RIMS Kôkyûroku Bessatsu
B44 (2013), 41–49

On a certain nilpotent extension over $\mathbb{Q}$ of degree 64 and the 4-th multiple residue symbol

By

Fumiya AMANO*

Abstract

This is the report of my talk at RIMS conference "Algebraic Number Theory and Related Topics". I would like to thank again the organizers for giving me an opportunity to participate in the conference.

§0 Background and main results

In this section, we review briefly the historical background on the subject with which we are concerned.

As is well known, for distinct odd prime numbers p_1 and p_2, the Legendre symbol $\left(\frac{p_1}{p_2}\right)$ describes the decomposition law of p_2 in the quadratic extension $\mathbb{Q}(\sqrt{p_1})/\mathbb{Q}$ as follows:

$$\left(\frac{p_1}{p_2}\right) = \begin{cases} 1 \cdots & \exists x \in \mathbb{Z} \quad \text{s.t.} \quad x^2 \equiv p_1 \pmod{p_2}, \\ -1 \cdots & \text{otherwise.} \end{cases}$$
$$= \begin{cases} 1 \cdots & p_2 \text{ is completely decomposed in } \mathbb{Q}(\sqrt{p_1})/\mathbb{Q}, \\ -1 \cdots & \text{otherwise.} \end{cases}$$

In 1939, Rédei ([R]) introduced a certain triple symbol, called the Rédei symbol, with the intention of a generalization of the Legendre symbol and Gauss' genus theory. For distinct prime numbers p_1, p_2 and p_3 satisfying

$$p_i \equiv 1 \pmod 4 \ (i = 1, 2, 3), \ \left(\frac{p_i}{p_j}\right) = 1 \ (1 \leqq i \neq j \leqq 3),$$

Received March 30, 2012. Revised April 12, 2012.
2000 Mathematics Subject Classification(s): 11R.
Key Words: Rédei symbol, multiple residue symbol.

*Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN.
e-mail: `f-amano@math.kyushu-u.ac.jp`

the Rédei symbol $[p_1, p_2, p_3]$ is defined as follows:

$$[p_1, p_2, p_3] = \begin{cases} 1 \cdots & p_3 \text{ is completely decomposed in a certain} \\ & D_8\text{-extension } K/\mathbb{Q}, \\ -1 \cdots & \text{otherwise.} \end{cases}$$

Here a D_8-extension means a Galois extension whose Galois group is the dihedral group of order 8. We will give the precise definition of the extension $K/\mathbb{Q}$ in §1. We note that all prime numbers ramified in $K/\mathbb{Q}$ are p_1 and p_2.

Although a meaning of the Rédei symbol had been obscure for a long time, in 2000, M. Morishita ([Mo1,2]) interpreted the Rédei symbol as an arithmetic analogue of a mod 2 triple linking number, following the analogies between knots and primes. In fact, he introduced arithmetic analogues $\mu_2(12\cdots r) \in \mathbb{Z}/2\mathbb{Z}$ of Milnor's link invariants (higher order linking numbers) for prime numbers $p_1, \cdots, p_r$ and showed

$$\left(\frac{p_1}{p_2}\right) = (-1)^{\mu_2(12)}, \ [p_1, p_2, p_3] = (-1)^{\mu_2(123)}.$$

Now, as we shall see in §2, the analogy with knot theory suggests the following problem (conjecture):

Problem. Introduce the multiple residue symbol $[p_1, p_2, \ldots, p_r]$, which should be $(-1)^{\mu_2(12\cdots r)}$ and describe the decomposition law of p_r in a certain

$$N_r(\mathbb{F}_2) = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \middle| \, * \in \mathbb{F}_2 \right\} - \text{extension } K/\mathbb{Q},$$

unramified outside $p_1, \ldots, p_{r-1}$ and ∞. (Note that $\mathbb{Z}/2\mathbb{Z} = N_2(\mathbb{F}_2)$ and $D_8 = N_3(\mathbb{F}_2)$)

My main result is to solve the above problem for the case $r = 4$, namely, we shall
(1) construct concretely an $N_4(\mathbb{F}_2)$-extension $K/\mathbb{Q}$, and
(2) introduce the 4-th multiple residue symbol $[p_1, p_2, p_3, p_4]$ and prove

$$[p_1, p_2, p_3, p_4] = (-1)^{\mu_2(1234)}.$$

§1 Rédei's D_8-extension and triple symbol

Let p_1 and p_2 be distinct prime numbers satisfying

$$p_i \equiv 1 \pmod 4 \ (i = 1, 2), \quad \left(\frac{p_i}{p_j}\right) = 1 \ (1 \leq i \neq j \leq 2). \tag{1.1}$$

By (1.1), there are integers x, y and z satisfying

$$\begin{cases} x^2 - p_1 y^2 - p_2 z^2 = 0. \\ \text{g.c.d}(x, y, z) = 1, \quad y \equiv 0 \pmod 2, \quad x - y \equiv 1 \pmod 4. \end{cases} \tag{1.2}$$

We fix such a triple $\boldsymbol{a} = (x, y, z)$ satisfying (1.2) and then set

$$k_{\boldsymbol{a}} := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}), \quad \alpha = x + y\sqrt{p_1}.$$

The following theorem is due to L. Rédei.

Theorem 1.3 ([R])**.** *The extension $k_{\boldsymbol{a}}/\mathbb{Q}$ is a D_8-extension where all ramified prime numbers are p_1 and p_2 with ramification index* 2.

The fact that $k_{\boldsymbol{a}}$ is independent of choice of $\boldsymbol{a} = (x, y, z)$ was also shown in [R] in an obscure manner. We proved this fact clearly.

Theorem 1.4 ([A1])**.** *A field $k_{\boldsymbol{a}}$ is independent of a choice of $\boldsymbol{a} = (x, y, z)$, namely, depends only on a set $\{p_1, p_2\}$.*

By Theorem 1.4, we denote $k_{\boldsymbol{a}}$ by $k_{\{p_1, p_2\}}$ and call it the *Rédei extension* associated to $\{p_1, p_2\}$.

The following theorem of mine characterizes the Rédei extension by the information on the Galois group and ramification data.

Theorem 1.5 ([A1])**.** *Let p_1 and p_2 be prime numbers satisfying* (1.1). *Then the following conditions on a number field K are equivalent*:
(1) *K is the Rédei extension $k_{\{p_1, p_2\}}$.*
(2) *K is a D_8-extension over $\mathbb{Q}$ such that all prime numbers ramified in $K/\mathbb{Q}$ are p_1 and p_2 with ramification index* 2.

Next, let p_1, p_2 and p_3 be distinct prime numbers satisfying

$$p_i \equiv 1 \pmod 4 \ (i = 1, 2, 3), \ \left(\frac{p_i}{p_j}\right) = 1 \ (1 \leq i \neq j \leq 3).$$

We then define the *Rédei triple symbol* $[p_1, p_2, p_3]$ by

$$[p_1, p_2, p_3] = \begin{cases} \;\;1 \cdots & \text{if } p_3 \text{ is completely decomposed in } k_{\{p_1,p_2\}}/\mathbb{Q}, \\ -1 \cdots & \text{otherwise.} \end{cases}$$

The following reciprocity law was shown by Rédei and we gave another simple proof.

Theorem 1.6 ([R], [A1]). *For any permutation i, j, k of $1, 2, 3$, we have*

$$[p_1, p_2, p_3] = [p_i, p_j, p_k].$$

§2 Milnor invariants

In this section, we recall the arithmetic Milnor invariants for primes, which are arithmetic analogues of Milnor invariants of a link, introduced by M. Morishita ([Mo1,2]). The underlying idea is based on the following analogies between knots and primes (cf. [Mo3]):

knot $\mathcal{K} : S^1 \hookrightarrow \mathbb{R}^3$	prime $\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$
link $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$	finite set of primes $S = \{p_1, \ldots, p_r\}$
$X_{\mathcal{L}} = \mathbb{R}^3 \setminus \mathcal{L}$	$X_S = \mathrm{Spec}(\mathbb{Z}) \setminus S$
link group $G_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$	Galois group with restricted ramification $G_S = \pi_1^{\text{ét}}(X_S) = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ $\mathbb{Q}_S$: maximal extension over $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$

2.1. Link case. Let $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$ be an r-component link in $\mathbb{R}^3$. Let $X_{\mathcal{L}} = \mathbb{R}^3 \setminus \mathcal{L}$ and $G_{\mathcal{L}} := \pi_1(X_{\mathcal{L}})$. Let F be the free group on the words $x_1 \ldots, x_r$ where x_i represents a meridian of $\mathcal{K}_i$. For a group G, we let $G^{(1)} := G, G^{(d+1)} := [G, G^{(d)}]$ $(d > 1)$. The following theorem is due to J. Milnor.

Theorem 2.1.1 ([Mi2]). *For each $d \in \mathbb{N}$, there is $y_i^{(d)} \in F$ such that*

$$\begin{aligned} G_{\mathcal{L}}/G_{\mathcal{L}}^{(d)} &= \langle x_1, \ldots, x_r \mid [x_1, y_1^{(d)}] = \cdots = [x_r, y_r^{(d)}] = 1,\ F^{(d)} = 1 \rangle, \\ y_j^{(d)} &\equiv y_j^{(d+1)} \mod F^{(d)}, \end{aligned}$$

where $y_j^{(d)}$ is a word representing a longitude of $\mathcal{K}_j$ in $G_{\mathcal{L}}/G_{\mathcal{L}}^{(d)}$.

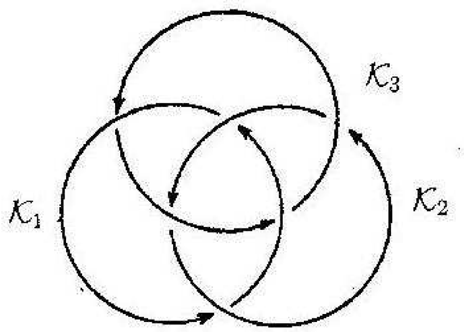
We define the *Milnor numbers* by

$$\mu(i_1 \cdots i_n j) := \epsilon\left(\frac{\partial^n y_j^{(d)}}{\partial x_{i_1} \cdots \partial x_{i_n}}\right).$$

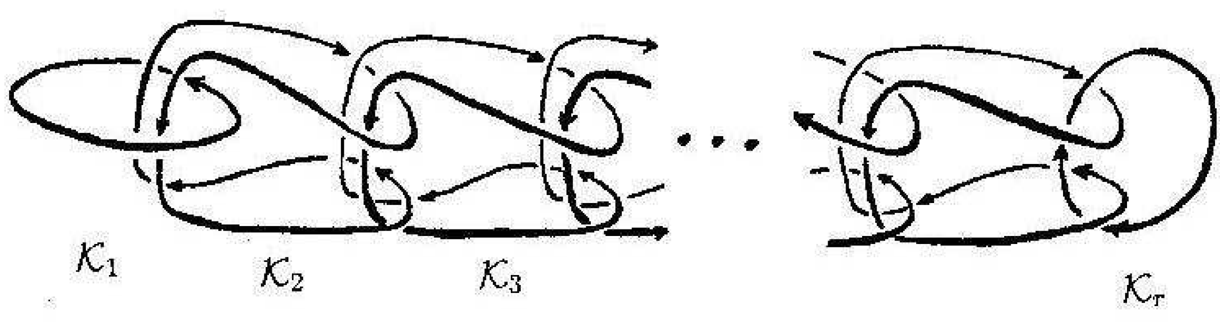
where $\partial/\partial x_i : \mathbb{Z}[F] \to \mathbb{Z}[F]$ is the Fox derivative ([F]) and $\epsilon_{\mathbb{Z}[F]} : \mathbb{Z}[F] \to \mathbb{Z}$ is the augmentation map. Note that the right hand side is independent of d for large enough d. We set $\mu(i) := 0$.

We have $\mu(ij) = \mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j)$ $(i \neq j)$, the linking number of $\mathcal{K}_i$ and $\mathcal{K}_j$, and it can be shown that $\mu(I)$ is an invariant of a link $\mathcal{L}$ if $\mu(J) = 0$ for any J with $|J| < |I|$.

Example 2.1.2. Let $\mathcal{L} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$ be the following *Borromean rings*:



Then $\mu(I) = 0$ if $|I| \leq 2$ and $\mu(123) = 1$. More generally, let $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$ be the following link, called the *Milnor link*:



Then $\mu(I) = 0$ if $|I| \leq r - 1$ and $\mu(12 \cdots r) = 1$.

A meaning of Milnor invariants in covering spaces is given as follows.

Theorem 2.1.3 ([Mo3, 8.2], [Mu]). *For $r \geq 2$, assume $\mu(J) = 0$ for any J with $|J| < r$. Then there is a Galois covering $M \to S^3$ ramified over $\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_{r-1}$ with Galois group*

$$N_r(\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z} & \cdots & \mathbb{Z} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{Z} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

such that $\mathcal{K}_r$ is completely decomposed in $M \to S^3$ if and only if $\mu(12\cdots r) = 0$.

This theorem suggests us to consider an $N_r(\mathbb{F}_2)$-extension in the arithmetic side as explained in §0.

2.2. Primes case. Let $S = \{p_1, \cdots, p_r\}$ be a set of r distinct odd prime numbers. Let $X_S = \mathrm{Spec}(\mathbb{Z}) \setminus S$ and $G_S(2)$ the maximal pro-2 quotient of $G_S := \pi_1^{\text{ét}}(\mathrm{Spec}(X_S))$. Let $\hat{F}$ denote the free pro-2 group on the words $x_1, \ldots, x_r$ where x_i represents a monodromy over p_i. The following theorem, which is due to H. Koch, may be regarded as an arithmetic analogue of Milnor's Theorem 2.1.1.

Theorem 2.2.1 ([K]). *We have*

$$G_S(2) = \langle x_1, \ldots, x_r \mid x_1^{p_1-1}[x_1, y_1] = \cdots = x_r^{p_r-1}[x_r, y_r] = 1 \rangle,$$

where $y_j \in \hat{F}$ is the pro-2 word representing a Frobenius auto. over p_j.

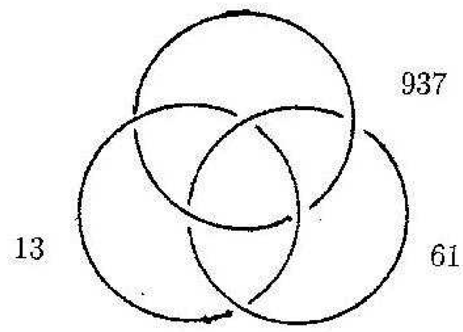
We then define the mod 2 *Milnor numbers* by

$$\mu_2(i_1 \cdots i_n j) := \hat{\epsilon}\left(\frac{\partial^n y_j}{\partial x_{i_1} \cdots \partial x_{i_n}}\right) \mod 2,$$

where $\partial/\partial x_i : \mathbb{Z}_2[[\hat{F}]] \to \mathbb{Z}_2[[\hat{F}]]$ is the pro-2 Fox derivative ([I], [O]) and $\hat{\epsilon} : \mathbb{Z}_2[[\hat{F}]] \to \mathbb{Z}_2$ is the augmentation map. We set $\mu_2(i) := 0$.

We have $(-1)^{\mu_2(ij)} = \left(\frac{p_i}{p_j}\right)$, and it can be shown that $\mu_2(I)$ is an invariant of S if $\mu_2(J) = 0$ for any J with $|J| < |I|$ and $2 \leq |I| \leq 2^{e_S}$ where $e_S := \max\{e \mid p_i \equiv 1 \mod 2^e \ (1 \leq i \leq r)\}$

Example 2.2.2 ([V]). Let $(p_1, p_2, p_3) = (13, 61, 937)$. Then we have $\mu_2(I) = 0$ if $|I| \leq 2$ and $\mu_2(123) = 1$. This triple of primes looks like Borromean rings in Example 2.1.2:



As in the link case, we have the following

Theorem 2.2.3 ([Mo1,2]). *For $2 \le r \le 2^{es}$, assume $\mu_2(J) = 0$ for any J with $|J| < r$. Then there is a Galois extension $K/\mathbb{Q}$ ramified over $p_1, \cdots, p_{r-1}$ with Galois group*

$$N_r(\mathbb{F}_2) = \begin{pmatrix} 1 & \mathbb{F}_2 & \cdots & \mathbb{F}_2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{F}_2 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

such that p_r is completely decomposed in $K/\mathbb{Q}$ if and only if $\mu_2(12\cdots r) = 0$.

For $r = 2$ and 3, K is given by $\mathbb{Q}(\sqrt{p_1})$ and the Rédei extension $k_{\{p_1,p_2\}}$ associated to $\{p_1, p_2\}$, respectively. In the next section, we give a concrete construction of $K/\mathbb{Q}$ for $r = 4$ and an arithmetic interpretation of $\mu_2(1234)$.

§3 $N_4(\mathbb{F}_2)$-extension and the 4-th multiple residue symbol

Let p_1, p_2, p_3 and p_4 be distinct odd prime numbers satisfying

$$\begin{cases} p_i \equiv 1 \pmod 4 \ (i = 1,2,3,4), \ \left(\dfrac{p_i}{p_j}\right) = 1 \ (1 \le i \ne j \le 4), \\ [p_i, p_j, p_k] = 1 \ (i,j,k: \text{ distinct}). \end{cases} \tag{3.1}$$

Let $k_{\{p_1,p_2\}} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha})$ (resp. $k_{\{p_3,p_2\}} = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_3}, \sqrt{\beta})$) be the Rédei extension associated to $\{p_1, p_2\}$ (resp. $\{p_3, p_2\}$).
By (3.1), we have a non-trivial integral solution (X, Y, Z) in $\mathbb{Q}(\sqrt{p_1})$ satisfying

$$X^2 - p_3 Y^2 - \alpha Z^2 = 0.$$

We then let

$$K := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\theta}) = k_{\{p_1,p_2\}} k_{\{p_3,p_2\}}(\sqrt{\theta}), \quad \theta := X + Y\sqrt{p_3}.$$

Theorem 3.2 ([A2]). *The extension $K/\mathbb{Q}$ is an $N_4(\mathbb{F}_2)$-extension unramified outside p_1, p_2, p_3 and ∞.*

The proof of the assertion on the ramification is hard. For the details, we refer to [A2].

We define the 4-*th multiple residue symbol* by

$$[p_1, p_2, p_3, p_4] = \begin{cases} 1 \cdots & p_4 \text{ is completely decomposed in } K/\mathbb{Q}, \\ -1 \cdots & \text{otherwise.} \end{cases}$$

Since $K \subset \mathbb{Q}_S$ for $S = \{p_1, p_2, p_3, p_4\}$ by Theorem 3.2, we can relate the Milnor invariant $\mu_2(1234)$ with our symbol $[p_1, p_2, p_3, p_4]$. As desired, we have the following.

Theorem 3.3 ([A2]). *We have*

$$(-1)^{\mu_2(1234)} = [p_1, p_2, p_3, p_4].$$

For the proof, we use a group presentation of $N_4(\mathbb{F}_2)$ which Y. Mizusawa kindly computed using GAP.

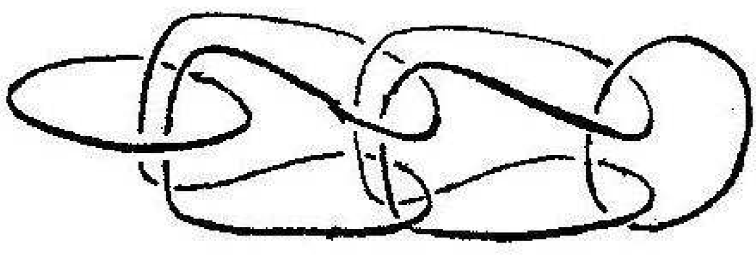
Example 3.4. Let $(p_1, p_2, p_3, p_4) := (5, 8081, 101, 449)$. Then we have

$$\begin{aligned} k_{\{p_1,p_2\}} &= \mathbb{Q}(\sqrt{5}, \sqrt{8081}, \sqrt{241 + 100\sqrt{5}}), \\ k_{\{p_3,p_2\}} &= \mathbb{Q}(\sqrt{8081}, \sqrt{101}, \sqrt{1009 + 100\sqrt{101}}), \\ K &= k_{\{p_1,p_2\}} \cdot k_{\{p_3,p_2\}}(\sqrt{25 + 2\sqrt{5} + 2\sqrt{101}}), \end{aligned}$$

and

$$\left(\frac{p_i}{p_j}\right) = 1 \ \ (1 \le i \ne j \le 4), \ \ [p_i, p_j, p_k] = 1 \ \ (i, j, k : \text{ distinct}),$$
$$[p_1, p_2, p_3, p_4] = -1.$$

In view of Example 2.1.2, this 4-tuple of primes looks like a Milnor link:



5 8081 101 449

Finally, we note that we can show the shuffle relation for $[p_1, p_2, p_3, p_4]$ ([Mo3, 8.4]) and $[p_1, p_2, p_3, p_4] = [p_3, p_2, p_1, p_4]$.

References

[A1] F. Amano, On Rédei's dihedral extension and triple reciprocity law, (2012), to appear in Proc. Japan Acad.

[A2] F. Amano, On a certain nilpotent extension over $\mathbb{Q}$ of degree 64 and the 4-th multiple residue symbol, (2012), to appear in Tohoku Math. J.

[F] R. H. Fox, Free differential calculus. I: Derivation in the free group ring, Ann. of Math., **57** (1953), 547-560.

[I] Y. Ihara, On Galois representations arising from towers of coverings of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, Invent. Math., **86** (1986), 427 - 459.

[K] H. Koch, On p-extension with given ramification, Appendix in: K. Haberland, Galois cohomology of algebraic number fields, VEB Deutscher Verlag der Wissenschaften, Berlin, (1978), 89-126.

[Mi1] J. Milnor, Link groups, Ann. of Math., **59** (1954), 177-195.

[Mi2] J. Milnor, Isotopy of links, Algebraic Geometry and Topology, In: A symposium in honor of S. Lefschetz (edited by R.H. Fox, D. C. Spencer and A. W. Tucker) Princeton Univ. Press, Princeton, (1957), 280-306.

[Mo1] M. Morishita, On certain analogies between knots and primes, J. Reine Angew. Math., **550** (2002), 141-167.

[Mo2] M. Morishita, Milnor invariants and Massey products for prime numbers, Compositio Math., **140** (2004), 69-83.

[Mo3] M. Morishita, Knots and Primes – An Introduction to Arithmetic Topology, Universitext, Springer, 2012.

[Mu] K. Murasugi, Nilpotent coverings of links and Milnor's invariant, In: Low-dimensional topology, London Math. Soc. Lecture Note Ser., **95**, Cambridge Univ. Press, Cambridge (1985), 106-142.

[O] T. Oda, Note on meta-abelian quotients of pro-l free groups, (1985), preprint.

[R] L. Rédei, Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper I, J. Reine Angew. Math., **180** (1939), 1-43.

[V] D. Vogel, On the Galois group of 2-extensions with restricted ramification, J. Reine Angew. Math., **581** (2005), 117-150.