) = \nu_n$. We often simply write $\varphi_{-1,n}$ for $(\varphi_{-1,n}, \nu(\varphi_{-1,n}))$. Then we define a pairing $a_{\Phi_n,\delta_n}\colon \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$ and a homomorphism $b_{\Phi_n,\delta_n}\colon \frac{1}{n}Y \to A^\vee[n]$ by requiring

$$a_{\Phi_n,\delta_n}(\frac{1}{n}y, \frac{1}{n}y') = \nu(\varphi_{-1,n})\bigl(\langle \varphi_{0,n}^{-1}(y), \varphi_{0,n}^{-1}(y')\rangle_{00,n}\bigr),$$
$$\mathrm{e}^{\lambda_A}(a, b_{\Phi_n,\delta_n}(\frac{1}{n}y)) = \nu(\varphi_{-1,n})\bigl(\langle \varphi_{-1,n}^{-1}(a), \varphi_{0,n}^{-1}(y)\rangle_{10,n}\bigr)$$

for $a \in A[n]$ and $y, y' \in Y$.

Definition 4.4. Let S be an $\mathcal{O}_{F_0,(\Box)}$-scheme. A degeneration data with a principal level n structure over S is a tuple

$$\bigl(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n)\bigr)$$

that satisfies the following:

1. $(A, \lambda_A, i_A, \varphi_{-1,n})$ is determined from some $(A, \lambda_A, i_A, (\alpha_n, \nu_n)) \in \mathsf{M}^{\mathsf{Z}_n}_n(S)$ as in Definition 4.3.

2. $c_n\colon \frac{1}{n}X \to A^\vee$ and $c_n^\vee\colon \frac{1}{n}Y \to A$ are $\mathcal{O}$-equivariant group homomorphisms satisfying the relation $\lambda_A c_n^\vee - c_n\phi_n = b_{\Phi_n,\delta_n}$ with $\phi_n\colon \frac{1}{n}Y \hookrightarrow \frac{1}{n}X$ induced by $\phi\colon Y \hookrightarrow X$. We write c for $c_n|_X$.

3. $\tau_n \colon \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} (c_n \times c)^* \mathcal{P}_A$ is a trivialization of biextensions over S which satisfies the relation

$$\tau_n(\frac{1}{n}y, \phi(y'))\tau_n(\frac{1}{n}y', \phi(y))^{-1} = a_{\Phi_n,\delta_n}(\frac{1}{n}y, \frac{1}{n}y') \in \boldsymbol{\mu}_n(S)$$

for $y, y' \in Y$ under the canonical isomorphism

$$(c_n^\vee(\frac{1}{n}y), c(\phi(y')))^* \mathcal{P}_A \otimes_{\mathscr{O}_S} (c_n^\vee(\frac{1}{n}y'), c(\phi(y)))^* \mathcal{P}_A^{\otimes -1} \simeq \mathscr{O}_S,$$

and satisfies the $\mathcal{O}$-compatibility $\tau_n(by, \chi) = \tau_n(y, b^\star \chi)$ for $y \in Y$ and $\chi \in X$ under the canonical isomorphism $(c_n^\vee(by), c(\chi))^* \mathcal{P}_A \simeq (c_n^\vee(y), c(b^\star \chi))^* \mathcal{P}_A$, where the two canonical isomorphisms are induced from the biextension structure of $\mathcal{P}_A$.

4. For each positive integer m such that $n|m$ and $\Box \nmid m$, suppose that we have lifted all the other data to some tuple

$$\big(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,m}, \varphi_{0,m}), (A, \lambda_A, i_A, \varphi_{-1,m}), \delta_m\big)$$

at level m, then the triple $(c_n, c_n^\vee, \tau_n)$ is also liftable to some $(c_m, c_m^\vee, \tau_m)$ that has the same kind of compatibility with other data as $(c_n, c_n^\vee, \tau_n)$ does.

Two degeneration data

$$\big(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n)\big),$$
$$\big(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n', (c_n', (c_n^\vee)', \tau_n')\big)$$

with principal level n structures over S are called equivalent if the following conditions are satisfied:

1. There is a liftable $\mathcal{O}$-linear isomorphism $\mathsf{z}_n \colon \mathrm{Gr}^{\mathsf{Z}_n} \xrightarrow{\sim} \mathrm{Gr}^{\mathsf{Z}_n}$ such that $\delta_n' = \delta_n \circ \mathsf{z}_n$ and

$$\mathsf{z}_{ij,n} = \begin{cases} \mathrm{id}_{\mathrm{Gr}_{-i}^{\mathsf{Z}_n}} & \text{if } i = j, \\ 0 & \text{if } i < j \end{cases}$$

for $0 \leq i, j \leq 2$, where $\mathsf{z}_{ij,n}$ is the composite

$$\mathrm{Gr}_{-j}^{\mathsf{Z}_n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}_n} \xrightarrow{\mathsf{z}_n} \mathrm{Gr}^{\mathsf{Z}_n} \twoheadrightarrow \mathrm{Gr}_{-i}^{\mathsf{Z}_n}.$$

2. If we consider the homomorphisms $d_n \colon \frac{1}{n}X \to A^\vee[n]$, $d_n^\vee \colon \frac{1}{n}Y \to A[n]$ and the pairing $e_n \colon \frac{1}{n}Y \times X \to \boldsymbol{\mu}_n(S)$ defined by the relations

$$\mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi)) = \nu(\varphi_{-1,n})\Big(\big((\varphi_{-2,n} \circ \mathsf{z}_{21,n} \circ \varphi_{-1,n}^{-1})(a)\big)(\chi)\Big),$$
$$d_n^\vee\Big(\frac{1}{n}y\Big) = (\varphi_{-1,n} \circ \mathsf{z}_{10,n} \circ \varphi_{0,n}^{-1})(y),$$
$$e_n\Big(\frac{1}{n}y, \chi\Big) = \nu(\varphi_{-1,n})\Big(\big((\varphi_{-2,n} \circ \mathsf{z}_{20,n} \circ \varphi_{0,n}^{-1})(y)\big)(\chi)\Big)$$

for each $a \in A[n]$, $\chi \in X$ and $y \in Y$, where $\mathrm{e}_{A[n]} \colon A[n] \times A^{\vee}[n] \to \boldsymbol{\mu}_n(S)$ is the Weil pairing, then $c_n' = c_n + d_n$, $(c_n^{\vee})' = c_n^{\vee} + d_n^{\vee}$ and the diagram

$$
\begin{tikzcd}
\mathcal{P}_A|_{((c_n^{\vee})'(\frac{1}{n}y),c(\chi))} \arrow[r, "e_n(\frac{1}{n}y,\chi)", "\sim"'] & \mathcal{P}_A|_{((c_n^{\vee})'(\frac{1}{n}y),c(\chi))} \\
 & \mathcal{P}_A|_{(c_n^{\vee}(\frac{1}{n}y),c(\chi))} \otimes_{\mathscr{O}_S} \mathcal{P}_A|_{(d_n^{\vee}(\frac{1}{n}y),c(\chi))} \arrow[u, "\wr"] \\
\mathscr{O}_S \arrow[uu, "\tau_n'(\frac{1}{n}y,\chi)", "\wr"'] & \mathscr{O}_S \otimes_{\mathscr{O}_S} \mathscr{O}_S \arrow[l, "\sim"'] \arrow[u, "\wr", "\tau_n(\frac{1}{n}y,\chi)\otimes r(d_n^{\vee}(\frac{1}{n}y),c_n(\frac{1}{n}\chi))"']
\end{tikzcd}
$$

is commutative for $\chi \in X$ and $y \in Y$, where $r(d_n^{\vee}(\frac{1}{n}y), c_n(\frac{1}{n}\chi))$ is the canonical isomorphism

$$\mathscr{O}_S \xrightarrow{\sim} \mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi))} = \mathcal{P}_A|_{(d_n^{\vee}(y),c_n(\frac{1}{n}\chi))} \xrightarrow{\sim} \mathcal{P}_A|_{(d_n^{\vee}(\frac{1}{n}y),c(\chi))}.$$

§ 5. Cone decomposition data

Let $\mathcal{H}$ be an open compact subgroup of $G(\widehat{\mathbb{Z}}^{\Box})$. In this section, let $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, where $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$. Then we can interpret $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ as a collection $\{(\mathsf{Z}_{H_n}, \Phi_{H_n}, \delta_{H_n})\}_n$ indexed by positive integers n such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$.

Definition 5.1. We choose an integer n such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, and a representative Z_n of the H_n-orbit Z_{H_n}. We put

$$P^{\mathrm{ess}}_{\mathsf{Z}_n} = \{(g_n, r_n) \in G^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z}) \mid g_n \mathsf{Z}_n = \mathsf{Z}_n\}.$$

We consider the map

$$p_{n,1} \colon P^{\mathrm{ess}}_{\mathsf{Z}_n} \to GL_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}_n}_{-1}) \times \mathbf{G}_{\mathrm{m}}(\mathbb{Z}/n\mathbb{Z});\ (g_n, r_n) \mapsto (\mathrm{Gr}^{\mathsf{Z}_n}_{-1}(g_n), r_n),$$

and we put $G^{\mathrm{ess}}_{h,\mathsf{Z}_n} = \operatorname{Im} p_{n,1}$. We take a PEL-type $\mathcal{O}$-lattice $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ as in Lemma 4.2. Then we have a natural identification $G^{\mathrm{ess}}_{h,\mathsf{Z}_n} \simeq G^{\mathrm{ess}}_{(L^{\mathsf{Z}_n},\langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n},h^{\mathsf{Z}_n})}(\mathbb{Z}/n\mathbb{Z})$. We put $H_{n,G^{\mathrm{ess}}_{h,\mathsf{Z}_n}} = p_{n,1}(H_n \cap P^{\mathrm{ess}}_{\mathsf{Z}_n})$. Let $\mathcal{H}_h$ be the preimage of $H_{n,G^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ under the surjection $G_{(L^{\mathsf{Z}_n},\langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n},h^{\mathsf{Z}_n})}(\mathbb{Z}^{\Box}) \twoheadrightarrow G^{\mathrm{ess}}_{h,\mathsf{Z}_n}$. Then we define an algebraic stack $\mathsf{M}^{Z_{\mathcal{H}}}_{\mathcal{H}}$ over $\mathcal{O}_{F_0,(\Box)}$ to be the moduli problem defined by $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ with level $\mathcal{H}_h$-structure. The isomorphism class of $\mathsf{M}^{Z_{\mathcal{H}}}_{\mathcal{H}}$ depends only on $Z_{\mathcal{H}}$.

Definition 5.2. Let S be an $\mathcal{O}_{F_0,(\Box)}$-scheme. A degeneration data with a level $\mathcal{H}$ structure over S is a tuple

$$\bigl(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}})\bigr),$$

which is a compatible system of étale locally defined H_n-orbits

$$\big\{\big(\mathsf{Z}_{H_n},(X,Y,\phi,\varphi_{-2,H_n},\varphi_{0,H_n}),(A,\lambda_A,i_A,\varphi_{-1,H_n}),\delta_{H_n},(c_{H_n},c^\vee_{H_n},\tau_{H_n})\big)\big\}$$

of equivalence classes of degeneration data with principal level structures over S indexed by positive integers n such that $\Box\nmid n$ and $\mathcal{U}^\Box(n)\subset\mathcal{H}$, where $(g_n,r_n)\in H_n$ sends a degeneration data

$$\big(\mathsf{Z}_n,(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n}),(A,\lambda_A,i_A,\varphi_{-1,n}),\delta_n,(c_n,c^\vee_n,\tau_n)\big)$$

with a principal level n structure to

$$\Big(g_n^{-1}\mathsf{Z}_n,\big(X,Y,\phi,r_n^{-1}(\varphi_{-2,n}\circ\mathrm{Gr}_{-2}(g_n)),\varphi_{0,n}\circ\mathrm{Gr}_0(g_n)\big),\\ (A,\lambda_A,i_A,\varphi_{-1,n}\circ\mathrm{Gr}_{-1}(g_n)),g_n^{-1}\circ\delta_n\circ\mathrm{Gr}(g_n),(c_n,c^\vee_n,\tau_n)\Big).$$

Proposition 5.3. *[Lan1, Proposition 6.2.4.7] We fix a choice of a representative $(\mathsf{Z}_\mathcal{H},\Phi_\mathcal{H},\delta_\mathcal{H})$ of a cusp label at level $\mathcal{H}$, where $\Phi_n=(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n})$. We consider the category $\mathsf{D}_\mathcal{H}$ fibered in groupoids over the category of locally Noetherian schemes over the algebraic stack $\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$ whose fiber over each locally Noetherian scheme S has as objects degeneration data*

$$\big(\mathsf{Z}_\mathcal{H},(X,Y,\phi,\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}}),(A,\lambda_A,i_A,\varphi_{-1,\mathcal{H}}),\delta_\mathcal{H},(c_\mathcal{H},c^\vee_\mathcal{H},\tau_\mathcal{H})\big)$$

with level $\mathcal{H}$ structures over S such that $(A,\lambda_A,i_A,\varphi_{-1,\mathcal{H}})$ is the pullback of the universal tuple over $\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$. Then we can construct a smooth separated relative scheme $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ over $\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$ with a natural action of $\Gamma_{\Phi_\mathcal{H}}$ such that the quotient $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}/\Gamma_{\Phi_\mathcal{H}}$ is isomorphic to $\mathsf{D}_\mathcal{H}$ as categories fibered in groupoids.

Further, we can construct a split torus $E_{\Phi_\mathcal{H}}$ over $\mathbb{Z}$ with a natural $\Gamma_{\Phi_\mathcal{H}}$-action, and the structural morphism $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$ factorizes as the composition $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to C_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$, where $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to C_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ is an $E_{\Phi_\mathcal{H}}$-torsor, and $C_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$ is a relative abelian scheme.

Remark 5.4. If $\mathcal{H}=\mathcal{U}^\Box(n)$ for a positive number n that is prime to $\Box$, then the character group of $E_{\Phi_\mathcal{H}}$ is the quotient group of

$$\Big(\Big(\frac{1}{n}Y\Big)\otimes_\mathbb{Z}X\Big)\Big/\Big(y\otimes\phi(y')-y'\otimes\phi(y),\ \Big(b\frac{1}{n}y\Big)\otimes\chi-\Big(\frac{1}{n}y\Big)\otimes(b^\star\chi)\Big)_{y,y'\in Y,\,\chi\in X,\,b\in\mathcal{O}}$$

by its torsion subgroup. In general, $E_{\Phi_\mathcal{H}}$ is a quotient of $E_{\Phi_{\mathcal{U}^\Box(n)}}$ for some positive integer n such that $\Box\nmid n$ and $\mathcal{U}^\Box(n)\subset\mathcal{H}$. Roughly speaking, $C_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$ parameterizes data $(c_\mathcal{H},c^\vee_\mathcal{H})$, and $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}\to C_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ parameterizes data $\tau_\mathcal{H}$.

We take $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}$, $C_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ and $E_{\Phi_\mathcal{H}}$ as in Proposition 5.3. We put $\mathbf{S}_{\mathbf{\Phi}_\mathcal{H}}=\underline{\mathbf{X}}(\mathbf{E}_{\mathbf{\Phi}_\mathcal{H}})$ and $\mathbf{S}^\vee_{\mathbf{\Phi}_\mathcal{H}}=\underline{\mathbf{X}}(\mathbf{E}_{\mathbf{\Phi}_\mathcal{H}})^\vee$. Let $(\mathbf{S}_{\mathbf{\Phi}_\mathcal{H}})^\vee_\mathbb{R}=\mathbf{S}^\vee_{\mathbf{\Phi}_\mathcal{H}}\otimes_\mathbb{Z}\mathbb{R}$.

Lemma 5.5. *The $\mathbb{R}$-vector space $(\mathbf{S}_{\mathbf{\Phi}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ is isomorphic to the space of Hermitian pairings $(\!|\cdot,\cdot|\!)\colon (Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}) \to B \otimes_{\mathbb{Q}} \mathbb{R}$. Here, a Hermitian pairing $(\!|\cdot,\cdot|\!)\colon (Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}) \to B \otimes_{\mathbb{Q}} \mathbb{R}$ means an $\mathbb{R}$-bilinear pairing such that $(\!|y,y'|\!) = (\!|y',y|\!)^{\star}$ and $(\!|y,by'|\!) = b(\!|y,y'|\!)$ for $b \in \mathcal{O}$.*

Proof. By the construction of $\mathbf{S}_{\mathbf{\Phi}_{\mathcal{H}}}$ in the proof of [Lan1, Proposition 6.2.4.7], the $\mathbb{R}$-vector space $(\mathbf{S}_{\mathbf{\Phi}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ is isomorphic to the space of symmetric $\mathbb{R}$-bilinear pairings $\langle\cdot,\cdot\rangle\colon (Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$ such that $\langle by, y'\rangle = \langle y, b^{\star}y'\rangle$ for $b \in \mathcal{O}$ (cf. [Lan1, (6.2.3.5) and Lemma 6.2.4.4]). The space of Hermitian pairings as in the claim is isomorphic to the space of symmetric $\mathbb{R}$-bilinear pairings as above by sending $(\!|\cdot,\cdot|\!)$ to the pairing $(y,y') \mapsto \mathrm{Tr}_{(B\otimes\mathbb{R})/\mathbb{R}}(\!|y,y'|\!)$ by [Lan1, Lemma 1.1.4.5]. □

Definition 5.6. We say that the radical of a positive semi-definite Hermitian pairing $(\!|\cdot,\cdot|\!)\colon (Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}) \to B \otimes_{\mathbb{Q}} \mathbb{R}$ is admissible if it is the $\mathbb{R}$-span of some admissible $\mathcal{O}$-submodule Y' of Y.

We define $\mathbf{P}_{\Phi_{\mathcal{H}}}$ to be the subset of $(\mathbf{S}_{\mathbf{\Phi}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ corresponding to positive semi-definite Hermitian pairings with admissible radicals. Let $\mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ be the subset of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ corresponding to positive definite Hermitian pairings. Then $\mathbf{P}_{\Phi_{\mathcal{H}}}$ and $\mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ are cones in $(\mathbf{S}_{\mathbf{\Phi}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$.

For a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ and $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ such that $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, we can construct $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ and $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ by Remark 3.7, and let $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ be the formal completion of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ along $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$.

Definition 5.7. Let σ be any nondegenerate rational polyhedral cone in $\mathbf{P}_{\Phi_{\mathcal{H}}}$. The group $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is defined as the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ consisting of elements that maps σ to itself under the natural action of $\Gamma_{\Phi_{\mathcal{H}}}$ on $\mathbf{P}_{\Phi_{\mathcal{H}}}$.

Remark 5.8. For a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ and $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ such that $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, the group $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ naturally acts on $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ by the construction.

Definition 5.9. Let $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ be representatives of cusp labels at level $\mathcal{H}$. Let σ and σ' be nondegenerate rational polyhedral cones in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ and $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ respectively. We say that the two triples $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ are equivalent if the following hold:

1. $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ are equivalent by $\gamma_X\colon X' \xrightarrow{\sim} X$ and $\gamma_Y\colon Y \xrightarrow{\sim} Y'$.
2. The isomorphism $\mathbf{P}_{\Phi'_{\mathcal{H}}} \xrightarrow{\sim} \mathbf{P}_{\Phi_{\mathcal{H}}}$ induced by γ_X and γ_Y sends σ' to σ.

Definition 5.10. Let $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ and $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ be equivalent classes of triples as in Definition 5.9. We say that $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ is a face of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ if there are representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ of $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ satisfying the following:

1. There is a surjection $(s_X, s_Y)\colon (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \to (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ between representatives of cusp labels.

2. The image of σ' under the embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ induced by (s_X, s_Y) is contained in the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of a face of σ.

Definition 5.11. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be representatives of cusp labels at level $\mathcal{H}$. Let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\mathbf{\Phi}'_{\mathcal{H}}}$). A surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \to (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ means a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \to (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ that induces an embedding $\mathbf{P}_{\mathbf{\Phi}'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ such that the restriction $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\mathbf{\Phi}'_{\mathcal{H}}}}$ of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ to $\mathbf{P}_{\mathbf{\Phi}'_{\mathcal{H}}}$ is the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\mathbf{\Phi}'_{\mathcal{H}}}$.

Definition 5.12. An admissible boundary component of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is the image of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ under the embedding $(\mathbf{S}_{\mathbf{\Phi}'_{\mathcal{H}}})^{\vee}_{\mathbb{R}} \hookrightarrow (\mathbf{S}_{\mathbf{\Phi}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ defined by some surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \to (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Condition 5.13. A cone decomposition $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j \in J}$ of $\mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ satisfies that, for each $j \in J$ if $\gamma\overline{\sigma}_j \cap \overline{\sigma}_j \neq \{0\}$ for some $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$ then γ acts as the identity on the smallest admissible boundary component of $\mathbf{P}_{\mathbf{\Phi}_{\mathcal{H}}}$ containing σ_j.

Definition 5.14. A compatible choice of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ means the following choices:

1. We choose a complete set of representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels at level $\mathcal{H}$.

2. We choose a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ satisfying Condition 5.13 for each $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ chosen above so that the cone decompositions $\Sigma_{\Phi_{\mathcal{H}}}$ and $\Sigma_{\Phi'_{\mathcal{H}}}$ defines a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \to (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ for every surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \to (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Proposition 5.15. *[Lan1, Proposition 6.3.3.5] There exists a compatible choice of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$.*

§ 6. Toroidal compactification

Definition 6.1. Let S be a normal locally Noetherian algebraic stack. A tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over S is called a degeneration family of type $\mathsf{M}_{\mathcal{H}}$ if the following hold:

1. There exists a dense sub-algebraic stack S_1 of S such that S_1 is defined over $\mathcal{O}_{F_0,(\Box)}$.

2. G is a relative semi-abelian scheme over S whose restriction G_{S_1} to S_1 is a relative abelian scheme.

3. $\lambda\colon G \to G^\vee$ is a homomorphism that induces a prime-to-$\Box$ polarization λ_{S_1} of G_{S_1} by restriction, where $G^\vee$ is the relative dual semi-abelian scheme of G.

4. $i\colon \mathcal{O} \to \mathrm{End}_S(G)$ is a homomorphism of algebra that defines an $\mathcal{O}$-structure $i_{S_1}\colon \mathcal{O} \to \mathrm{End}_{S_1}(G_{S_1})$ of (G_{S_1}, λ_{S_1}) by restriction.

5. $\underline{\mathrm{Lie}}_{G_{S_1}/S_1}$ satisfies the determinant condition defined by $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\cdot\,,\,\cdot\rangle, h)$.

6. $\alpha_{\mathcal{H}}$ is a level $\mathcal{H}$ structure of $(G_{S_1}, \lambda_{S_1}, i_{S_1})$ of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Box}, \langle\cdot\,,\,\cdot\rangle)$.

Theorem 6.2. *[Lan1, Theorem 6.4.1.1] We assume that $(L, \langle\cdot\,,\,\cdot\rangle, h)$ satisfies Condition 4.1. To each compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$, there is a proper smooth algebraic stack $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over $\mathcal{O}_{F_0,(\Box)}$ with a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ of type $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ satisfying the following:*

1. *$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ contains $\mathsf{M}_{\mathcal{H}}$ as an open dense sub-algebraic stack.*

2. *The restriction $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}})$ of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ is the universal tuple over $\mathsf{M}_{\mathcal{H}}$.*

3. *$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ has a stratification*
$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$$
by locally closed sub-algebraic stacks, where $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ run through the equivalence classes of triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ with $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ such that $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$. In this stratification, the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ lies in the closure of the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ if and only if $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ is a face of $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$. Further, $\mathsf{M}_{\mathcal{H}}$ is an open dense stratum in this stratification corresponding to the unique class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ such that $X = Y = 0$.

4. *The complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ with its reduced structure is a relative Cartier divisor with normal crossings.*

5. *The formal completion of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ along its $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is canonically isomorphic to the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ for any representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$.*

Further, if $\mathcal{H}$ is neat, then $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is an algebraic space.

Remark 6.3. In Theorem 6.2, if $\mathcal{H}$ is neat, we can take a compatible choice Σ so that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is a projective scheme by [Lan1, Proposition 7.3.1.4 and Theorem 7.3.3.4].

§ 7. Applications

We assume the following conditions:

– $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.

– $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

– $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ satisfies Condition 4.1.

We take a compact open neat subgroup $\mathcal{H}^p \subset G(\widehat{\mathbb{Z}}^p)$. Let m be a non-negative integer. We put $\mathcal{U}_p(m) = \mathrm{Ker}(G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^m\mathbb{Z}))$ and $\mathcal{H}(m) = \mathcal{H}^p\mathcal{U}_p(m) \subset G(\widehat{\mathbb{Z}})$. We choose a place v of F_0 dividing p. Then we can construct an integral model $\mathcal{M}_{\mathcal{H}(m)}$ of $\mathsf{M}_{\mathcal{H}(m)} \otimes_{F_0} F_{0,v}$ over $\mathcal{O}_{F_{0,v}}$ as in [Man, §6]. Let κ_v be the residue field of $\mathcal{O}_{F_{0,v}}$. We write $\mathcal{M}_{\mathcal{H}(m),\overline{v}}$ for $\mathcal{M}_{\mathcal{H}(m)} \otimes_{\mathcal{O}_{F_{0,v}}} \overline{\kappa}_v$. We take a prime number ℓ different from p. Let ξ be an algebraic representation of the algebraic group $G_{\mathbb{Q}_\ell}$ on a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Let $\mathcal{L}_{\xi,m}$ be the smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathsf{M}_{\mathcal{H}(m)}$ associated to ξ. Then we have the following theorem.

Theorem 7.1. *The kernel and cokernel of the canonical homomorphism*

$$\varinjlim_m H^i_c(\mathcal{M}_{\mathcal{H}(m),\overline{v}}, R\psi\mathcal{L}_{\xi,m}) \longrightarrow \varinjlim_m H^i_c(\mathsf{M}_{\mathcal{H}(m)} \otimes_{F_0} \overline{F}_{0,v}, \mathcal{L}_{\xi,m})$$

have no supercuspidal subquotient as $G(\mathbb{Q}_p)$-representations for any non-negative integer i.

In [IM], we construct potentially good reduction loci of pre-abelian Shimura varieties. For $\mathsf{M}_{\mathcal{H}(m)}$, the potentially good reduction locus $\mathsf{M}^{\mathrm{pg}}_{\mathcal{H}(m)}$ coincides with the Raynaud generic fiber of the formal completion of $\mathcal{M}_{\mathcal{H}(m)}$ along its special fiber. A main theorem of [IM] says that kernel and cokernel of the canonical homomorphism

$$\varinjlim_m H^i_c(\mathsf{M}^{\mathrm{pg}}_{\mathcal{H}(m)} \otimes_{F_{0,v}} \overline{F}_{0,v}, \mathcal{L}_{\xi,m}) \longrightarrow \varinjlim_m H^i_c(\mathsf{M}_{\mathcal{H}(m)} \otimes_{F_0} \overline{F}_{0,v}, \mathcal{L}_{\xi,m})$$

have no supercuspidal subquotient as $G(\mathbb{Q}_p)$-representations for any non-negative integer i. Therefore we obtain Theorem 7.1 because we have a natural isomorphism

$$H^i_c(\mathcal{M}_{\mathcal{H}(m),\overline{v}}, R\psi\mathcal{L}_{\xi,m}) \cong H^i_c(\mathsf{M}^{\mathrm{pg}}_{\mathcal{H}(m)} \otimes_{F_{0,v}} \overline{F}_{0,v}, \mathcal{L}_{\xi,m})$$

for any non-negative integer i.

To prove the main theorem of [IM], we use toroidal compactifications of some Shimura varieties. In fact, in a proof given in [IM], we use only toroidal compactifications of Siegel modular varieties by considering embeddings of Shimura varieties of Hodge type into Siegel modular varieties. However, in the situation of Theorem 7.1, we can also argue directly using toroidal compactifications of Shimura varieties of PEL type. We explain an outline of a proof using toroidal compactifications of Shimura varieties of PEL type.

A main point of the proof is to construct a partition of $\mathsf{M}_{\mathcal{H}(m)} \setminus \mathsf{M}^{\mathrm{pg}}_{\mathcal{H}(m)}$. First, we construct a partition of $\mathsf{M}_{\mathcal{H}(m)} \setminus \mathsf{M}^{\mathrm{pg}}_{\mathcal{H}(m)}$ according to degeneration of abelian varieties. More explicitly, we make a partition using the inverse images in $\mathsf{M}_{\mathcal{H}(m)}$ of the tubular neighborhoods of the special fibers of the strata in a stratification of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}(0),\Sigma}$ as in Theorem 6.2.3, where Σ is taken so that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}(0),\Sigma}$ is a projective scheme (cf. Remark 6.3). Next, we make a finer partition using level structures. Then we can show that a difference of limits of cohomology of $\mathsf{M}_{\mathcal{H}(m)}$ and $\mathsf{M}^{\mathrm{pg}}_{\mathcal{H}(m)}$ with respect to m comes from proper parabolic inductions of limits of cohomology of some loci in the constructed partition.

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